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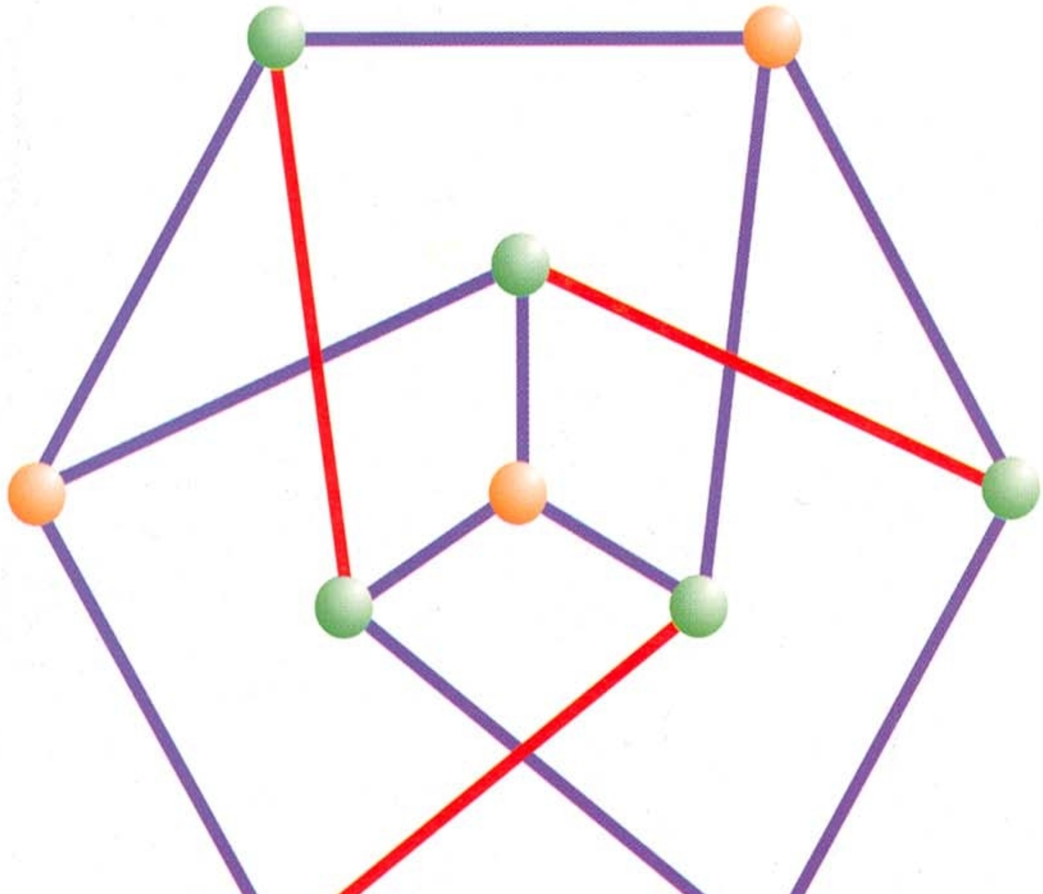
COMBINATORICS AND GRAPH THEORY



C. Vasudev

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I
Dedicated this book
to
'OM'
'Maha Kaali Maata'
&
my
"Parents"

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PREFACE

This text has been carefully designed for flexible use. It is primarily designed to provide an introduction to some fundamental concepts in COMBINATORICS AND GRAPH THEORY for post-graduate (Master of Computer Applications – M.C.A.) students.

Each topic is divided into sections of approximately the same length, and each section is divided into subsections that form natural blocks of material for teaching. Instructors can easily pace their lectures using these blocks.

All definitions and theorems in this text are stated extremely carefully so that students will appreciate the precision of language and rigor needed in mathematical sciences. Proofs are motivated and developed slowly, their steps are all carefully justified. Recursive definitions are explained and used extensively.

The writing style in this book is direct and pragmatic. Precise mathematical language is used without excessive formalism and abstraction. Care has been taken to balance the mix of notation and words in mathematical statements.

Over 1000 problems are used to illustrate concept, related to different topics, and introduce applications. In most examples, a question is first posed, then its solution is presented with appropriate details. The applications included in this text demonstrate the utility of combinatorics and Graph Theory in the solution of real world problem. This text includes applications to wide-variety of areas, including computer science and engineering.

There are over 900 exercises in the text with many different types of questions posed. There is an ample supply of straight forward exercises that develop basic skills, a large number of intermediate exercises and many challenging (problem sets) exercise sets. Problem sets are stated clearly and unambiguously, and all are carefully graded for various levels of difficulty. It will be honest on my part to accept that it is not possible to include every thing in one book.

Many people contributed directly or indirectly to the completion of this book. Thanks are due to my friends who were able to convince me that I should write this book.

I am grateful to my students, who always encouraged me and many times thanked me for writing this book.

Special thanks to my teachers, who made me realize that I can indeed write a book on "Combinatorial Mathematics". Some pulled me down, some encouraged me and some gave me constructive suggestions. I am grateful to all of them.

I specially thank to Mr. K. Krishna (president); Mr. R. Rajagopal (secretary); Mr. Lokanath (Treasurer); Dr. Rukhmangada (Chairman); Mr. Kuppaswamy (Convener); Mr. Subramanyam (Manager); K.S. Institute of Technology, Bangalore; who always encouraged me and gave me constructive suggestions. I am grateful to all of them.

I an greateful to my parents; Grandmother; maternal uncle; elder sister, elder brothers; who tolerated me all along while I devoted my time to completing this book.

I express my sincere thanks to the chairman Mr. R.K. Gupta, the Managing Director Mr. Saumya Gupta, the Marketing Manager Mr. V.R. Babu and Mr. R. Srinath and Lucknow Branch Manager Mr. L.N. Mishra of M/s New Age International (P) Limited, publishers, New Delhi, for their responsible work-done at every level in the publication of the book with high production standards.

Healthy criticism and suggestions to improve the quality and standards of the text are most welcome.

Bangalore, March 2007

C. Vasudev

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Unit-1 Counting Principles and Generating Functions

1.1 THE RULES OF SUM AND PRODUCT

1.1.1 Sum Rule. (The Principle of disjunctive Counting)

If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do one of these tasks.

In other words. If a set X is the union of disjoint non empty subsets $S_1, S_2, ..., S_n$, then $|X| = |S_1| + |S_2| + ... + |S_n|$.

1.1.2 Product Rule. (The principle of Sequential Counting)

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

In other words, If $S_1, S_2, ..., S_n$ are non empty sets, then the number of elements in the cartesian product $S_1 \times S_2 \times ...... \times S_n$ is the product $\prod_{i=1}^{n} |S_i|$. That is, $|S_1 \times S_2 \times ...... \times S_n| = \prod_{i=1}^{n} |S_i|$.

Problem 1.1. *A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from ?*

Solution. The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways.

Hence, there are 23 + 15 + 19 = 57 projects to choose from.

Problem 1.2. *Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors ?*

Solution. The first task, choosing a member of the mathematics faculty, can be done in 37 ways.

The second task, choosing a mathematics major, can be done in 83 days.

From the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Problem 1.3. *What is the value of k after the following code has been executed ?*

```
      k : = 0
   for i1 : = 1 to n1
      k : = k + 1
   for i2 : 1 to n2
      k : = k + 1
........................................
........................................
   for im : = 1 to nm
      k : = k + 1.
```

Solution. The initial value of k is zero. This block of code is made up of m different loops.

Each time a loop is traversed, 1 is added to k.

Let T_i be the task of traversing the i^{th} loop.

The task T_i can be done in n_i ways, since the i^{th} loop is traversed n_i times.

Since no two of these tasks can be done at the same time, the sum rule shows that the final value of k, which is the number of ways to do one of the tasks T_i, $i = 1, 2, ..., m$, is $n_1 + n_2 + ... + n_m$.

Problem 1.4. *In a version of the computer language BASIC, the name of a variable is a string of one or two alpha numeric characters, where upper case and lower case letters are not distinguished. (An alpha numeric character is either one of the 26 English letters or one of the 10 digits).*

Moreover, a variable name must begin with a letter and must be different from the five string of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC ?

Solution. Let V equal the number of different variable names in this version of BASIC.

Let V_1 be the number of these that are one character long and V_2 be the number of these that are two character long.

Then by the sum rule, $V = V_1 + V_2$.

Note that $V_1 = 26$, since a one-character variable name must be a letter.

Furthermore, by the product rule there are 26 . 36 strings of length two that begin with a letter and end with an alphanumeric character.

However, five of these are excluded, so that

$$V_2 = 26 \cdot 36 - 5 = 931.$$

Hence, there are $V = V_1 + V_2 = 26 + 931 = 957$ different names for variables in this version of BASIC.

Problem 1.5. *Each user on a computer system has a password, which is six to eight characters long, where each character is an upper case letter or a digit. Each password must conatin at least one digit. How many possible passwords are there ?*

Solution. Let P be the total number of possible passwords, and let P_6, P_7 and P_8 denote the number of possible passwords of length 6, 7, and 8 respectively.

By the sum rule, $P = P_6 + P_7 + P_8$

We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult.

To find P_6 it is easier to find the number of strings of upper case letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits.

By the product rule, the number of strings of six characters is 36^6 and the number of strings with no digits is 26^6.

Hence, $P_6 = 36^6 - 26^6 = 2, 176, 782, 336 - 308, 915, 776$

$= 1, 867, 866, 560.$

Similarly, it can be shown that

$P_7 = 36^7 - 26^7 = 78, 364, 164, 096 - 8, 031, 810, 176$

$= 70, 332, 353, 920.$

and $P_8 = 36^8 - 26^8 = 2, 821, 109, 907, 456 - 208, 827, 064, 576$

$= 2, 612, 282, 842, 880.$

Consequently,

$P = P_6 + P_7 + P_8$

$= 2, 684, 483, 063, 360.$

Problem 1.6. *In how many ways can we draw a heart or a spade from an ordinary deck of playing cards ? A heart or an ace ? An ace or a king ? A card numbered 2 through 10 ? A numbered card or a king ?*

Solution. Since there are 13 hearts and 13 spades we may draw a heart or a spade in
13 + 13 = 26 ways.

We may draw a heart or an ace in 13 + 3 = 16 ways, since there are only 3 aces that are not hearts.

We may draw an ace or a king in 4 + 4 = 8 ways.

These are 9 cards numbered 2 through 10 in each of 4 suits, clubs, diamonds, hearts, or spades. So we may choose a numbered card in 36 ways.

Thus, we may choose a numbered card or a king in 36 + 4 = 40 ways.

Problem 1.7. *How many ways can we get a sum of 4 or of 8 when two distinguishable dice (say one die is red and the other is white) are rolled ? How many ways can we get an even sum ?*

Solution. Let us label the outcome of a 1 on the red die and a 3 on the white die as the ordered pair (1, 3).

Then we see that the out comes (1, 3), (2, 2), and (3, 1) are the only ones whose sum is 4.

Thus, there are 3 ways to obtain the sum.

Likewise, the obtain the sum 8 from the outcomes (2, 6), (3, 5), (4, 4), (5, 3), and (6, 2).

Thus, there are 3 + 5 = 8 outcomes whose sum is 4 or 8.

The number of ways to obtain an even sum is the same as the number of ways to obtain either the sum 2, 4, 6, 8, 10 or 12.

There is 1 way to obtain the sum 2, 3 ways to obtain the sum 4, 5 ways to obtain 6, 5 ways to obtain an 8, 3 ways to obtain a 10, and 1 way to obtain a 12.

Therefore, there are 1 + 3 + 5 + 5 + 3 + 1 = 18 ways to obtain an even sum.

Problem 1.8. *How many ways can we get a sum of 8 when two indistinguishable dice are rolled ? An even sum ?*

Solution. Had the dice been distinguishable, we should obtain a sum of 8 by the outcomes (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2), but since the dice are similar, the outcomes (2, 6) and (6, 2) and, as well, (3, 5) and (5, 3) cannot be differentiated and thus we obtain the sum of 8 with the roll of two similar dice in only 3 ways. Likewise, we can get an even sum in 1 + 2 + 3 + 3 + 2 + 1 = 12 ways.

Problem 1.9. *If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student as class representative.*

Solution. Using sum rule, there are 14 + 12 = 26 ways of selecting one student (either a boy or a girl) as class representative.

Problem 1.10. *If a student is getting admission in 4 different Engineering Colleges and 5 Medical Colleges, find the number of ways of choosing one of the above colleges.*

Solution. Using sum rule, there are 4 + 5 = 9 ways of choosing one of the colleges.

Problem 1.11. *In how many ways can you get a total of six when rolling two dice ?*

Solution. The event "get a six" is the union of the mutually exclusive subevents.

A_1 : "two 3's"

A_2 : "a 2 and a 4"

A_3 : "a 1 and a 5"

Event A_1 can occur in one way, A_2 can occur in two ways (depending on which die lands 4), and A_3 can occur in two ways, so the number of ways to get a six is 1 + 2 + 2 = 5.

Problem 1.12. *The chairs of an auditorium are to be labeled with a letters and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently* ?

Solution. The procedure of labeling a chair consists of two tasks, namely, assigning one of the 26 letters and then assigning one of the 100 possible integers to the seat. The product rule shows that there are 26.100 = 2600 different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

Problem 1.13. *There are 32 micro computers in a computer center. Each micro computer has 24 ports. How many different ports to a micro computer in the center are there ?*

Soltuion. The procedure of choosing a port consists of two tasks, first picking a micro computer and then picking a port on this micro computer.

Since there are 32 ways to choose the micro computer and 24 ways to choose the port no matter which micro computer has been selected, the product rule shows that there are 32.24 = 768 ports.

Problem 1.14. *How many different bit strings are there of length seven ?*

Solution. Each of the seven bits can be chosen in two ways, since each bit is either 0 or 1.

Therefore, the product rule shows there are a total of 2^7 = 128 different bit strings of length seven.

Problem 1.15. *How many different license plates are available if each contains a sequence of three letters followed by three digits (and no sequences of letters are prohibited, even if they are abscence) ?*

Solution. There are 26 choices for each of the three letters and ten choices for each of the three digits.

Hence, by the product rule there are a total of 26 . 26 . 26 . 10 . 10 . 10 = 17,576,000 possible license plates.

Problem 1.16. *How many functions are there from a set with m elements to one with n elements ? (Counting Functions).*

Solution. A function corresponds to a choice of one of the n elements in the co-domain for each of the m elements in the domain.

Hence, by the product rule there are $n \,.\, n \,......\, n = n^m$ functions from a set with m elements to one with n elements.

For example, there are 5^3 different functions from a set with three elements to a set with 5 elements.

Problem 1.17. *How many one-to-one functions are there from a set with m elements to one with n elements ?*

(Counting one-to-one Functions)

Solution. First note when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$.

Suppose the elements in the domain are $a_1, a_2, ..., a_m$.

There are n ways to choose the value of the function at a_1.

Since the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways.

(Since the value used for a_1 cannot be used again).

In general, the value of the function at a_k can be choosen in $n - k + 1$ ways.

By the product rule, there are $n(n-1)(n-2) \,......\, (n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

For example, there are 5.4.3 = 60 one-to-one functions from a set with elements to a set with 5 elements.

Problem 1.18. *The format of telephone numbers in North America is specified by a numbering plan. A telephone number consists of 10 digits, which are split into a 3-digit area code, a 3-digit office code, and a 4-digit station code. Because of signating considerations, these are certain restrictions on some of these digits. To specify the allowable format, let X denote a digit that can take any of the values 0 through 9, let N denote a digit that can take any of the values 2 through 9, and let Y denote a digit that must be a 0 or a 1. Two numbering plans, which will be called the old plan and the new plan, will be discussed. (The old plan, in use in the 1960s, has been replaced by the new plan, but the recent rapid growth in demand for new numbers will make even this new plan obsolete). As will be shown, the new plan allows the use of more numbers.*

In the old plan, the formats of the area code, office code, and station code are NYX, NNX and XXXX, respectively, so that telephone numbers had the form NYX-NNX-XXXX. In the new plan, the *formats of these codes are NXX, NXX and XXXX, respectively, so that telephone numebrs have the form NXX-NXX-XXXX. How many different North American telephone numbers are possible under the old plan and under the new plan ? (The telephone Numbering plane)*

Solution. By the product rule, there are 8.2.10 = 160 area codes with format NYX and 8.10.10 = 800 area codes with format NXX.

Similarly, by the product rule, there are 8.8.10 = 640 office codes with formats NNX.

The product rule also shows that there are

10.10.10.10 = 10,000 station codes with format XXXX.

Consequently, applying the product rule again, it follows that under the old plan there are

160.640.10,000 = 1,024,000,000

different numbers available in North America.

Under the new plan there are

800.800.10,000 = 6,400,000,000

different numbers available.

Problem 1.19. *What is the value of k after the following code has been executed ?*

```
    k : = 0
  for i1 : = 1 to n1
  for i2 : = 1 to n2
  ....................................
  ....................................
  for im : 1 to nm
      k : = k + 1.
```

Solution. The initial value of k is zero.

Each time the nested loop is traversed, 1 is added to k.

Let T_i be the task of traversing the i^{th} loop.

Then the number of times the loop is traversed is the number of ways to do the tasks $T_1, T_2, \ldots, T_m$.

The number of ways to carry out the task T_j, $j = 1, 2, \ldots, m$, is n_j, since the j^{th} loop is traversed once for each integer i_j with $1 \le i_j \le n_j$.

By the product rule, it follows that the nested loop is traversed $n_1 n_2 \ldots n_m$ times.

Hence, the final value of k is $n_1 n_2 \ldots n_m$.

Problem 1.20. *Use the product rule to show that the number of different subsets of a finite set S is* $2^{|S|}$.

(Counting subsets of a Finite Set).

Solution. Let S be a finite set. List the elements of S in arbitrary order.

Recall that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.

Namely, a subset of S is associated with the bit string with a 1 in the i^{th} position if the i^{th} element in the list is in the subset, and a 0 in this position otherwise.

By the product rule, there are $2^{|S|}$ bit strings of length $|S|$.

Hence, $|P(S)| = 2^{|S|}$.

Problem 1.21. *Licence plates in the canadian province of Ontario consist of four letters followed by three of the digits 0 – 9 (not necessarily distinct). How many different licence plates can be made in ontario ?*

Solution. There are 26 ways in which the first letter can be chosen, 26 ways in which the second can be chosen. Similarly, for the third and fourth.

By the multiplication rule, the number of ways in which the three letters can be chosen is

$$26 \times 26 \times 26 \times 26 = 26^4.$$

By the same reasoning there are 10^3 ways in which the final three digits of an ontario licence plate can be selected and, all in all.

$26^4 \times 10^3 = 456{,}976{,}000$ different licence plates which can be manufactured by the government of ontario under its current system.

Problem 1.22. *How many numbers in the range 1000—9999 do not have any repeated digits ?*

Solution. Imagine enumerating all numbers of the desired type in the spirit of Fig. 1.1.

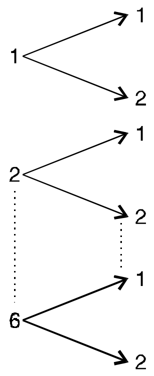


Fig. 1.1.

There are nine choices for the first digit (any of 1—9). Once this has been chosen, there remains still nine choices for the second (the chosen first digit cannot be repeated but 0 can now be used).

There are now eight choices for the third digit and seven for the fourth.

Altogether, there are $9 \times 9 \times 8 \times 7 = 4536$ possible numbers.

Problem 1.23. *How many even numbers in the range 100—999 have no repeated digits ?*

Solution. The question is equivalent to asking for the number of ways in which one can write down an even number in the range 100—999 without repeating digits.

This event can be partitioned into two mutually exclusive cases.

Case 1. The number ends in 0.

In this case, there are nine choices for the first digit (1—9) and then eight for the second (since 0 and the first digit must be excluded).

So there are $9 \times 8 = 72$ numbers of this type.

Case 2. The number does not end in 0.

Now there are four choices for the final digit (2, 4, 6 and 8), then eight choices for the first digit (0 and the last digit are excluded), and eight choices for the second digit (the first and last digits are excluded). There are $4 \times 8 \times 8 = 256$ numbers of this type.

By the addition rule, there are $72 + 256 = 328$ even numbers in the range 100—999 with no repeated digits.

Problem 1.24. *A typesetter (long ago) has before him 26 trays, one for each letter of the alphabet. Each tray contains ten copies of the same letter. In how many ways can he form a three letter "word" which requires atmost two different letters ? By "word", we mean any sequence of three letters—x pt, for example—not necessarily a real word from the dictionary. Two "ways" are different unless they use the identical pieces of type.*

Solution. The event "atmost two different letters" is comprised of two mutually exclusive cases :

Case 1. The first two letters are the same. Here, the third letter can be arbitrary, that is, any of the 258 letters which remain after the first two are set can be used.

So the number of ways in which this case can occur is

$$260 \times 9 \times (260 - 2) = 603{,}720.$$

Case 2. The first two letters are different.

In this case, the third letter must match one of the first two, so it must be one of the 18 letters remaining in the two trays used for the first two letters.

The number of ways in which this case occurs is

$$260 \times 250 \times 18 = 1{,}170{,}000.$$

By the addition rule, the number of ways to form a word using at most two different letters is

$$603{,}720 \times 1{,}170{,}000 = 1{,}773{,}720.$$

Theorem 1.1. *A set of cardinality n contains 2^n subsets (including the empty set and the entire set itself).*

Proof. There are several ways to prove this fundamental result. We present one here which uses the ideas of this section.

Given n objects $a_1, a_2, ..., a_n$, each subset corresponds to a sequence of choices. Is a_1 in the subset. Is a_2 in the subset. Finally, is a_n in the subset.

There are two answers to the first question, two for the second, and so on.

In all, there are

$$\underbrace{2 \times 2 \times ...... \times 2}_{n \text{ factors}} = 2^n$$

Ways in which all n choices can be made.

Thus, these are 2^n subsets.

Problem 1.25. *If 2 distinguishable dice are rolled, in how many ways can they fall ? If 5 distinguishable dice are rolled, how many possible outcomes are there ? How many if 100 distinguishable dice are tossed ?*

Solution. The first dice can fall (event E_1) in 6 ways and the second can fall (event E_2) in 6 ways.

Thus, there are $6 . 6 = 6^2 = 36$ outcomes when 2 dice are rolled.

Also, the third, fourth, and fifth die each have 6 possible outcomes so there are $6.6.6.6.6 = 6^5$ possible outcomes when all 5 dice are tossed.

Likewise there are 6^{100} possible outcomes when 100 dice are tossed.

Problem 1.26. *Suppose that the licence plates of a certain state require 3 English letters followed by 4 digits.*

(*a*) *How many different plates can be manufactured if repetition of letters and digits are allowed ?*

(*b*) *How many plates are possible if only the letters can be repeated ?*

(*c*) *How many are possible if only the digits can be repeated ?*

(*d*) *How many are possible if no repetitions are allowed at all ?*

Solution. (*a*) $26^3 . 10^4$ since there are 26 possibilities for each of the 3 letters and 10 possibilities for each of 4 digits.

(*b*) $26^3 .10.9.8.7$

(*c*) $26.25.24.10^4$

(*d*) 26.25.24.10.9.8.7.

Problem 1.27. (*a*) *How many 3-digit numbers can be formed using the digits 1, 3, 4, 5, 6, 8 and 9 ?*

(*b*) *How many can be formed if no digit can be repeated ?*

Solution. There are 7^3 such 3-digit numbers in

(*a*) Since each of the 3-digits can be filled with 7 possibilities. Likewise, the answer to question.

(*b*) is 7.6.5 since these are 7 possibilities for the hundreds digit but once one digit is used it is not available for the tens digit (since no digit can be repeated in this problem).

Thus, there are only 6 possibilities for the tens digit, and then for the same reason there are only 5 possibilities for the units digit.

Problem 1.28. *How many different licence plates are there that involve 1, 2 or 3 letters followed by 4 digits ?*

Solution. We can form plates with 1 letter followed by 4 digits in 26.10^4 ways, plates with 2 letters followed by 4 digits in $26^2 . 10^4$ ways, and plates with 3 letters followed by 4 digits in $26^3 . 10^4$ ways.

These separate events are mutually exclusive so we can apply the sum rule to conclude that there are $26 . 10^4 + 26^2 . 10^4 + 26^3 . 10^4 = (26 + 26^2 + 26^3)\ 10^4$ plates with 1, 2 or 3 letters followed by 4-digits.

Problem 1.29. *How many different plates are there that involve 1, 2 or 3 letters followed by 1, 2, 3 or 4 digits ?*

Solution. We see that there are $(26 + 26^2 + 26^3)$ 10 ways to form plates of 1, 2 or 3 letters followed by 1 digit. $(26 + 26^2 + 26^3)\ 10^2$ plates of 1, 2 or 3 letters followed by 2 digits,

$(26 + 26^2 + 26^3)10^3$ plates of 1, 2 or 3 letters followed by 3 digits, and $(26 + 26^2 + 26^3)10^4$ plates of 1, 2 or 3 letters followed by 4 digits.

Thus, we can apply the sum rule to conclude that there are

$$(26 + 26^2 + 26^3)10 + (26 + 26^2 + 26^3)10^2 + (26 + 26^2 + 26^3)10^3 + (26 + 26^2 + 26^3)10^4$$
$$= (26 + 26^2 + 26^3)(10 + 10^2 + 10^3 + 10^4)$$

ways to form plates of 1, 2 or 3 letters followed by 1, 2, 3, or 4 digits.

Problem 1.30. (*a*) *How many 2 digit or 3-digit numbers can be formed using the digits 1, 3, 4, 5, 6, 8 and 9 if no repetition is allowed ?*

(*b*) *How many numbers can be formed using the digits 1, 3, 4, 5, 6, 8 and 9 if no repitition are allowed ?*

Solution. (*a*) There are 7.6.5, three-digit numbers possible. Likewise, we can apply the product rule to see that these are 7.6 possible 2-digit numbers.

Hence, these are 7.6 + 7.6.5 possible two-digit or three-digit numbers.

(*b*) The number of digits are not specified in this problem so we can form one-digit numbers, two-digit numbers, or three digit numbers, etc.

But since no repetitions are allowed and we have only the 7 integers to work with, the maximum number of digits would have to be 7.

Applying the product rule, we see that we may form 7 one-digit numbers, 7.6 = 42 two digit numbers 7.6.5 three digit numbers, 7.6.5.4 four digit numbers, 7.6.5.4.3 five digit nubmers, 7.6.5.4.3.2 six-digit numbers, and 7.6.5.4.3.2.1 seven-digit numbers.

The events of forming one-digit numbers, two digit numbers, three digit numbers, etc., are mutually exclusive events so we apply the sum rule to see that there are 7 + 7.6 + 7.6.5 + 7.6.5.4 + 7.6.5.4.3 + 7.6.5.4.3.2 + 7.6 + 7.6.5.4.3.2.1 different numbers we can form under the restrictions of this problem.

Problem 1.31. *How many three-digit numbers are there which are even and have no repeated digits ? (Here we are using all digits 0 through 9).*

Solution. For a number to be even it must end in 0, 2, 4, 6 or 8. There are two cases to consider.

First, suppose that the number ends in 0 ; then there are 9 possibilities for the first digit and 8 possibilities for the second since no digit can be repeated. Hence there are 9.8 three-digit numbers that end in 0. Now suppose the number does not end in 0.

Then there are 4 choices for the last digit (2, 4, 6 or 8).

When this digit is specified, then there are only 8 possibilities for the first digit, since the number cannot begin with 0. Finally, there are 8 choices for the second digit and therefore there are 8.8.4 numbers that do not end in 0. Accordingly since these two cases are mutually exclusive, the sum rule gives 9.8 + 8.8.4 even three-digit numbers with no repeated digits.

Problem 1.32. *Suppose that we draw a card from a deck of 52 cards and replace it before the next draw. In how many ways can 10 cards be drawn so that the tenth card is a repetition of a previous draw ?*

Solution. First we count the number of ways, we can draw 10 cards so that the 10^{th} card is not a repetition.

First choose what the 10^{th} card will be. This can be done in 52 ways.

If the first 9 draws are different from this, then each of the 9 draws can be chosen from 51 cards.

Thus there are 51^9 ways to draw the first 9 cards different from the 10^{th} card.

Hence, there are $(51^9)(52)$ ways to choose 10 cards with the 10^{th} card different from any of the previous 9 draws.

Hence, there are $52^{10} – (51^9)(52)$ ways to draw 10 cards where the 10^{th} is a repetition since there are 52^{10} ways to draw 10 cards with replacements.

Problem 1.33. *In how many ways can 10 people be seated in a row so that a certain pair of them are not next to each other ?*

Solution. There are 10 ! ways of seating all 10 people. Thus, by indirect counting, we need only count the number of ways of seating the 10 people where the certain pair of people (say, A and B) are seated next to each other. If we treat the pair AB as one entity, then there are 9 total entities to arrange in 9 ! ways.

But A and B can be seated next to each other in 2 different orders, namely AB and BA.

Thus, there are (2)(9 !) ways of seating all 10 people where A and B are next to each other.

The answer to our problem then is 10 ! – (2)(9 !).

Problem 1.34. *In how many ways can one select two books from different subjects from among six distinct computer science books, three distinct mathematics books, and two distinct chemistry books ?*

Solution. Using product rule one can select two books from different subjects as follows :

(*i*) one from computer science and one from mathematics in 6.3 = 18 ways.

(*ii*) one from computer science and one from chemistry in 6.2 = 12 ways.

(*iii*) one from mathematics and one from chemistry in 3.2 = 6 ways.

Since these sets of selections are pairwise disjoint one can use the sum rule to get the required number of ways which is 18 + 12 + 6 = 36.

Problem 1.35. *For a set of six true or false questions, find the number of ways of answering all questions.*

Solution. The number of ways of answering the first question is 2.

The second question can also be answered in 2 ways and similarly for other 4 questions.

Hence, the total number of ways of answering all the questions is $2^6 = 64$.

Problem 1.36. *Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats ?*

Solution. The first person has a choice of 5 seats and can sit in any one of those 5 seats.

So, there are 5 ways of occupying the first seat. The second person has a choice of 4 seats.

Similarly, the third person has a choice of 3 seats. Hence, the required number of ways in which all the three persons can seat is $5 \times 4 \times 3 = 60$.

Problem 1.37. *There are four roads from city X to Y and five roads from city Y to Z, find*

(*i*) *how many ways is it possible to travel from city X to city Z via city Y.*

(*ii*) *how different round trip routes are there from city X to Y to Z to Y and back to X.*

Solution. (*i*) In going from city X to Y, any of the 4 roads may be taken.

In going from city Y to Z, any of the 5 roads may be taken.

So by the product rule, there are 5.4 = 20 ways to travel from city X to Z via city Y.

(*ii*) A round trip journey can be performed in the following four ways.

(1) From city X to Y

(2) From city Y to Z

(3) From city Z to Y

(4) From city Y to X.

(1) Can be performed 4 ways, 5 ways to perform 2, 5 ways to perform 3 and 4 ways to perform 4. By product rule, there are 4.5.5.4 = 400 round trip routes.

Problem 1.38. *How many bit strings of length eight either start with a 1 bit or end with the two bits 00 ?*

Solution. The first task, constructing a bit string of length eight beginning with a 1 bit, can be done in $2^7 = 128$ ways.

This follows by the product rule, since the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways.

The second task, constructing a bit string of length eight ending with the two bits 00, can be done in $2^6 = 64$ ways.

This follows by the product rule, since each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

Both tasks, constructing a bit string of length eight that begins with a 1 and ends with 00, can be done in $2^5 = 32$ ways.

This follows by the product rule, since the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way.

Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to do either the first task or the second task, equals 128 + 64 – 32 = 160.

1.2 PERMUTATIONS

Suppose that, we have three letters *a, b* and *c*. Then, all possible arrangements of any two letters out of these three letters can be enumerated as : *ab, ac, bc, ba, ca* and *cb.* If we make an arrangement of all three letters out of these three, then we have *abc, acb, bac, bca, cab* and *cba* as possible arrangements.

Each of the distinct order of arrangements of a given set of distinct objects, taking some or all of them at a time (with or without repetition), is called a **permutation** of the objects. The total number of permutations of n distinct objects, taken n at a time ($r \le n$), is equal to the total nubmer of ways of placing n objects in r boxes. This is denoted as $P(n, r)$ or ^nP_r. This number is equal to $n(n-1)$ ......
$(n-r+1)$.

Notation ^nP_r :

We know that ^nP_r is the number of permutation of n distinct objects taken r at a time, and this is equal to $n(n-1)(n-2)$ ...... $(n-r+1)$.

Therefore, $$^nP_r = \frac{n(n-1)(n-2) \ldots\ldots (n-r+1) * (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Therefore $^{n}P_r = \dfrac{n!}{(n-r)!}$

$$\left({}^{1}P_1 = \frac{1!}{(1-1)!} = \frac{1}{0!} = 1 \quad \because\ 0! = 1 \right)$$

The number of permutations of n distinct objects, taken n at a time, is given by $^{n}P_n = n!$.

- Permutation of objects when all are not distinct, in this case, the number of permutations is given by

$$x = \frac{n!}{p!\,q!\,r!}$$

- The number of distinguishable permutations that can be formed from a collection of n objects, taken all n at a time, in which the first object appears k_1 times, the second object k_2 times, and so on, is given by

$$\frac{n!}{k_1!\,k_2!\,k_3!\,......\,k_r!}$$

  where $k_1 + k_2 + ... + k_r = n$.
- The number of permutations of n distinct objects, taken r at a time, when repetitions are allowed, is given by n^r.
- The number of ways of arranging objects under some restriction = the number of arrangements of the same number of object without restriction – the number of arrangements of the same number of arrangements with the opposite restriction.
- Generating function for permutation :

  The coefficient of $\dfrac{x^r}{r!}$ in a polynomial $P(x)$ is $^{n}P_r$ and $^{n}P_r$ is the number of permutations of n objects taken r at a time.

Theorem 1.2. *The number of r-permutations of a set with n distinct elements is*

$$P(n, r) = n(n-1)(n-2)\ ......\ (n-r+1) = {}^{n}P_r = \frac{n!}{(n-r)!}$$

Proof. The first element of the permutation can be chosen in n ways, since there are n elements in the set. There are $n - 1$ ways to choose the second element of the permutation, since there are $n - 1$ elements left in the set after using the element picked for the first position. Similarly, there are $n - 2$ ways to choose the third element and so on, until there are exactly $n - (r - 1) = n - r + 1$ ways to choose the r^{th} element.

Consequently, by the product rule, there are

$$n(n-1)(n-2)\ ......\ (n-r+1)$$

r-permutations of the set.

It follows that

$$P(n, r) = n(n-1)(n-2) \ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}.$$

Example 1. Let A be {1, 2, 3, 4}. Then the sequences 124, 421, 341 and 243 are same permutations of A taken 3 at a time. The sequences 12, 43, 31, 24, and 21 are examples of different permutations of A taken two at a time.

The total number of permutations of A taken three at a time is $4P_3$ or 4.3.2 or 24.

The total number of permutations of A taken two at a time is $4P_2$ or 4.3 or 12.

Note. When $r = n$, we are counting the distinct arrangements of the elements of A, with $|A| = n$, into sequences of length n. Such a sequence is simply called a permutation of A.

The number of permutations of A is thus ${}^nP_n$ or

$n \,.\, (n-1) \,.\, (n-2) \ldots\ldots 2.1$, if $n \geq 1$. This number is also written n ! and is read n factorial.

Both ${}^nP_r$ and n ! are built in functions on many calculators.

Example 2. Let A be $\{a, b, c\}$. Then the possible permutations of A are the sequences *abc, acb, bac, bca, cab and cba*.

For convenience, we define 0 ! to be 1. Then for every $n \geq 0$ the number of permutations of n objects is n !.

If $n \geq 1$ and $1 \leq r \leq n$.

Example 3. Let A consist of all 52 cards in an ordinary deck of playing cards. Suppose that this deck is shuffled and a hand of five cards is dealt. A list of cards in this hand, in the order in which they were dealt, is a permutation of A taken five at a time. Examples would include AH, 3D, 5C, 2H, JS, 2H, 3H, 5H, QH, KD ; JH, JD, JS, 4H, 4C ; and 3D, 2H, AH, JS, 5C.

Note that the first hand and last hands are the same, but they represent different permutations since they were dealt in a different order.

The number permutations of A taken five at a time is

$${}^{52}P_5 = \frac{52!}{47!} \text{ or } 52.51.50.49.48 \text{ or } 311,875,200.$$

This is the number of five-card hands that can be dealt if we consider the order in which they were dealt.

Exercise 4. *The number of distinguishable "words" that can be formed from the letters of MISSISSIPPI is* $\frac{11!}{1!\,4!\,4!\,2!}$ *or 34, 650.*

Theorem 1.3. *Suppose that two tasks T_1 and T_2 are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in n_1n_2 ways. (Multiplication principle of counting).*

Proof. Each choice of a method of performing T_1 will result in a different way of performing the task sequence. There are n_1 such methods, and for each of these we may choose n_2 ways of performing T_2.

Thus, in all, there will be n_1n_2 ways of performing the sequence T_1T_2. See Fig. 1.2 for the case where n_1 is 3 and n_2 is 4.

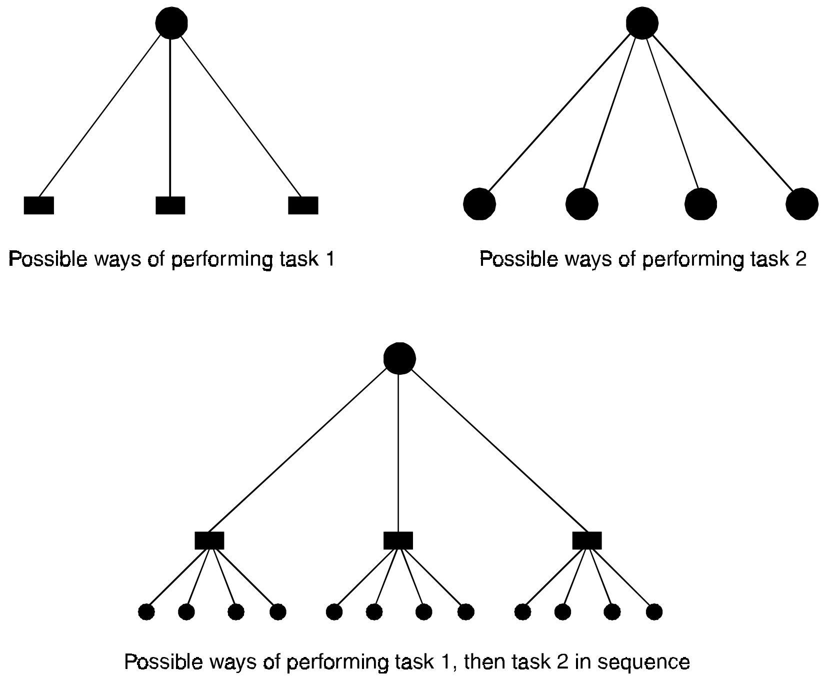


Fig. 1.2.

Theorem 1.4. *Suppose that tasks T_1, T_2, ... T_k are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, and for each of these n_1n_2 ways of performing T_1T_2 in sequence, T_3 can be performed in n_3 ways, and so on, then the sequence T_1T_2 ...... T_k can be performed in exactly n_1n_2 ......n_k ways.*

(Extended Multiplication principle of counting).

Theorem 1.5. *Given natural numbers r and n with $r \le n$, the number of ways to place r marbles of different colours into n numbered boxes, at most one marble to a box, is P(n, r).*

Notice that $\underbrace{n(n-1)(n-2)\ ......\ (n-r+1)}_{P(n,\, r)}\ \underbrace{(n-r)(n-r-1)\ ......\ (3)(2)(1)}_{(n-r)\,!} = n\,!$

Thus, $P(n, r) = \dfrac{n\,!}{(n-r)\,!}$ a formula which holds also for $r = 0$ and $r = n$ because $P(n, 0) = 1$ and $0\,! = 1$.

Theorem 1.6. *The number of permutations of n symbols is n !. The number of r-permutations of n symbols is P(n, r).*

Problem 1.39. *How many pairs of dance partners can be selected from a group of 12 women and 20 men ?*

Solution. The first woman can be paired with any of 20 men, the second woman with any of the remaining 19 men, the third with any of the remaining 18, and so on.

These are 20.19.18......9 = P(20, 12) possible couples.

Problem 1.40. *There are 7 ! = 5040 ways in which seven people can form a line. In how many ways can seven people form a circle ?*

Solution. A circle is determined by the order of the people to the right of any one of the individuals, say Eric. There are six possibilities for the person in Eric's right, then five possibilities for the next person, four for the next, and so on. The number of possible circles is 6 ! = 720.

Problem 1.41. *In how many ways can the letters of the English alphabet be arranged so that there are exactly ten letters between a and z ?*

Solution. There are P(24, 10) arrangements of the letters of the alphabet (excluding *a* and *z*) taken ten at a time, and hence 2.P(24, 10) strings of 12 letters, each beginning and ending with an *a* and *az*(either letter coming first in a string).

For each of these strings, there are 15 ! ways to arrange the 14 remaining letters and the string.

So there are altogether 2.P(24, 10) . 15 ! arrangements of the desired type.

Problem 1.42. *In how many ways can ten adults and five children stand in a line so that no two children are next to each other ?*

Solution. Imagine a line of ten adults named

A, B, ..., *j*, XD XJ XH XC XI XE XB XA XG XF X, then X's representing the 11 possible locations for the children.

For each such line, the first child can be positioned in any of the 11 spots, the second child in any of the remaining 10, and so on.

Hence, the children can be positioned in 11.10.9.8.7 = P(11, 5) ways.

For each such positioning, there are 10 ! ways of ordering the adults A, ..., J, so, by the multiplication rule, the number of lines of adults and children is 10 ! P(11, 5).

Problem 1.43. *A label identifier, for a computer system, consists of one letter followed by three digits. If repetitions are allowed, how many distinct label identifiers are possible ?*

Solution. There are 26 possibilities for the beginning letter and there are 10 possibilities for each of three digits.

Thus, by the extended multiplication principle, there are 26 × 10 × 10 × 10 or 26,000 possible label identifiers.

Problem 1.44. *Let A be a set with n elements. How many subsets does A have ?*

Solution. We know that, each subset of A is determined by its characteristic function, and if A has *n* elements, this function may be described as an array of 0's and 1's having length *n*.

The first element of the array can be filled in two ways (with a 0 or a 1), and this is true for all succeeding elements as well.

Thus, by the extended multiplication principle, there are

$$\underbrace{2\,.\,2\ldots\ldots 2}_{n \text{ factors}} = 2^n$$

ways of filling the array, and therefore 2^n subsets of A.

Problem 1.45. *How many different sequences, each of length r, can be formed using elements from A if*

(*a*) *elements in the sequence may be repeated ?*

(*b*) *all elements in the sequence must be distinct ?*

Solution. First we note that any sequence of length r can be formed by filling r boxes in order from left to right with elements of A.

In case (*a*), we may use copies of elements of A.

			...		
box 1	box 2	box 3		box $r-1$	box r

Let T_1 be the task "fill box 1", let T_2 be the task "fill box 2", and so on. Then combined task T_1T_2 ... T_r represents the formation of the sequence.

Case (*a*). T_1 can be accomplished in n ways, since we may copy any element of A for the first position of the sequence. The same is true for each of the tasks T_2, T_2, ... T_r.

Then by the extended multiplication principle, the number of sequences that can be formed is

$$\underbrace{n \, . \, n \, . \ldots\ldots n}_{r \text{ factors}} = n^r$$

Now we consider case (*b*), here also T_1 can be performed in n ways, since any element of A can be chosen for the first position. Which ever element is chosen, only $(n - 1)$ elements remain, so that T_2 can be performed in $(n - 1)$ ways, and so on, until finally T_r can be performed in

$$n - (r - 1) \quad \text{or} \quad (n - r + 1) \text{ ways.}$$

Thus, by the extended principle of multiplication, a sequence of r distinct elements from A can be formed in $n(n - 1)(n - 2)$ ...... $(n - r + 1)$ ways.

Problem 46. *How many three-letter "words" can be formed from letters in the set {a, b, y, z}. If repeated letters are allowed ?*

Solution. Here n is 4 and r is 3, so the number of such words is 4^3 or 64.

Problem 47. *How many "words" of three distinct letters can be formed from the letters of the word MAST ?*

Solution. The number is ${}^4P_3 = \dfrac{4\,!}{(4-3)\,!}$ or $\dfrac{4\,!}{1\,!}$ or 24.

Problem 48. *How many distinguishable permutations of the letters in the word BANANA are there ?*

Solution. We begin by tagging the A's and N's in order to distinguish between them temporarily.

For the letters B, A_1, N_1, A_2, N_2, A_3, there are 6 ! or 720 permutations. Some of these permutations are identical except for the order in which the N's appear.

For example, $A_1A_2A_3BN_1N_2$ and $A_1A_2A_3BN_2N_1$.

In fact, the 720 permutations can be listed in pairs whose members differ only in the order of the two N's. This means that if the tags are dropped from the N's only $\frac{720}{2}$ or 360 distinguishable permutations remain.

Reasoning in a similar way we see that these can be gruoped in groups of 3 ! or 6 that differ only in the order of the three A's.

For example, one group of 6 consists of $BNNA_1A_2A_3$, $BNNA_1A_3A_2$, $BNNA_2A_1A_3$, $BNNA_2A_3A_1$, $BNNA_3A_1A_2$, $BNNA_3A_2A_1$.

Dropping the tags would change these 6 into the single permutation BNNAAA.

Thus, there are $\frac{360}{6}$ or 60 distinguishable permutations of the letters of BANANA.

Problem 49. *A man, a woman, a boy, a girl, a dog, and a cat are walking down a long and winding road one after the other.*

(*a*) *In how many ways can this happen ?*

(*b*) *In how many ways can this happen if the dog comes first ?*

(*c*) *In how many ways can this happen if the dog immediately follows the boy ?*

(*d*) *In how many ways can this happen if the dog (and only the dog) is between the man and the boy ?*

Solution. (*a*) There are 6 ! = 720 ways for six creatures to form a line.

(*b*) If the dog comes first, the others can form = 5 ! lines behind.

(*c*) If the dog immediately follows the boy, then the dog-boy pair should be thought of as a single object to be put into a line with four others.

There are 5 ! = 120 such lines.

(*d*) If the man, dog, and boy appear in this order, then thinking of man-dog-boy as a single object to be put into a line with three others, we see that there are 4 ! possible lines. Similarly, there are 4 ! lines in which the boy, dog, and man appear in this order.

So, by the addition rule, there are 4 ! + 4 ! = 48 lines in which the dog (and only the dog) is between the man and the boy.

Problem 1.50. *In how many ways can ten adults and five children stand in a circle so that no two children are next to each other ?*

Solution. Arrange the adults into a circle in one of 9 ! ways. There are then 10 locations for the first child, 9 for the second, 8 for the third, 7 for the fourth, and 6 for the fifth. The answer is 9 ! (10.9.8.7.6) = 9 ! P(10, 5).

Problem 1.51. *Find (a) P(5, 3), P(4, 4) and P(7, 2)*

(*b*) $\frac{20!}{17!}$, $\frac{100!}{98!}$ *and P(7, 0).*

Solution. (*a*) P(5, 3) = 5.4.3 = 60,

P(4, 4) = 4.3.2.1 = 24,

P(7, 2) = 7.6 = 42.

(*b*) $\dfrac{20\,!}{17\,!} = 20.19.18 = 6840,$

$\dfrac{100\,!}{98\,!} = 100.99 = 9900$

$P(7, 0) = \dfrac{7\,!}{7\,!} = 1.$

Problem 1.52. *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

Solution. A card can be send to any one friend by any one of the three servants.

Let us take the tasks of sending cards to six friends as

$$T_1, T_2, T_3, T_4, T_5 \text{ and } T_6.$$

Each of the tasks can be completed in three distinct ways according to the number of servants to carry the cards.

Thus, by the multiplication principle of counting the tasks $T_1T_2T_3T_4T_5T_6$ can be performed in $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$ ways.

Problem 1.53. *How many numbers of three digits can be formed with the digits 1, 2, 3, 4 and 5 if the digits in the same number are not repeated ? How many such numbers are possible between 100 and 10,000 ?*

Solution. Here we have to find the number of permutations of 5 distinct objects (digits) taken 3 at a time.

This is given by $^5P_3 = 5 \times 4 \times 3 = 60$.

The numbers between 100 and 10,000 are numbers of three digits and of four digits.

The total number of three digits numbers, formed with the given digits, is calculated above and it is equal to 60.

Similarly, the total number of four digits numbers, formed with 1, 2, 3, 4 and 5, is given by 0.

$$^5P_4 = 5 \times 4 \times 3 \times 2 = 120.$$

Thus, the required number is 60 + 120 = 180.

Problem 1.54. *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made ?*

Solution. There are five arms say T_1, T_2, T_3, T_4 and T_5.

Each arm can be in any one of the four positions.

For each of the position of arm T_1, there are four possible positions for the second arm T_2, for each of the possible positions for T_1T_2, there are four possible positions for the third arm T_3 and so on.

Thus, by the multiplication principle of counting the total possible positions for $T_1T_2T_3T_4T_5$ is $4 \times 4 \times 4 \times 4 \times 4 = 1024$.

Since each distinct position is a distinct signal, so total number of possible signals is 1024 including the signal (meaningless) corresponding to the situation when all arms are in rest position.

Therefore, the total number of signals that can be generated is 1024 – 1 = 1023.

Problem 1.55. *Find the number of positive integers greater than a million that can be formed with the digits 2, 3, 0, 3, 4, 2 and 3.*

Solution. The numbers greater than a million must be of 7 digits.

In the given set of digits, 2 appear twice, 3 appear thrice and all others are distinct.

Thus, the total number of seven digit numbers that can be formed with given digits is $\frac{7\,!}{2\,!\,3\,!} = 420$.

The set of these 420 positive integers, include some numbers which begin with 0.

Clearly, these nubmers are less than a million and they must not be counted in our answer.

The number of such numbers is given by the permutations of 6 non-zero digits and is equal to $\frac{6\,!}{2\,!\,3\,!} = 60$.

Therefore, the number of positive integers greater than a million that can be formed with given digits is equal to 420 – 60 = 360.

Problem 1.56. *How many distinguishable permutations of the letters in the word BANANA are there ?*

Solution. The word "BANANA" has 6 letters. All the letters are not distinct. Let us use subscript to distinguish them temporarily.

Let the letters be B, A_1, N_1, A_2, N_2, A_3.

Thus, the number of permutations are 6 ! = 720.

Some of the permutations are identical like $A_1A_2A_3BN_1N_2$ and $A_1A_2A_3BN_2N_1$ except the order in which the N's appear.

This means that if we drop the subscripts, the total number of permutations will be $\frac{6\,!}{2\,!} = 360$.

Similarly, if we drop subscript with A's then total number of distinguishable permutations are $\frac{360}{3\,!} = 60$.

Problem 1.57. *There are 10 stalls for animals in an exhibition. Three animals ; lion, pussycat and horse are to be exhibited. Animals of each kind are not less than 10 in number. What is the possible number of ways of arranging the exhibition.*

Solution. There are three types of animals and 10 stalls.

One stall can be filled by any of the three animals. Once the first stall is filled, the second stall can be filled again in three ways by placing any of the three animals in it.

We have to fill 10 such stalls and number of each animal is greater than equal to 10, so we have $3^{10} = 59049$ ways to fill the stalls.

Thus, we can arrange the exhibition in 59049 ways.

Problem 1.58. *In how many ways can 10 different examination papers be scheduled so that*

(*i*) *the best and the worst always come together*

(*ii*) *the best and the worst never come together ?*

Solution. (*i*) Let us consider the best and the worst paper together and consider them as one object.

We have, now, 9 objects (papers).

These 9 objects can be arranged in 9 ! ways.

And in each of these 9 ! arrangements, the best and the worst papers can be arranged in 2 ! ways.

Therefore, the number of ways in which the 10 papers can be scheduled in this situation

$$= 2\,! * 9\,! = 725760.$$

(*ii*) Without any restriction, the 10 papers can be scheduled in 10 ! ways.

We have just calculated in part (*i*) that total number of ways in which the 10 papers can be scheduled so that the best and the worst always come together = 725760.

Therefore, the number of ways of scheduling 10 papers so that the best and the worst never come together = 10 ! – 725760 = 3628800 – 725760 = 2903040.

Problem 1.59. *In how many different ways can 5 men and 5 women sit around a table, if*

(*i*) *there is no restriction*

(*ii*) *no two women sit together ?*

Solution. The problem is related to circular permutation of 10 objects (5 men and 5 women). If there is no restriction then the number of permutations is (10 – 1) ! = 9 ! = 362880.

Notice here the difference in arrangement between clockwise and anticlockwise.

In the second case, there is a restriction that no two women are allowed to sit side by side.

To meet this restriction each woman should occupy a sit between two men.

The number of ways five men can sit around a table is 4 ! = 24.

Once these five men have sat on alternate chairs, the five women can occupy the 5 empty chairs in 5 ! ways.

Thus, total number of ways, in this case, will be 24 * 5 ! = 24 * 120 = 2880.

Problem 1.60. *Find the sum of all the four-digit numbers that can be formed with the digits 3, 2, 3 and 4.*

Solution. One thing is worth noticing here that a four-digit number so formed does not conatin a repeated digit except the digit 3.

This is implied from the question, because if it were not so, 3 should not have been repeated in the list of the digits.

To find the sum of the four-digit numbers formed with 3, 2, 3 and 4, we have to calculate the sum of digits at unit place in all such numbers.

The sum of the digits at ten, hundred and thousand place will be the same, only theirs place value will change.

The number of four-digit numbers in which 2 appears at unit place is determined by the number of permutations of digits 3, 3 and 4 to fill ten, hundred and thousand place. And this is $\frac{3\,!}{2\,!} = 3$.

Similarly, the number of four-digit numbers in which 3 appears at unit place is 3 ! = 6.

The number of four-digit numbers in which 4 appears at unit place is $\frac{3\,!}{2\,!} = 3$.

Therefore, sum of the digits in the unit place of all the numbers = $3 \times 2 + 6 \times 3 + 3 \times 4 = 36$.

As stated above, the sum of the digits in all such numbers at ten, hundred and thousand places is 36 each.

Thus, the sum of all such numbers

$$= 36 \times 1000 + 36 \times 100 + 36 \times 10 + 36$$

$$= 39996.$$

Problem 1.61. *We are asked to make slips for all numbers up to five-digit. Since the digits 0, 1, 6, 8 and 9 can be read as 0, 1, 9, 8 and 6 when they are read upside down, there are pairs of numbers that can share same slip if the slips are read upside down or right sideup (e.g. 89166 and 99168).*

Find the number of slips required for all five-digit numbers.

Solution. We have 10 digits. We have to make all five digit numbers.

The total such numbers is equal to 10^5.

Here we have to make slips for these many numbers. The numbers made of digits 0, 1, 6, 8 and 9 can be read upside down or right side up.

And, there are 5^5 many such five-digit numbers (all those five-digit numbers made of digits 0, 1, 6, 8 and 9).

Out of these 5^5 many numbers, however, there are some numbers that read the same either upside down or right side up.

For example, 91816, and there are 3×5^2 such numbers (center place filled with 0, 1 or 8).

Thus, there are $5^5 - 3 \times 5^2$ numbers that can be read upside down or right side up.

And, for these numbers we need only

$$\frac{1}{2}(5^5 - 3 * 5^2) \text{ number of slips.}$$

Therefore, number of slips required to be made is $10^5 - \frac{1}{2}(5^5 - 3 * 5^2)$.

Problem 1.62. *How many binary sequences of r-bits long have even number of 1's ?*

Solution. There will be 2^r possible binary sequences of r-bits long.

This can be verified by the permutation of objects when repetitions are allowed.

There are r places and two objects (0 and 1).

The first place can be filled in 2 ways, for each of these, the second place can be filled in 2 ways, so we have 2×2 ways to fill first two places. Extending the sequence upto the r^{th} steps, we have 2^r possible arrangements and hence 2^r possible binary sequences.

We can now make pairs of binary sequences in such a way that two sequences differ only at r^{th} place.

There will 2^{r-1} such pairs, and in each pair one sequence will have even number of 1's.

Thus, number of binary seqeunces of r bits long having even number of 1's = 2^{r-1}.

Problem 1.63. *How many different words can be made from letters of the word 'committee'.*

Solution. The word committee contains letters *c*, *o* and *i* once and *m, t* and *e* twice.

When a word is formed from these letters, a letter may appear at the most the number of time it appear in the word committee or not at all.

So generating function for *c, o* and *i* is given by $(1 + x)$ each, whereas, for *c, m* and *e* is given by $\left(1+x+\dfrac{x^2}{2!}\right)$ each.

Thus, generating function for the problem is then given by

$$(1+x)^3\left(1+x+\frac{x^2}{2!}\right)^3 = (1+3x+3x^2+x^3)\left(1+x+\frac{x^2}{2!}\right)^3$$

If words are to be formed taking all the letters at once, then the numbers of such words is given by the coefficient of $\dfrac{x^9}{9!}$ and this is equal to $\dfrac{9!}{2!2!2!}$.

Problem 1.64. *A fair six-sided die is tossed four times and the numbers shown are recorded in a sequence. How many different sequences are there ?*

Solution. Let us assume, here, that each toss is an object and the number appearing on the face of a die is the number of times it occurs. Thus each object occurs at least once and at the most 6 times.

Also the order of appearing of different 1's (conceptually) to make it 2 or 3 or 4 or 5 or 6 is fixed and is one and only one way.

The generating function for first toss is given as $x + x^2 + x^3 + \ldots\ldots + x^6$.

The sum of all the coefficients, here, is 6 and number of possible sequences of numbers, one face of a die in one toss, is 6 only, die is tossed four times,

Therefore, generating function for the problem is

$$\{x + x^2 + x^3 + \ldots\ldots + x^6\}^4.$$

The number of sequence is the sum of coefficients of all the terms in this generating function. This is equal to 6^4.

Problem 1.65. *Find the generating function, also called enumerator, for permutations of n objects with the following specified conditions :*

(*a*) *each object occurs at the most twice*

(*b*) *each object occurs at least twice*

(*c*) *each object occurs at least once and at the most k times.*

Solution. (*a*) Each object occurs at the most twice implies that an object may occur 0, 1 or 2 times. The exponential generating function for an object under this condition is given as

$$1+x+\frac{x^2}{2!}.$$

There are n objects, so the generating function for the problem is written as

$$\left(1+x+\frac{x^2}{2!}\right)^n.$$

(b) In this case, an objcet occurs at least twice. This implies that an object may appear 2 or more times. The exponential generating function for an object under this condition is $\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+......$

Therefore, for the problem dealing with n objects, the generating function can be written as

$$\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+......\right)^n.$$

(c) Here, each object occurs at least once and at the most k times.

That is to say that an object may occur 1 or 2 or 3 or ...... or k times.

The exponential generating function for an object under this condition is

$$x+\frac{x^2}{2!}+\frac{x^3}{3!}+......+\frac{x^k}{k!}$$

Since the problem for n objects, the generating function for the problem is written as

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+......+\frac{x^k}{k!}\right)^n.$$

Problem 1.66. *How many ways are there to select a first prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest ?*

Solution. Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations iof a set of 100 elements.

Consequently, the answer is

P(100, 3) = 100.99.98 = 970,200.

Problem 1.67. *Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities ?*

Solution. The number of possible paths between the cities is the number of permutations of seven elements, since the first city is determined, but the remaining seven can be ordered arbitrarily.

Consequently, there are 7 ! = 7.6.5.4.3.2.1 = 5040 ways for the saleswoman to choose her tour.

If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths.

Problem 1.68. *Suppose that there are eight runners in a race. The winner receives a gold medal the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties ?*

Solution. The number of different ways to award the medals is the number of 3-permutations of a set with eight elements.

Hence, there are P(8, 3) = 8.7.6 = 336 possible ways to award the medals.

Problem 1.69. *How many permutations of the letters ABCDEFGH contain the string ABC ?*

Solution. Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G and H.

Because these six objects can occurs in any order, there are 6 ! = 720 permutations of the letters ABCDEFGH in which ABC occurs as a block.

1.3 COMBINATIONS

Let A be a set with $| A | = n$, and let $1 \leq r \leq n$. Then the number of combinations of the elements of A, taken r at a time that is the number of r-element subsets of A is

$$\frac{n\,!}{r\,!(n-r)\,!}.$$

The number of combinations of A, taken r at a time, does not depend on A, but only on n and r. This number is often written nC_r and is called the number of combinations of n objects taken r at a time.

We have $\quad {}_nC_r = \dfrac{n\,!}{r\,!(n-r)\,!}.$

• Suppose k selections are to be made from n items without regard to order and that repeats are allowed, assuming at least k copies of each of the n items. The number of ways these selections can be made is $(n + k - 1)\ C_k$.

Problem 1.70. *Show that* ${}_nC_r = {}_nC_{n-r}$.

Solution. We have ${}^nC_r = \dfrac{n\,!}{r\,!(n-r)\,!} = \dfrac{n\,!}{(n-(n-r))\,!(n-r)\,!} = {}_nC_{n-r}.$

Problem 1.71. *Compute the number of distinct five-card hands than can be dealt from a deck of 52 cards.*

Solution. This number is ${}_{52}C_5$ because the order in which the cards were dealt is irrelevant.

$${}_{52}C_5 = \frac{52\,!}{5\,!47\,!} \quad \text{or} \quad 2{,}598{,}960.$$

Problem 1.72. *In how many ways can a prize winner choose three CDs from the top ten list if repeats are allowed ?*

Solution. Here n is 10 and k is 3.

There are $(10 + 3 - 1)C_3$ or ${}_{12}C_3$ ways to make the selections.

The prize winner can make the selection in 220 ways.

Problem 1.73. *How many different seven-person committees can be formed each containing three women from an available set of 20 women and four men from an available set of 30 men ?*

Solution. In this case a committee can be formed by performing the following two tasks in succession :

Task 1 : Choose three women from the set of 20 women.

Task 2 : Choose four men from the set of 30 men.

Here order does not matter in the individual choices, so we are merely counting the number of possible subsets.

Thus task 1 can be performed in ${}_{20}C_3$ or 1140 ways and task 2 can be performed in ${}_{30}C_4$ or 27, 405 ways.

By the multiplication principle, there are (1140) (27405) or 31,241,700 different committees.

Theorem 1.7. *Let n and r be integers with $n \geq 0$ and $0 \leq r \leq n$. The number of ways to choose r objects from n is* $\binom{n}{r}$.

Proof. If $r = 0$, the result is true, because there is just one way to choose 0 objects (do nothing !), while $\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$, because $0! = 1$.

Thus, we may assume that $r \geq 1$ and hence $n \geq 1$.

Let N be the number we are seeking, that is, there are N ways to choose r objects from the n given objects.

Notice that for each way of choosing r objects, there are $r!$ ways to order them.

By the multiplication rule, the number of r-permutations of n objects (which we know is $P(n, r)$) is the number of ways to choose r objects multiplied by $r!$, the number of ways to order the r objects.

$$P(n, r) = N \times r!$$

Therefore, $N = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$.

Corollary. The number of r-combinations of n objects is $\binom{n}{r}$.

Problem 1.74. *Wandana is going to toss a coin eight times. In how many ways can she get five heads and three tails ?*

Solution. Wandana might get a string of five heads followed by three tails (denote this possibility HHHHHTTT), or a string of three tails followed by five heads, TTTHHHHH, or the sequence HTHHTHTH, and so on.

The number of such sequences is the number of ways of selecting five occasions (from the eight) on which the heads should arise or, equivalently, the number of ways of selecting the three occasions on which tails should come up.

The answer is $\binom{8}{5} = \binom{8}{3} = 56$.

Problem 1.75. *Explain why* $\binom{n}{r} = \binom{n}{n-r}$.

Solution. Suppose we have n white marbles and we wish to point r of them black.

Choosing the r marbles is equivalent to choosing the $n - r$ marbles which are to remain white.

Thus, each choice of r marbles from n corresponds to a choice of the remaining $n - r$, so the numbers of choices are the same.

There are $\binom{n}{r}$ and $\binom{n}{n-r}$, respectively.

Problem 1.76. *Mr. Hiscock has ten children but his car holds only five people (including driver). When he goes to the circus, in how many ways can he select four children to accompany him ?*

Solution. The question involves choosing, not order.

$$\text{There are } \binom{10}{4} = \frac{10!}{4!6!} = 210 \text{ different ways.}$$

Theorem 1.8. *C(n, r) = C(n, n – r).*

Proof. (Algebraic) :

We have that $C(n, n - r) = \dfrac{n!}{[n-(n-r)]!(n-r)!}$

Simplifying, we find that

$$C(n, n - r) = \frac{n!}{r!(n-r)!},$$ which in turn is equal to $C(n, r)$, as was to be shown.

Theorem 1.9. *C(n, k) = C(n – 1, k) + C(n – 1, k – 1) for n > k > 0.*

Proof. (Combinatorial)

Let A be a set of cardinality n and let k be an integer such that $0 < k < n$.

There are $C(n, k)$ different subsets of A of cardinality k.

Let y be an element of A. Every subset of A either includes y or does not.

There are $C(n - 1, k)$ different subsets of A of cardinality k that do not include y.

(We form such a subset from the $(n - 1)$ elements of A that are not equal to y).

There are $C(n - 1, k - 1)$ ways to choose a subset of A of cardinality k that includes y.

(To form such a subset, we must choose $(k - 1)$ elements in addition to y from the $(n - 1)$ elements in A that are not equal to y).

Adding, we find that there are $C(n - 1, k) + C(n - 1, k - 1)$ ways in which to choose a subset of cardinality k from A.

Thus, $C(n, k) = C(n - 1, k - 1) + C(n - 1, k)$.

Algebraic Proof. By definition, we have C(n – 1, k) + C(n – 1, k – 1)

$$= \frac{(n-1)!}{[(n-1)-k]!\,k!} + \frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!}$$

Simplifying, finding a common denominator, and adding, we find that the sum is equal to

$$\frac{(n-k)\,(n-1)!+(n-1)!\,k}{(n-k)!\,k!}.$$

Using the distributive law, we find that the latter sum is in turn equal to $\frac{n!}{(n-k)!\,k!}$, as was to be proved.

- In general, combinations of n items taking r at a time where r lies between a and b where $1 \le a$ and $b \le n$ is given by the sum

  $${}^nC_a + {}^nC_{a+1} + {}^nC_{a+2} + \ldots\ldots + {}^nC_b.$$

- The number of ways of distributing $p + q$ different objects between two distinguishable groups in such a ways that one group gets P objects and other gets q objects is given by $\frac{(p+q)!}{p!\,q!}$.
- If the two groups are indistinguishable in the above case then, the number of ways of distribution is given by

  $$\frac{(p+q)!}{2!\,p!\,q!}.$$

- In general, if n different objects are to be distributed among m distinguishable groups containing $P_1, P_2, P_3, \ldots\ldots P_m$ objects, where $P_1 + P_2 + \ldots\ldots + P_m = n$.

  The number of ways in which this takes can be completed is given by

  $$\frac{n!}{P_1!\,P_2!\,P_3!\ldots\ldots P_m!}.$$

- In the previous result if the m groups are indistinguishable, the number of ways of distribution is given by $\frac{n!}{m!\,P_1!\,P_2!\,P_3!\ldots\ldots P_m!}$.

Problem 1.77. *How many committees of five people can be chosen from 20 men and 12 women.*

(*a*) *if exactly three men must be on each committee ?*

(*b*) *if at least four women must be on each committee ?*

Solution. (*a*) We must choose three men from 20 and then two women from 12.

The answer is $\binom{20}{3}\binom{12}{2}$ = 1140(66) = 75,240.

(*b*) We calculate the case of four women and five women separately and add the results (using the addition rule).

The answer is $\binom{12}{4}\binom{20}{1} + \binom{12}{5}\binom{20}{0}$ = 495(20) + 792 = 10,692.

Problem 1.78. *In how many ways can 20 students out of a class of 32 be chosen to attend class on a late Thursday afternoon (and take notes for the others) if*

(*a*) *Paul refuses to go to class ?*

(*b*) *Michelle insists on going ?*

(*c*) *Jim and Michelle insist on going ?*

(*d*) *either Jim or Michelle (or both) go to class ?*

(*e*) *just one of Jim and Michelle attend ?*

(*f*) *Paul and Michelle refuse to attend class together ?*

Solution. (*a*) The answer is $\binom{31}{20}$ = 84,672,315, since, in effect, it is necessary to select 20 students from the 31 students excluding Paul.

(*b*) Now the number of possibilities is $\binom{31}{19}$ = 141,120,525 since 19 students must be chosen from 31.

(*c*) The answer is $\binom{30}{18}$ = 86,493,225, it being necessary to choose the remaining 18 students from a group of 30.

(*d*) Let J be the set of classes of 20 which contain Jim and M the set of classes of 20 which contain Michelle. The question asks for $|J \cup M|$.

Using the principle of Inclusion-Exclusion, we obtain,

$$|J \cup M| = |J| + |M| - |J \cap M| = \binom{31}{19} + \binom{31}{19} - \binom{30}{18} = 195{,}747{,}825.$$

(*e*) Using the formula $|J \oplus M| = |J| + |M| - 2\,|J \cap M|$

We obtain, $\binom{31}{19} + \binom{31}{19} - 2\binom{30}{18}$ = 109,254,600.

(*f*) The number of classes containing Paul and Michelle is $\binom{30}{18}$. So the number which do not contain both is

$$\binom{32}{20} - \binom{30}{18} = 139{,}299{,}615.$$

Problem 1.79. *A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job) ?*

Solution. The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter.

The number of such combinations is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30.29.28.27.26.25}{6.5.4.3.2.1} = 593{,}775.$$

Problem 1.80. *How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consists of three faculty members from the mathematics department and four from the computer science department, if there are nine faculty members of the mathematics department and 11 of the computer science department ?*

Solution. By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements.

The number of ways to select the committee is

$$C(9, 3) . C(11, 4) = \frac{9!}{3!6!} . \frac{11!}{4!7!}$$

$$= 84.330 = 27{,}720.$$

Problem 1.81. *How many ways are there to select five players from a 10-members tennis team to make a trip to a match at another school ?*

Solution. The answer is given by the number of 5-combinations of a set with ten elements.

The number of such combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Problem 1.82. *How many bit strings of length n contain exactly r 1 s ?*

Solution. The positions of r 1s in a bit string of length n form an r-combination of the set

$$\{1, 2, 3, ..., n\}$$

Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s.

Problem 1.83. *A person has 8 children of them he takes 3 at a time to a circus. He does not take the same three children twice to the circus. How many times he will have to go to circus to ensure that every three children have seen the circus together ? In this case find the number of times a particular child has visited the circus.*

Solution. Here we have to find the number of combinations of 8 children taken 3 at a time.

Note that order of selection of child is not important in this case.

This selection can be made in $^8C_3$ ways.

We can make $^8C_3 = 56$ distinct groups of three children, and for each such group, the person will have to go to circus once.

Therefore, the person will have to visit circus 56 times.

A particular child goes to circus with every possible pair of two children out of remaining 7 children.

Number of such possible pair is $^7C_2 = 21$.

Therefore, a particular child goes to circus 21 times.

Problem 1.84. *In a party of 30 people, each shakes hand with others. How many handshakes took place in the party ?*

Solution. In a normal case, a handshake involves two persons. This case is of counting 2-elements subsets of a set containing 30 elements.

And this count is $^{30}C_2 = 30 \times \frac{29}{2} = 435$.

Problem 1.85. *From 8 men and 4 women and team of 5 is to be formed. In how many ways can this be done so as to include at least one woman ?*

Solution. This is a case of restricted combination.

Total number of persons = 8 men + 4 women = 12 persons.

A team of 5 has to be made, and this can be made in $^{12}C_5$ ways.

This count includes the case of teams containing all five men (*i.e.,* no women in the team) which is equal to $^8C_5$.

Thus, number of ways in which the specified team can be selected = $^{12}C_5 - {}^8C_5$

$$= \frac{12!}{5!7!} - \frac{8!}{5!3!}$$

$$= \frac{12\times11\times10\times9\times8}{5\times4\times3\times2} - \frac{8\times7\times6}{3\times2}$$

$$= 792 - 56 = 736.$$

Problem 1.86. *There are 10 points in a 2-D plane. Four of these are co-linear. Find the number of different straight lines that can be drawn by joining these points.*

Solution. Any two points are always co-linear. So, a line can be drawn between any two points.

If there are three non-co-linear points (a single line cannot be drawn joining all these three points).

We can draw $3 = {}^3C_2$ distinct lines.

A triangle is an example of this. Thus, we can draw $^{10}C_2$ distinct lines joining 10 points. Out of these 10 points, 4 are co-linear. So $^4C_2$ lines will be same and we consider them as one line.

Therefore, actual number of lines that can be drawn = $^{10}C_2 - {}^4C_2 + 1 = 45 - 6 + 1 = 40$.

Problem 1.87. *What is the number of diagonals that can be drawn in a polygon of n sides ?*

Solution. A polygon having n sides has n vertices.

A diagonal is a line between two points, which are not adjacent to each other.

The total number of line that can be drawn in a polygon of n vertices = $^nC_2$.

There are already n sides (lines) which are not diagonal,

the remaining lines will be diagonals.

Therefore, the number of diagonals that can be drawn in a polygon of side n is equal to

$$^{n}C_2 - n = \frac{n!}{2!(n-2)!} - n = \frac{n(n-1)}{2} - n$$

$$= \frac{n^2 - n - 2n}{2} = \frac{n(n-3)}{2}.$$

Problem 1.88. *If three dice are rolled, and we make a set of numbers shown on the three dice. How many different sets are possible.*

Solution. Rolling three dice is equivalent to selecting three numbers from the list of six numbers 1, 2, 3, 4, 5 and 6 with repetitions allowed. Because sequence 111, 121 etc. are possible.

Thus the different possible combinations is

$$^{6+3-1}C_3 = {^8C_3} = 56.$$

Problem 1.89. *A bookstore allows the recipient of a gift coupon to choose 6 books from the combined list of 10 best-selling fiction books and 10 best-selling non-fiction books. In how many different ways can the selection of 6 books be made ?*

Solution. The number of different types of books is 10 + 10 = 20.

A gift coupon recipient may select any 6 books, possible 6 copies of a single book.

This is a case of selection of 6 objects from 20 objects with repetitions allowed.

The number of ways the selection can be made is

$$^{20+6-1}C_6 = {^{25}C_6} = 177100.$$

Problem 1.90. *In an election the number of candidates is one more than the number of vacancies. If a voter can vote in 30 different ways, find the number of candidates.*

Solution. Let the number of candidates be x. An elector may vote for any one or, any two or, any three up to maximum of any $x - 1$ candidates from total of 4, because number of vacancies is $x - 1$ only.

Therefore, number of ways in which an elector can cast his vote is

$^{x}C_1 + {^xC_2} + {^xC_3} + \ldots\ldots + {^xC_{x-1}}$ and this value is given to be 30.

Thus, $^{x}C_1 + {^xC_2} + {^xC_3} + \ldots\ldots + {^xC_{x-1}} = 30.$

or $^{x}C_0 + {^xC_1} + {^xC_2} + {^xC_3} + \ldots\ldots + {^xC_{x-1}} + {^xC_x} - {^xC_0} - {^xC_x} = 30.$

$\Rightarrow \quad 2^x - 2 = 30$

$\Rightarrow \quad x = 5.$

Therefore, the number of candidates is 5.

Problem 1.91. *In an election, there are four candidates contesting for three vacancies, an elector can vote for any number of candidates not exceeding the number of vacancies. In how many ways can one cast his votes ?*

Solution. An elector may vote for any one or, any two, or any three candidates out of total 4.

Therefore, an elector may vote in $^4C_1 + {}^4C_2 + {}^4C_3$

$= 4 + 6 + 4 = 14$ different possible ways.

Problem 1.92. *Find the total number of selections of at least one red ball from 4 red balls and 3 green balls, if*

(*a*) *the balls of the same colour are different*

(*b*) *the balls of the same colour are identical.*

Solution. (*a*) From 4 different red balls and 3 different green balls, we have to find number of selections taking at least one red ball and any number of (including 0) 3 green balls.

The total number of ways of selecting at least one red ball from 4 different red balls

$= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15.$

Corresponding to each of these selections, the number of ways of selecting green balls

$= {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 8.$

Therefore, total number of different ways of selection

$= 15 \times 8 = 120.$

Problem 1.93. *In an examination a candidate has to pass in each of the 5 papers. How many different combinations of papers are there so that a student may fail ?*

Solution. For a student to pass the examination, he/she will have to pass in each of the five papers.

To fail, a student may fail in any one or, in any two or so on including in all the five papers.

Thus, a student may fail in as many as

$^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5 - 1 = 31$ ways.

Problem 1.94. *In how many ways can a pack of 52 cards be equally divided into four groups ? If the cards are to be distributed equally among four players, then find the number of ways of this distribution.*

Solution. **First part :**

When 52 cards are distributed equally among four groups, each group contains 13 cards.

Since groups are indistinguishable, the number of ways of distribution is given by

$$\frac{52!}{4!\,13!\,13!\,13!}$$

Second part :

Here four groups (players) are distinguishable, thus number of ways of distribution is given by

$$\frac{52!}{13!\,13!\,13!\,13!}.$$

Problem 1.95. *A library has 5 black books, 4 red books and 3 yellow books, all with different titles. How many distinguishable ways can a student take home 6 books, 2 of each colour ?*

Solution. In this case, the books of a particular colour are distinguishable by their title.

Each book can be either selected or not, giving the possible number of selection for each book as 0 or 1.

This can be written as, in polynomial, form, as $1 + x$.

Thus, the generating function for 5 black books is $(1 + x)^5$.

Similarly, the generating function for 4 red books is $(1 + y)^4$ and for 3 yellow books is $(1 + z)^3$.

The count of number of ways 6 books, 2 of each colour, can be selected, we take coefficient of $x^2y^2z^2$ in the generating of function

$$f(x) = (1 + x)^5(1 + y)^4(1 + z)^3.$$

The coefficient of $x^2y^2z^2$ is ${}^5C_2\,{}^4C_2\,{}^3C_2$

$$= 10 \times 6 \times 3 = 180.$$

Problem 1.96. *A library has 5 indistinguishable black books,*

4 indistinguishable red books and 3 indistinguishable yellow books. In how many distinguishable ways can a student take home (*a*) *6 books ?* (*b*) *6 books taking atleast 1 of each colour ?* (*c*) *6 books taking 2 of each colour ?*

Solution. Here all books of a particular color are indistinguishable. A black (or red or yellow) book can be selected or, not selected.

If selected then maximum number of black books that can be selected is 5,

i.e., possible ways of selections for black books are 0, 1, 2, 3, 4 or 5.

Similarly, the possible ways of selections for red books are 0, 1, 2, 3 or 4 and for yellow books are 0, 1, 2 or 3.

If we take three variables x, y and z for the numbers of black, red and yellow books selected by a student, respectively then the number of solutions to the equations $x + y + z = 6$ where $0 \le x \le 5$,

$0 \le y \le 4$ and $0 \le z \le 3$ is the required number of ways in which a student can take home 6 books.

The generating function for x is $(1 + x + x^2 + x^3 + x^4 + x^5)$, for y is $(1 + y + y^2 + y^3 + y^4)$ and for z is $(1 + z + z^2 + z^3)$.

If we replace y and z by x, we get the generating function for the above problem as

$$f(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3)$$

The coefficient of x^6 in $f(x)$ is the required number, and this number is 18.

In the second part, at least one book of each colour has to be selected, so generating function $f(x)$ is given as

$$f(x) = (x + x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3 + x^4)(x + x^2 + x^3)$$

The coefficient of x^6 in $f(x)$ is the required number, and this number is 9.

In the third part, two books of each colour are to be selected.

So generating function $f(x)$ is given as

$$f(x) = x^2 * x^2 * x^2.$$

The coefficient of x^6 in $f(x)$ is the required number, and this number is 1.

Problem 1.97. *In how many ways can one choose n pieces of fruit, assuming there are an infinitely large number of apples, bananas, oranges and pears, and he(she) wants an even number of apples, an odd number of bananas, not more than 4 oranges and atleast two pears ?*

Solution. The generating function for the selection of apple can be written as $(1 + x^2 + x^4 + x^6 + ...)$, for the selection of bananas can be written as $(x + x^3 + x^5 + x^7 + ...)$, for the selection of oranges can be written as $(1 + x + x^2 + x^3 + x^4)$ and for pears it can be written as $(x^2 + x^3 + x^4 + ...)$.

Therefore, the generating function for this problem is

$$f(x) = (1 + x^2 + x^4 + x^6 + ...)(x + x^3 + x^5 + x^7 + ...)(1 + x + x^2 + x^3 + x^4)(x^2 + x^3 + x^4 + ...)$$

Problem 1.98. *In how many ways 2n + 1 items can be distributed among 3 persons so that the sum of the number of items received by any two persons should exceed the number of items received by the other ?*

Solution. Let us take three variables *x, y* and *z* for the number of items received by the three persons. Then number of ways $2n + 1$ items can be distributed among three persons is equal to the number of positive integer solutions to the equation

$$x + y + z = 2n + 1 \text{ where } 1 \le x, y, z \le n.$$

To ensure that the sum of the two variables must exceed the third one, it is important to define the range of values for each variable as above.

Therefore, the generating function for the problem is written as

$$\begin{aligned} f(x) &= (x + x^2 + x^3 + ... + x^n)(x + x^2 + x^3 + ... + x^n)(x + x^2 + x^3 + ... + x^n) \\ &= (x + x^2 + x^3 + ... + x^n)^3 \\ &= \left\{\frac{x(1-x^n)}{1-x}\right\}^3 \\ &= x^3(1 - 3x^n + 3x^{2n} - x^{3n})(1 - x)^{-3}. \end{aligned}$$

In this function $f(x)$, the coefficient of x^{2n+1} is the count we are looking for.

And, this coefficient is equal to the coefficient of x^{2n-2} in $(1 - 3x^n + 3x^{2n} - x^{3n})(1 - x)^{-3}$.

This value is ${}^{3+2n-2-1}C_{2n-2} - 3 \times {}^{3+n-2-1}C_{n-2}$

or,
$$\begin{aligned} {}^{2n}C_{2n-2} - 3 \times {}^{n}C_{n-2} &= \frac{2n(2n-1)}{2\,!} - 3\,\frac{n(n-1)}{2\,!} \\ &= \frac{n}{2}(4n - 2 - 3n + 3) \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

Problem 1.99. *A valid password consists of seven symbols. Symbols are chosen from digits and Roman capital alphabets. The first symbol of the password must be a Roman capital alphabet. How many different passwords are possible ?*

Solution. Number of symbols = 26 + 10 = 36

The first place of seven characters password can be chosen in 26 ways.

The remaining 6 places can be filled in 36 ways each *i.e.,* the second symbol for a password can be chosen in 36 ways, and for each of this, the third place can be filled in 36 ways and so on.

Thus, the number of different possible passwords

$$= 26 \times (36)^6 = 56596340736.$$

1.4 PERMUTATIONS AND COMBINATIONS WITH REPETITIONS

Factorials :

Frequently it is useful to have a simple notation for products such as

4.3.2.1, 6.5.4.3.2.1 or 7.6.5.4.

For each positive integer we define $n\,! = n(n-1)(n-2)\ ......\ 3.2.1$

= the product of all integers from 1 to n.

Also define $0\,! = 1$, note that $1\,! = 1$

Thus, $4! = 4.3.2.1$, $6\,! = 6.5.4.3.2.1$

and $$7.6.5.4 = \frac{7\,!}{3\,!} = \frac{7.6.5.4.3.2.1}{3.2.1}$$

We read $n\,!$ as "n factorial".

It is true that $4\,! = 24$ and $6\,! = 720$ but frequently we leave our answers in factorial form rather than evaluating the factorials.

Nevertheless, the relation $n\,! = n[(n-1)\,!]$ enables us to compute the values of $n\,!$ for small n fairly quickly.

For example :

$0\,! = 1$,	$1\,! = 1$,	$2\,! = 2$
$3\,! = 6$,	$4\,! = 24$,	$5\,! = 120$
$6\,! = 720$,	$7\,! = 5040$,	$8\,! = 40320$
$9\,! = 362880$,	$10\,! = 3628800$,	$11\,! = 39916800$.

Theorem 1.10. *Enumerating r-permutations without repetitions*

$$P(n, r) = n(n-1)\ ......\ (n-r+1) = \frac{n\,!}{(n-r)\,!}\ .$$

Proof. Since there are n distinct objects, the first position of an r-permutation may be filled in n ways.

This done, the second position can be filled in $n-1$ ways since no repetitions are allowed and there are $n-1$ objects left to choose from.

The third can be filled in $n-2$ ways and so on until the r^{th} position is filled in $n-r+1$ ways . (See figure below).

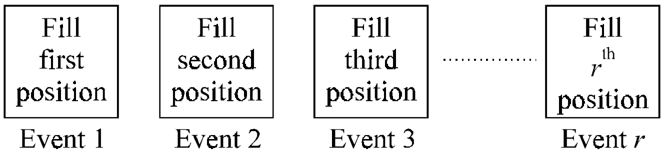


Fig. 1.3.

By applying the product rule, we conclude that

$$P(n, r) = n(n-1)(n-2)\ldots\ldots(n-r+1)$$

From the definition of factorials, it follows that

$$P(n, r) = \frac{n!}{(n-r)!}.$$

When $r = n$, this formula becomes

$$P(n, n) = \frac{n!}{0!} = n!.$$

Corollary

There are n ! permutations of n distinct objects.

Example 1. *There are* 3 ! = 6 permutations of $\{a, b, c\}$. There are 4 ! = 24 permutations of $\{a, b, c, d\}$. The number of 2-permutations of $\{a, b, c, d, e\}$ is $P(5, 2) = \frac{5!}{(5.2)!} = 5.4 = 20$.

The number of 5-letters words using the letters a, b, c, d and e at most once is P(5, 5) = 120.

Example 2. *There are P(10, 4) = 5040, 4-digit numbers that contain no repeated digits since each such number is just an arrangement of four of the digits 0, 1, 2, 3, ..., 9*

(*leading zeros are allowed*)

There are P(26, 3), 3-letters words formed from the English alphabet with no repeated letters.

Thus, there are P(26, 3) P(10, 4) license places formed by 3-distinct letters followed by 4-distinct digits.

Theorem 1.11. *Enumerating r-permutations with unlimited repetitions*

$$U(n, r) = n^r.$$

Proof. Each of the r positions can be filled in n ways and so by the product rule, $U(n, r) = n^r$.

Theorem 1.12. (Enumerating n-Permutations with constrained repetitions)

$$P(n\,;\, q_1, \ldots\ldots, q_t) = \frac{n!}{q_1!\,q_2!\ldots\ldots q_t!}$$

$$= C(n, q_1)\, C(n-q_1, q_2)\, C(n-q_1-q_2, q_3) \ldots\ldots C(n-q_1-q_2 \ldots\ldots - q_{t-1}, q_t).$$

Proof. Let $x = P(n\,;\, q_1, q_2, \ldots\ldots, q_t)$

If the q_1 a_1's were all different there would be $(q_1 !)\,x$ permutations since each old permutation would give rise of q_1 ! new permutations corresponding to the number of ways of arranging the q_1 distinct objects in a row. If the q_2 a_2's were all replaced by distinct objects, then by similar reasoning there would be $(q_2 !)(q_1 !\, x)$ permutations.

If we repeat this procedure until all the objects are distinct. We will have $(q_t !) \ldots\ldots (q_2 !)(q_1 !)x$ permutations.

However, we know that there are n ! permutations of n distinct objects.

Equating these two quantities and solving for x gives the first equality of the theorem.

The second equality is obtained as follows :

First choose the q_1 positions for the a_1's ; then from the remaining $n - q_1$ positions, choose q_2 positions for the a_2's and so on.

Note that at the last we will have left $n - q_1 - q_2 \ldots\ldots - q_{t-1} = q_t$

Positions to fill with the q_t a_t's, so

$$C(n - q_1 - q_2 \ldots\ldots - q_{t-1}, q_t) = C(q_t, q_t).$$

The last equality of the theorem follows because both numbers represent the same number of permutations.

Example :

The number of arrangements of letters in the word TALLAHASSEE is

$$P(11 ; 3, 2, 2, 2, 1, 1) = \frac{11!}{3!2!2!2!1!1!}$$

Since this equals the number of permutations of {3.A, 2.E, 2.L, 2.S, 1.H, 1.T}.

The number of arrangements of these letters that begin with T and end with E is $\frac{9!}{3!1!2!2!1!}$.

Theorem 1.13. Enumerating r-combinations wtih unlimited repetitions

V(n, r) = the number of r-combinations of n distinct objects wtih unlimited repetitions

= the number of nonnegative integral solutions to $x_1 + x_2 + \ldots\ldots + x_n = r$

= the number of ways of distributing r similar balls into n numbered boxes

= the number of binary numbers with $n - 1$ one's and zeros.

$= C(n - 1 + r, r) = C(n - 1 + r, n - 1)$

$$= \frac{(n + r - 1)!}{[r!(n - 1)!]}.$$

Remark. The number of r-combinations of $\{\infty . a_1, \infty . a_2, \ldots\ldots, \infty . a_n\}$ is the same as the number of r-combinations of $\{r . a_1, r . a_2, \ldots\ldots, r . a_n\}$.

Theorem 1.14. The number of integral solutions of $x_1 + x_2 + \ldots\ldots + x_n = r$ where each $x_i > 0$

= the number of ways of distributing r similar balls into n numbered boxes with at least one ball in each box

$= C(n - 1 + (r - n), r - n) = C(r - 1, r - n)$

$= C(r - 1, n - 1)$.

Likewise, suppose that $r_1, r_2, \ldots\ldots, r_n$ are integers. Then the number of integral solutions of $x_1 + x_2 + \ldots\ldots + x_n = r$, where $x_1 \geq r_1, x_2 \geq r_2, \ldots\ldots$, and $x_n \geq r_n$.

= the number of ways of distributing r similar balls into n numbered boxes where there are atleast r_1 balls in the first box, atleast r_2 balls in the second box, ..., and atleast r_n balls in the n^{th} box.

$= C(n - 1 + r - r_1 - r_2 - \ldots\ldots - r_n, r - r_1 - r_2 - \ldots\ldots - r_n)$

$= C(n - 1 + r - r_1 - r_2 - \ldots\ldots - r_n, n - 1)$.

Problem 1.100. *The results of 50 football games (win, lose or, tie) are to be predicted. How many different forecasts can contain exactly 28 correct results ?*

Solution. Choose 28 correct results C(50, 28) ways.

Each of the remaining 22 games has 2 wrong forecasts. Thus, there are C(50, 28). 2^{22} forecasts with exactly 28 correct predictions.

Problem 1.101. *A telegraph can transmit two different signals : a dot and a dash. What length of these symbols is needed to encode the 26 letters of the English alphabet and the ten digits 0, 1, ..., 9 ?*

Solution. Since there are two choices for each character, the number of different sequences of length k is 2^k.

The number of non trivial sequences of length n or less is

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2.$$

If $n = 4$ this total is 30, which is enough to encode the letters of the English alphabet, but not enough to also encode the digits.

To encode the digits we need to allow sequences of length upto 5 for then there are possibly $2^{5+1} - 2 = 62$ total sequences.

Problem 1.102. *How many 10-digit binary numbers are there with exactly six 1's ?*

Solution. The key to this problem is that we can specify a binary number by choosing the subset of 6 positions where the 1's go (or the subset of 4 positions for the 0's)

Thus, there are C(10, 6) = C(10, 4) = 210 such binary numbers.

Problem 1.103. *There are 21 consonants and 5 vowels in the English alphabet. Consider only 8-letter words with 3 different vowels and 5 different consonants.*

(*a*) *How many such words can be formed ?*

(*b*) *How many such words contain the letter a ?*

(*c*) *How many contain the letters a and b ?*

(*d*) *How many contain the letters b and c ?*

(*e*) *How many contain the letters a, b and c ?*

(*f*) *How many begin with a and end with b ?*

(*g*) *How many begin with b and end with c ?*

Solution. (*a*) C(5, 3) C(21, 5) 8 !

(Choose the vowels, choose the consonants, and then arrange the 8 letters.)

(*b*) C(4, 2) C(21, 5) 8 !

(*c*) C(4, 2) C(20, 4) 8 !

(*d*) C(5, 3) C(19, 3) 8 !

(*e*) C(4, 2) C(19, 3) 8 !

(*f*) C(4, 2) C(20, 4) 6 !

(*g*) C(5, 3) C(19, 3) 6 !.

Problem 1.104. *There are 30 females and 35 males in the junior class while there are 25 females and 20 male in the senior class.In how many ways can a committee of 10 be chosen so that there exactly 5 females and 3 juniors on the committee ?*

Solution. Let us draw a chart illustrating the possible male-female and junior senior constitution of the committee.

Juniors		*Seniors*		
Female	*Male*	*Female*	*Male*	*Number of ways of selecting*
0	3	5	2	C(30, 0) C(35, 3) C(25, 5) C(20, 2)
1	2	4	3	C(30, 1) C(35, 2) C(25, 4) C(20, 3)
2	1	3	4	C(30, 2) C(35, 1) C(25, 3) C(20, 4)
3	0	2	5	C(30, 3) C(35, 0) C(25, 2) C(20, 5)

Thus, the total number of ways is the sum of the terms in the last column:

C(30, 0) C(35, 3) C(25, 5) C(20, 2) + C(30, 1) C(35, 2) C(25, 4) C(20, 3) + C(30, 2) C(35, 1) C(25, 3) C(20, 4) + C(30, 3) C(35, 0) C(25, 2) C(20, 5).

Problem 1.105. *There are 25 true or false questions on an examination. How many different ways can a student do the examination if he or she can also choose to leave the answer blank ?*

Solution. 3^{25}.

Problem 1.106. (*a*) *How many different outcomes are possible by tossing 10 similar coins ?*

(*b*) *How many different outcomes are possible from tossing 10 similar dice ?*

(*c*) *How many ways can 20 similar books be placed on 5 different shelves ?*

(*d*) *Out of a large supply of pennies, nickels, dimes, and quarter, in how many ways can 10 coins be selected ?*

(*e*) *How many ways are there to fill a box with a dozen dough nuts chosen from 8 different varieties of dough nuts ?*

Solution. (*a*) This is the same as placing 10 similar balls into two boxes labeled "heads" and "tails".

$$C(2 - 1 + 10, 10) = C(11, 10) = 11.$$

(*b*) This is the same as placing 10 similar balls into 6 numbered boxes.

Therefore there are C(15, 10) = 3,003 possibilities.

(*c*) C(5 – 1 + 20, 20) = C(24, 20).

(*d*) C(4 – 1 + 10, 10) = C(13, 10) since this is equivalent to placing 10 similar balls in 4 numbered boxes labeled "pennies", "nickels", "dimes", and "quarters".

(*e*) First, we observe that relative positions in the box are immaterial so that order does not count.

Therefore, this is a combination problem

Secondly, a box might consist of a dozen of one variety of doughnut, so that we see that this problem allows unlimited repetitions.

The answer then is C(8 – 1 + 12, 12) = C(19, 12).

Problem 1.107. *In how many ways can the letters of the word attention be rearranged ?*

Solution. The word attention has nine letters, three of one kind, two of another, and four other different letters.

The number of rearrangements of this word is $\frac{9\,!}{3\,!\,2\,!}$ = 30,240.

Problem 1.108. *Suppose there are ten players to be assigned to three teams. The Xtreme, the Maniax, and the Enforcers. The Xtreme and the Maniax are to receive four players each and the Enforcers are to receive two. In how many ways can this be done ?*

Solution. The assignment of players is accomplished by choosing four players from ten for the Xtreme, then choosing four players from the remaining six for the Maniax, and assigning the remaining two players to the Enforcers. The number of possible teams is $\binom{10}{4} \times \binom{6}{4}$ = 210(15) = 3150.

Problem 1.109. *In how many ways can 14 men be divided into six named teams, two with three players and four with two ? In how many ways can 14 men be divided into two unnamed teams of three and four teams of two ?*

Solution. If the teams are named, the answer is

$$\binom{14}{3} \times \binom{11}{3} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} = 364(165)(28)(15)(6)$$

$$= 151,\, 351,\, 200.$$

This number is 2 ! 4 ! = 48 times the number of divisions into unnamed teams of sizes 3, 3, 2, 2, 2 and 2, because for each division into unnamed teams, there are two ways to name the teams of three and 4 ! ways to name the teams of two.

The answer in the case of unnamed teams is

$$\frac{151,\, 351,\, 200}{48} = 3,\, 153,\, 150.$$

Problem 1.110. *In how many ways can the letters of the word REARRANGE be rearranged ?*

Solution. There are $\frac{9\,!}{3\,!\,2\,!\,2\,!}$ = 15,120 rearrangements of the letters of the word REARRANGE.

Problem 1.111. (*a*) *In how many ways can the letters of the word easy be rearranged ?*

(*b*) *In how many ways can the letters of the word ease be rearranged ?*

Solution. (*a*) The question just asks for the number of permutations of four different letters.

The answer is 4 ! = 24.

(*b*) This is a slightly different Problem because of the repeated *e's*, when the first and last letters of easy are interchanged, we get two different arrangements of the four letters, but when the first and last letters of ease are interchanged we get the same word.

To see how to count the ways in which the four letters of ease can be arranged, we imagine the list of all these arrangements.

s e a e

a e e s

e e s a

⋮

Pretending for a moment that the two e's are different (say one is a capital E), then each "word" in this list will produce two different arrangements of the letters easE (See Fig. (1.3(a)))

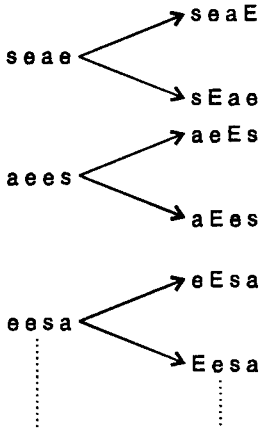


Fig. 1.3(a).

The list on the right contains the $4\,! = 24$ arrangements of the four letters $e\ a\ s$ E, so the list on the left contains half as many.

There are $\frac{4!}{2!} = 12$ ways in which the letters of ease can be arranged.

Theorem 1.15. *The number of ways to put r identical marbles into n boxes is* $\binom{n+r-1}{r}$.

Problem 1.112. *Doughnuts come in 30 different varieties and catherine wants to buy a dozen. How many choices does she have ?*

Solution. Imagine that the 30 varieties are in $n = 30$ boxes labeled "chocolate white", "Boston creme", "peanut crunch", and so on.

Catherine can indicate her choice by dropping $r = 12$ (identical) marbles into the boxes.

∙So there are $\binom{30+12-1}{12} = 7{,}898{,}654{,}920$ possibilities.

Problem 1.113. *David wants to buy 30 doughnuts and finds just 12 varieties available. In how many ways can he make his selection ?*

Solution. We imagine the 12 varieties in boxes.

Each of David's possible decisions can be indicated by dropping 30 identical marbles into these boxes.

There are $\binom{12+30-1}{30}$ = 3,159,461,968 possibilities (r = 30, n = 12).

Problem 1.114. *Consider the set {a, b, c, d}. In how many ways can we select two of these letters (repetition is not allowed) when (i) order matters (ii) order does not matter.*

Solution. (i) If order matters but repetition is not allowed , n = 4 and r = 2 and hence the number of ways of selecting two letters from four letters is

$$P(4, 2) = \frac{4\,!}{(4-2)\,!} = 12 \text{ and 12 possibilities are}$$

ab	*ba*	*ac*	*da*
ac	*bc*	*cb*	*db*
ad	*bd*	*cd*	*dc*

(ii) If order does not matter and repetition is not allowed then $C(4, 2) = \frac{4\,!}{2\,!(4-2)\,!}$ = 6 and 6 possibilities

are

ab	*bc*	*cd*
ac	*bd*	*ad*

Problem 1.115. *A man 7 relatives, 4 of them are ladies and 3 gentlemen, his wife has 7 relatives and 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of wife's relatives ?*

Solution. They can invite in four possible ways :

(i) 3 ladies from husband's side and three from wife's side.

Number of ways = C(4, 3) × C(4, 3) = 16.

(ii) 3 gents from husband's side and 3 ladies from wife's side.

Number of ways = C(3, 3) × C(3, 3) = 1.

(iii) 2 ladies and 1 gent from husband's side and one lady and 2 gents from wife's side.

Number of ways in this case

= {C(4, 2) × C(3, 1)} × {C(3, 1) × C(4, 2)}

= 324.

(iv) One lady and 2 gents from husband's side and 2 ladies and 1 gent from wife's side.

Number of ways in this case

= {C(4, 1) × C(3, 2)} × {C(3, 2) × C(4, 1)}

= 144

The total number of ways = 16 + 1 + 324 + 144 = 485.

Problem 1.116. *In how many ways can the letters of the word MONDAY be arranged ? How many of them begin with M and end with Y ? How many of them do not begin with M but end with ?*

Solution. The word MONDAY consists of six letters which can be arranged in P(6, 6) = 6 ! = 720 ways.

If M occupies the first place and Y the last place, then there are 4 letters (O, N, D, A) left to be arranged in 4 places in between M and Y.

This can be done in 4 ! = 24 ways.

If M does not occupy the first place but Y occupies the last place, the first place can be occupied in 4 ways by any one of the letters O, N, D, A.

For the second place, again 4 letters are available including M.

The third, fourth, and fifth places can be filled by 3, 2, 1 ways.

Hence, by product rule, the required number of arrangements are $4 \times 4 \times 3 \times 2 \times 1 = 96$.

Problem 1.117. *In how many ways can a cricket team of eleven be chosen out of batch of 15 players ? How many of them will*

(*a*) *include a particular player*

(*b*) *exclude a particular player ?*

Solution. The number of ways of selecting 11 players out of 15 is C(15, 11) = 1365

(*a*) The number of ways in which a particular player is included is C(14, 10) = 1001

(*b*) The number of ways in which a particular player is excluded is C(14, 11) = 364.

Problem 1.118. *A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find (a) the total number of password that can be formed, and (b) the number of passwords in which no digit repeats.*

Solution. (*a*) Since there are 26 alphabets and 10 digits and the digits can be repeated, by product rule the number of 4-character password is 26.10.10.10 = 26000.

Similarly the number of 5-character password is

$$26.10.10.10.10 = 260000$$

Hence the total number of password is

$$26000 + 260000 = 286000.$$

(*b*) Since the digits are not repeated, the first digit after alphabet can be taken from any one out of 10, the second digit from remaining 9 digits and so on. Thus the number of 4-character password is

26.10.9.8 = 18720 and the number of 5-character password is

26.10.9.8.7 = 131040 by the product rule.

Hence, the total number of passwords is 149760.

Problem 1.119. *A bit is either 0 or 1 : a byte is a sequence of 8 bits.*

Find (*a*) *the number of bytes that can be formed*

(*b*) *the number of bytes that begin with 11 and end with 11*

(*c*) *the number of bytes that begin with 11 and do not end with 11 and*

(*d*) *the number of bytes that begin with 11 or end with 11.*

Solution. (*a*) Since the bits 0 or 1 can repeat, the eight positions can be filled up either by 0 or 1 in 2^8 ways.

Hence the number of bytes that can be formed is 256.

(*b*) Keeping two positions at the beginning by 11 and the two positions the end by 11, there are four open positions which can be filled up in $2^4 = 16$ ways.

Hence the required number is 16.

(*c*) Keeping two positions at the beginning by 11, the remaining six open positions can be filled up by $2^6 = 46$ ways.

Hence the required number is 64 – 16 = 48.

(*d*) 64 bytes begin with 11, likewise, 64 bytes end with 11. In the sum of these numbers, 64 + 64 = 128, each byte that both begins and ends with 11 is counted twice.

Hence the required number is 128 – 16 = 112 bytes.

Problem 1.120. *Consider the set {a, b, c, d}. In how many ways can we select two of these letters when repetition is allowed.*

Solution. If order matters and repetition is allowed, there are $2^4 = 16$ possible selections and they are

aa	*ba*	*ca*	*da*
ab	*bb*	*cb*	*db*
ac	*bc*	*cc*	*dc*
ad	*bd*	*cd*	*dd*

If order does not matter but repetitions are allowed, there are C(4 + 2 – 1, 2) = C(5, 2) = 10 possibilities and these possibilities are

aa	*bb*	*cc*	*dd*		
ab	*bc*	*cd*	*ad*	*ac*	*bd*

Problem 1.121. *How many solutions are there of $x + y + z = 17$ subject to the constraints $x \geq 1$, $y \geq 2$ and $z \geq 3$.*

Solution. Put $x = 1 + u$, $y = 2 + v$, and $z = 3 + w$.

The given equation becomes $u + v + w = 11$ and we seek in non negative integers *u, v, w*.

The number of solutions is therefore

$$C(11 + 3 - 1, 11) = C(13, 11) = C(13, 2) = 78.$$

Problem 1.122. *How many solutions does the equation $x + y + z = 17$ have, where x, y, z are non negative integers ?*

Solution. Each solution of the given equation is equivalent to selecting 17 items from the set $\{x, y, z\}$, repetitions allowed.

Hence, the required number of solutions

$$= C(17 + 3 - 1, 17) = C(19, 2) = 171.$$

Problem 1.123. *Find the number of ways in which 7 different beads can be arranged to form a necklace.*

Solution. Fixing the position of one bead, the remaining beads can be arranged in 6 ! ways.

But this is a ring permutation, so the required number of arrangements is $\left(\frac{1}{2}\right)(6\,!) = 360$.

Problem 1.124. *In how many ways can 7 persons form a ring ? In how way can 7 gentlemen and 7 ladies sit down at a round table, no two ladies being together ?*

Solution. Seven persons can be seated along a circle in 6 ! ways.

First, let all the gentlemen be seated along the round table in 6 ! ways. Between any two men let a women be seated.

Hence all the seven ladies can be seated in 7 intermediate places in 7 ! ways.

Therefore, 7 gentlemen and 7 ladies can be seated along a round table in 6 ! × 7 ! ways.

Problem 1.125. *Find the number of unordered samples of size five (repetition allowed) from the set {a, b, c, d, e, f}.*

(*a*) *No further restrictions*

(*b*) *a occurs at least twice*

(*c*) *a occurs exactly twice.*

Solution. (*a*) Here $n = 6$, $r = 5$

So, the required member C(6 + 5 – 1, 5) = C(10, 5)

$$= \frac{10.9.8.7.6}{5.4.3.2} = 252.$$

(*b*) Since *a* occurs at least twice, we have to find the number of unordered samples of size 3 (repetitions allowed) from 6-element set.

So, the required number is C(6 + 3 – 1, 3) = C(8, 3)

$$= \frac{8.7.6}{3.2} = 56.$$

The form of the samples are *a a x y z, a a a y z* etc.

(*c*) Since *a* occurs exactly twice, we have to find the number of unordered samples of size 3(repetition allowed) from 5-element set {*b, c, d, e, f*}.

So, the required number is

$$C(5 + 3 - 1, 3) = C(7, 3) = \frac{7.6.5}{3.2} = 35.$$

Problem 1.126. *In how many ways can 12 balloons be distribuetd at a Birth day party among 10 children ?*

Solution. This is an unordered selection with repetition, of 12 objects from 10 types.

Hence the number of selection is C(10 + 12 – 1, 12) = C(21, 12).

If we want to ensure that every child gets at least one balloon, we must give a balloon to each child, then distribute the remaining two balloons which can be done is C(10 + 2 – 1, 2) = C(11, 2) = 55 ways.

Problem 1.127. *In how many ways can the letters of the English alphabet be arranged so that there are exactly 5 letters between the letters a and b ?*

Solution. There are P(24, 5) ways to arrange the 5 letters between *a* and *b*, 2 ways to place *a* and *b*, and then 20 ! ways to arrange any 7-letter word treated as one unit along with the remaining 19 letters. The total is P(24, 5)(20 !)(2).

Problem 1.128. *In how many ways can 7 women and 3 men be arranged in a row if the 3 men must always stand next to each other ?*

Solution. There are 3 ! ways of arranging the 3 men.

Since the 3 men always stand next to each other, we treat them as a single entity, which we denote by X. Then if $W_1, W_2, ..., W_7$ represents the women, we next are interested in the number of ways of arranging $\{X, W_1, W_2, W_3, ..., W_7\}$.

There are 8 ! permutations of these 8 objects.

Hence there are (3 !)(8 !) permutations altogether.

Problem 1.129. *How many 6-digit numbers without repetition of digits are there such that the digits are all non zero and 1 and 2 do not appear consecutively in either order ?*

Solution. We are asked to count cetain 6-permutations of the 9 integers 1, 2, ..., 9.

In the following table we separate these 6-permutations into 4 disjoint classes and count the number of permutations in each class.

Class	*Number of permutations in the class*
(*i*) Neither 1 nor 2 appears as a digit	7 !
(*ii*) 1, but not 2, appears as a digit	6P(7, 5)
(*iii*) 2, but not 1, appears	6P(7, 5)
(*iv*) Both 1 and 2 appear	(2)(7)(4) P(6, 3) + (4)(7)(6)(3) P(5, 2)
Total	7 ! + (2)(6) P(7, 5) + (56) P(6, 3) + (504) P(5, 2)

Let us explain how to count the elements in class (*iv*).

1. The hundred thousands digit is 1 (and thus the ten thousands digit is not 2). The second digit can be chosen in 7 ways. Choose the position for 2 in 4 ways ; then fill the other 3 positions P(6, 3) ways.

   Hence, there are (7) 4P(6, 3) numbers in this category.
2. The units digit is 1 (and hence the tens digit is not 2). Likewise, there are (7) 4P(6, 3) numbers in this category.
3. The integer 1 appears in a position different from the hundred thousands digit and the units digit. Hence, 2 cannot appear immediately to the left or to the right of 1. Since 1 can be any one of the digits from the tens digit up to the ten thousands digit, 1 can be placed in 4 ways. The digit immediately to the left of 1 can be filled in 7 ways, while the digit immediately to the right of 1 can be filled in 6 ways. The integer 2 can be placed in any of the remaining positions in 3 ways and then the other 2 digits are a 2-permutation of the remaining 5 integers.

Hence, there are (4)(7)(6)(3) P(5, 2) numbers in this category.

Thus, there are (2)(7)(4) P(6, 3) + (4)(7)(6)(3) P(5, 2) numbers in class (*iv*).

Then, by the sum rule, there are

P(7, 6) + (2)(6) P(7, 5) + (56) P(6, 3) + (504) P(5, 2) elements in all four classes.

Problem 1.130. *In how many ways can 5 children arrange themselves in a ring ?*

Solution. Here, the 5 children are not assigned to particular places but are only arranged relative to one another.

Thus, the arrangements (see figure below) are considered the same if the children are in the same order clockwise.

Hence, the position of child C_1 is immaterial and it is only the position of the 4 other children relative to C_1 that counts.

Therefore, keeping C_1 fixed in position, there are 4 ! arrangements of the remaining children.

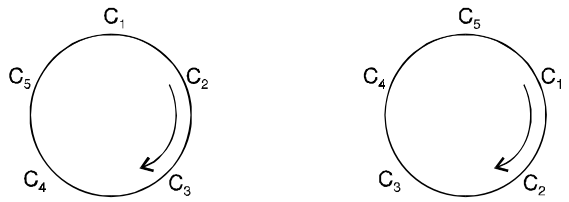


Fig. 1.4.

Problem 1.131. *In how many ways can a hand of 5 cards be selected from a deck of 52 cards ?*

Solution. Each hand is essentially a 5-combination of 52 cards.

Thus there are $C(52, 5) = \dfrac{52\,!}{5\,!\,47\,!} = \dfrac{52.51.50.49.48}{5.4.3.2.1}$

$= 52.51.10.49.2 = 2{,}598{,}960$ such hands.

Problem 1.132. (*a*) *How many 5-card hands consist only of hearts ?*

(*b*) *How many 5-card hands consist of cards from a single suit ?*

(*c*) *How many 5-card hands have 2 clubs and 3 hearts ?*

(*d*) *How many 5-card hands have 2 cards of one suit and 3-cards of a different suit ?*

(*e*) *How many 5-card hands contain 2 aces and 3 kings ?*

(*f*) *How many 5-card hands contain exactly 2 of one kind and 3 of another kind ?*

Solution. (*a*) Since there are 13 hearts to choose from, each such hand is a 5-combination of 13 objects.

Thus, there is a total of

$$C(13, 5) = \frac{13\,!}{5\,!\,8\,!} = \frac{13.12.11.10.9}{5.4.3.2.1} = 13.11.9 = 1{,}287.$$

(*b*) For each of the 4 suits, spades, hearts, diamonds, or clubs, there are C(13, 5) 5-card hands.

Hence, there are a total of 4C(13, 5) such hands.

(*c*) C(13, 2), C(13, 3).

(*d*) For a fixed choice of 2 suits there are 2C(13, 2), C(13, 3) ways to choose 2 from one of the suits and 3 from the other. We can choose the 2 suits in C(4, 2) ways.

Thus, there are 2C(13, 2) C(13, 3) C(4, 2) such 5-card hands. Recall that two of a kind means 2 aces, 2 kings, 2 queens etc. Similarly, 3 tens are called three of a kind.

Thus, there are 13 kinds in a deck of 52 cards.

(*e*) C(4, 2) C(4, 3)

(*f*) Choose the first kind 13 ways, choose 2 of the first kind C(4, 2) ways, choose the second kind 12 ways and choose 3 of the second kind in C(4, 3) ways.

Hence there are (13)C(4, 2) (12)C(4, 3), 5-card hands with 2 of one kind and 3 of another kind.

Problem 1.133. (*a*) *In how many ways can a committee of 5 be chosen from 9 people ?*

(*b*) *How many committees of 5 or more can be chosen from 9 people ?*

(*c*) *In how many ways can a committee of 5 teachers and 4 students be chosen from 9 teachers and 15 students ?*

(*d*) *In how many ways can the committee in (C) be formed if teacher A refuses to serve if student B is on the committee ?*

Solution. (*a*) C(9, 5) ways.

(*b*) C(9, 5) + C(9, 6) + C(9, 7) + C(9, 8) + C(9, 9)

(*c*) The teachers can be selected in C(9, 5) ways while the students can be chosen in C(15, 4) ways so that the committee can be formed in C(9, 5) C(15, 4) ways.

(*d*) We answer this question by counting indirectly. First we count the number of committees where both A and B are on the committee. Thus, there are only 8 teachers remaining from which 4 teachers are to be chosen.

Likewise, there are only 14 students remaining from which 3 more students are to be chosen.

There are C(8, 4), C(14, 3) committees containing both A and B, and hence there are

C(9, 5) C(15, 4) – C(8, 4) C(14, 3) committees that do not have both A and B on the committee.

Problem 1.134. *How many strings of length n can be formed from the English alphabet ?*

Solution. By the product rule, since there are 26 letters, and since each letter can be used repeadtly, we see that there are 26^n strings of length n.

Problem 1.135. *How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl ?*

Solution. To solve this problem we list all the ways possible to select the fruit. There are 15 ways :

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange

2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three element set, {apple, orange, pear}.

Problem 1.136. *Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen ? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.*

Solution. The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.

We have, there are $C(n + r - 1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

This equals $C(4 + 6 - 1, 6) = C(9, 6)$

Since $C(9, 6) = C(9, 3) = \frac{9.8.7}{1.2.3} = 84.$

There are 84 different ways to choose the six cookies.

Problem 1.137. *What is the value of k after the following pseudocode has been executed ?*

$k := 0$

for $i_1 = 1$ *to* n

for $i_2 = 1$ *to* i_1

....

....

for $i_m = 1$ *to* i_{m-1}

$k := k + 1$

Solution. Note that the initial value of k is 0 and that 1 is added to k each time the nested loop is traversed with a sequence of integers $i_1, i_2, ..., i_m$ such that

$$1 \le i_m \le i_{m-1} \le ...... \le i_1 \le n.$$

The number of such sequences of integers is the number of ways to choose m integers from {1, 2, ..., n}, with repetition allowed.

Note that once such a sequence has been selected, if we order the integers in the sequence in non decreasing order, this uniquely defines an assignment of $i_m, i_{m-1}, ..., i_1$.

Conversely, every such assignment corresponds to a unique unordered set.

Hence, it follows that $K = C(n + m - 1, m)$ after this code has been executed.

Problem 1.138. *How many solutions does the equation* $x_1 + x_2 + x_3 = 11$ *have, where* x_1, x_2 *and* x_3 *are non negative integers ?*

Solution. To count the number of solutions, we note that a solution corresponding to a way of selecting 11 items from a set with three elements, so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen.

Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements.

We have, there are C($n + r - 1$, r) r-combination from a set with n elements when repetition of elements is allowed, it follows that there are

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13.12}{1.2} = 78 \text{ solutions.}$$

The number of solutions of this equation can also be found when the variables are subject to constraints.

For instance, we can find the number of solutions where the variables are integers with $x_1 \geq 1$, $x_2 \geq 2$ and $x_3 \geq 3$.

A solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three where, in addition, there is atleast one item of type one, two items of type two, and three items of type three. So, choose one item of type one, two of type two, and three of type three. Then select five additional items this can be done in

$$C(3 + 5 - 1, 5 = C(7, 5) = C(7, 2) = \frac{7.6}{1.2} = 21 \text{ ways.}$$

Thus, there are 21 solutions of the equation subject to the given constraints.

Problem 1.139. *How many ways are there to place ten indistinguishable balls into eight distinguishable bins ?*

Solution. The number of ways to place ten indistinguishable balls into eight bins equals the number of 10 combinations from a set with eight elements when repetition is allowed. Consequently, there are

$$C(8 + 10 - 1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448.$$

Problem 1.140. *How many different strings can be made by reordering the letters of the word SUCCESS ?*

Solution. Because some of the letters of SUCCESS are the same, the answer is not given by the number of permutations of seven letters.

This word contains three Ss, two Cs, one U, and one E. To determine the number of different strings that can be made by reordering the letters, first note that the three Ss can be placed among the seven positions in C(7, 3) different ways, leaving four positions free.

Then the two Cs can be placed in C(4, 2) ways, leaving two free positions. The U can be placed in C(2, 1) ways, leaving just one position free.

Hence E can be placed in C(1, 1) way.

Consequently, from the product rule, the number of different strings that can be made is

$$C(7, 3)\, C(4, 2)\, C(2, 1)\, C(1, 1)$$

$$= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!}$$

$$= \frac{7!}{3!2!1!1!} = 420.$$

Problem 1.141. *How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards ?*

Solution. We will use the product rule to solve this problem.

Note that, the first player can be dealt 5 cards in C(52, 5) ways.

The second player can be dealt 5 cards in C(47, 5) ways, since only 47 cards are left.

The third player can be dealt 5 cards in C(42, 5) ways.

Finally, the fourth player can be dealt 5 cards in C(37, 5) ways.

Hence, the total number of ways to deal four players 5 cards each is

$$C(52, 5)\, C(47, 5)\, C(42, 5)\, C(37, 5)$$

$$= \frac{52!}{47!5!} \cdot \frac{47!}{42!5!} \cdot \frac{42!}{37!5!} \cdot \frac{37!}{32!5!}$$

$$= \frac{52!}{5!5!5!5!32!}.$$

Problem 1.142. *What is the next largest permutation in lexicographic order after 362541 ?*

Solution. The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$.

Hence, 4 is placed in the third position.

Then the integers 2, 5 and 1 are placed in order in the last three positions, giving 125 as the last three positions of the permutation.

Hence, the next permutation is 364125.

Problem 1.143. *Generate the permutations of the integers 1, 2, 3 in Lexicographic order.*

Solution. Begin with 123. The next permutation is obtained by interchanging 3 and 2 to obtain 132.

Next, since 3 > 2 and 1 < 3, permute the three integers in 132.

Put the smaller of 3 and 2 in the first position, and then put 1 and 3 in increasing order in positions 2 and 3 to obtain 213.

This is followed by 231, obtained by interchanging 1 and 3, since 1 < 3.

The next largest permutation has 3 in the first position followed by 1 and 2 in increasing order, namely, 312.

Finally, interchange 1 and 2 to obtain the last permutation 321.

Problem 1.144. *Find the next largest bit string after 1000100111.*

Solution. The first bit from the right that is not a 1 is the fourth bit from the right.

Change this bit to *a* 1 and change all the following bits to 0s.

This produces the next largest bit string, 1000101000.

1.5 PROBABILITY

1.5.1 Random Experiment.

If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment.

For example ; Tossing a coin, selecting a card from a pack of playing cards, throwing a die, selecting a family out of a given group of families etc.

1.5.2 Event and Trial.

Any particular performance of a random experiment is called a trial and combination of outcomes are called events.

1.5.3 Outcome.

The result of a random experiment will be called an outcome. **For example** ; If a coin is tossed repeatedly, the result is not unique, we may get any of the two forces, head or tail. Thus tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

1.5.4 Exhaustive Event.

An event consisting of al the various possibilities is called an exhaustive event.

1.5.5 Mutually exclusive events.

Two or more events are said to be mutually exclusive if the happening of one event prevent the simultaneous happening of the others.

For example ;

(*i*) In tossing a coin, getting head and tail are mutually exclusive in view of the fact that if head is the turn out, getting tail is not possible.

(*ii*) In throwing a cubical 'die', getting any of the number 1, 2, 3, 4, 5, 6 are mutually exclusive as the turn out of any number rules out the possibility of the turn out of other numbers.

1.5.6 Independent events.

Two or more events are said to be independent if the happending or non-happening of one event does not prevent the happening or non-happening of the others.

For example ;

(*i*) When two coins are tossed the event of getting head is an independent event as both the coins can turn out heads.

(*ii*) When a card is drawn at random from a pack of 52 cards and if the card is repeated, the result of second draw is independent of the first. But if the card is not replaced then the result of the second depends on the result of the first draw.

1.5.7 Probability : (Mathematical form).

If the outcome of a trial consists n exhaustive, mutually exclusive equally possible cases, of which m of them are favourable cases to an event E, then the probability of the happening of the event E, usually denoted by P(E) or simply p is defined to be equal to $\frac{m}{n}$ *i.e.,* $P(E) = p \ \frac{\text{Number of favourable cases}}{\text{Number of possible cases}}$

$= \frac{m}{n}$.

The probability can atmost be equal to 1, because the number of favourable cases and the number of possible cases can atmost coincide with each other. Since m cases are favourable to the event, it follows that $(n - m)$ cases are not favourable to the event. This set of unfavourable events is denoted by $\overline{E}$ or E′.

Probability of the non happening of the event usually denoted by q is given by

$$q = \frac{n-m}{n} \text{ or } P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

i.e., $$q = 1 - \frac{m}{n} = 1 - P(E) = 1 - p$$

$\Rightarrow \quad q = 1 - p \quad \Rightarrow \quad p + q = 1 \quad \Rightarrow \quad P(E) + P(\bar{E}) = 1$

Here p is the probability of success q is the probability of failure.

Sum of p and q (*i.e.,* $p + q = 1$) is always equal to 1.

If P(E) = 1 ; E is called a sure event and

If P(E) = 0 ; E is called an impossible event.

1.5.8 Probability Function.

P(A) is the probability function defined on a σ-field B of events if the following properties hold

(*i*) For each $A \in B$, P(A) is defined, is real and $P(A) \geq 0$

(*ii*) P(S) = 1

(*iii*) If $\{A_n\}$ is any finite or infinite sequence of disjoint events in B then $P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} P(A_i)$.

Theorem 1.16. *Probability of the complementary event* $\bar{A}$ *of A is given by* $P(\bar{A}) = 1 - P(A)$.

Proof. A and $\bar{A}$ are mutually disjoint events, so that

$$A \cup \bar{A} = S \quad \Rightarrow \quad P(A \cup \bar{A}) = P(S)$$

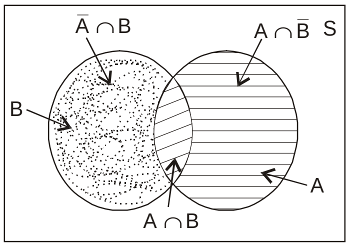
We have $\quad P(A) + P(\bar{A}) = P(S) = 1$

$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$

1.5.9 Addition theorem of probability 1.17.

If A and B are any two events and are not disjoint then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. From the venn diagram, we have



$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually disjoint

$\therefore \quad P(A \cup B) = P[A \cup (\bar{A} \cap B)]$

$= P(A) + P(\bar{A} \cap B)$

$= P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P[(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B)$

$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.5.10 Multiplication theorem of probability 1.18.

For two events A and B

$$P(A \cap B) = P(A) \,.\, P(B/A),\ P(A) > 0$$
$$= P(B) \,.\, P(A/B),\ P(B) > 0$$

where P(B/A) represents conditional probability of occurrence of B when the event A has already happened and P(A/B) is the conditional probability of happening of A, given that B has already happened.

Proof. We have $P(A) = \dfrac{n(A)}{n(S)}$; $P(B) = \dfrac{n(B)}{n(S)}$ and $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)}$ ...(1)

For the conditional event $\dfrac{A}{B}$, the favourable outcomes must be one of the sample points of B.

i.e., for the event $\dfrac{A}{B}$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurrence of the event A.

Hence $P(A/B) = \dfrac{n(A \cap B)}{n(B)}$

Rewriting (1), we get

$$P(A \cap B) = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)} = P(B) \,.\, P(A/B) \qquad ...(2)$$

Similarly, we get from (1)

$$P(A \cap B) = \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)} = P(A) \,.\, P(B/A) \qquad ...(3)$$

From (2) and (3), we get the result

$$P(A \cap B) = P(A) \,.\, P(B/A),\ P(A) > 0$$
$$= P(B) \,.\, P(A/B),\ P(B) > 0$$

1.5.11 Baye's theorem 1.19.

If E_1, E_2, E_3 ..... E_n are mutually disjoint events with $P(E_i) \neq 0$ ($i = 1, 2, ... n$) then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} = \frac{P(E_i)\, P(A/E_i)}{P(A)} \ ;\ i = 1, 2, ... n$$

Proof. Since $A \subset \bigcup_{i=1}^{n} E_i$, we have $A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i)$

Since $(A \cap E_i) \subset E_i$; (i = 1, 2, ... n) are mutually disjoint events, we have by addition theorem of probability,

$$P(A) = P\left\{\bigcup_{i=1}^{n} (A \cap E_i)\right\} = \sum_{i=1}^{n} P(A \cap E_i)$$

$$= \sum_{i=1}^{n} P(E_i)\, P(A/E_i)$$

by multiplification theorem of probability.

Also we have $P(A \cap E_i) = P(A)\, P(E_i/A)$

$$\Rightarrow \quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

Problem 1.145. *What is the probability of getting 9 cards of the same suit in one hand at a game of bridge ?*

Solution. Since one hand in a bridge game consists of 13 cards, the exhaustive number of cases is ${}^{52}C_{13}$.

The number of ways in which 9 cards of a suit can come out of 13 cards of the suit = ${}^{13}C_9$.

The number of ways in which balance 13 – 9 = 4 cards can come in one hand out of a balance of 39 cards of other suits is ${}^{39}C_4$.

Since there are four different suits and 9 cards of any suit can come, by the principle of counting, the total number of favourable cases of getting 9 cases of suit = ${}^{13}C_4 \times {}^{39}C_4 \times 4$.

$$\therefore \quad \text{Required probability} = \frac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}.$$

Problem 1.146. *What is the probability that at least two out of n people have the same birthday ? Assume 365 days in a year and that all days are equally likely.*

Solution. Since the birthday of any person can fall on any one of the 365 days, the exhaustive number of cases for the birthday of n persons is 365^n.

If the birthdays of all n persons fall on different days, then the number of favourable cases is :

365(365 – 1)(365 – 2) ...... [365 – (n – 1)], because is this case the birthday of the first person can fall on any one of 365 days, the birthday of the second person can fall on any of the remaining 364 days, and so on.

Hence, the probability (p) that birthdays of all the n persons are different is given by :

$$p = \frac{365(365-1)(365-2)\ ......\ [365-(n-1)]}{365^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)......\left(1-\frac{n-1}{365}\right)$$

Hence, the required probability that at least two persons have same birthday is

$$1-p = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)......\left(1-\frac{n-1}{365}\right)$$

Problem 1.147. *A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.*

Solution. Let us define the following events :

A : the card drawn is a king ; B : the card drawn is a heart ; C : the card drawn is a red card

Then A, B and C are not mutually exclusive

$A \cap B$: the card drawn is the king of hearts

$\Rightarrow$ $n(A \cap B) = 1$

$B \cap C = B$: the card drawn a heart ($\because$ $B \subset C$)

$\Rightarrow$ $n(B \cap C) = 13$

$C \cap A$: the card drawn is a red king

$\Rightarrow$ $n(C \subset A) = 2$

$A \cap B \cap C = A \cap B$: the card drawn is the king of hearts

$\Rightarrow$ $n(A \cap B \cap C) = 1$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}\ ;\ P(B) = \frac{13}{52}\ ;\ P(C) = \frac{26}{52}$$

$$P(A \cap B) = \frac{1}{52}\ ;\ P(B \cap C) = \frac{13}{52}\ ;\ P(C \cap A) = \frac{2}{52}$$

$$P(A \cap B \cap C) = \frac{1}{52}$$

The required probability of getting a king or heart or a red card is given by :

$$= \frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52} = \frac{28}{52} = \frac{7}{13}.$$

Problem 1.148. *A problem in statistics is given to three students A, B and C whose chances of solving it are* $\frac{1}{2}$, $\frac{3}{4}$ *and* $\frac{1}{4}$ *respectively. What is the probability that the problem will be solved if all of them try independently ?*

Solution. Let A, B, C denote the events that the problem is solved by the students A, B, C respectively. Then

$$P(A) = \frac{1}{2} \text{ ; } P(B) = \frac{3}{4} \text{ and } P(C) = \frac{1}{4}$$

The problem will be solved if at least one of them solves the problem. Thus we have to calculate the probability of occurrence of atleast one of the three events A, B, C *i.e.,* $P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C)$$

$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2} \cdot \frac{3}{4} - \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{29}{32}.$$

Problem 1.149. *A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced ; a second drawing is then made, what is the probability that :*

(*i*) *the first number drawn is even and the second odd ?*

(*ii*) *the first number is odd and the second even ?*

How will you results in (*i*) *and* (*ii*) *be effected if the first counter drawn is not replaced ?*

Solution. (*i*) Let A denote the event of getting even numbered counter on the first draw and B denote the event of getting odd numbered counter on the second draw. Since the counter drawn is replaced, events A and B are independent.

Now from 1 to 17, the even numbers are 2, 4, 6, 8, 10, 12, 14 and 16 ; *i.e.,* 8 and odd numbers are 9.

$$\therefore \quad P(A) = \frac{8}{17} \text{ and } P(B) = \frac{9}{17}$$

Using multiplication theorem of probability, the probability of getting even number on the first draw and odd number on the second draw is given by

$$P(A \cap B) = P(A) - P(B) = \frac{8}{17} \cdot \frac{9}{17} = \frac{72}{289}$$

However, if the first counter drawn is not replaced before the second counter is drawn, the events A and B are not independent. In this case

$$P(A \cap B) = P(A) \,.\, P(B/A) = \frac{8}{17} \cdot \frac{9}{16} = \frac{9}{34}$$

(*ii*) The probabilities of the first counter drawn being odd and the second counter drawn being even are :

$$\frac{9}{17} \cdot \frac{8}{17} = \frac{72}{289}, \text{ if replacemetn is made and}$$

$$\frac{9}{17} \cdot \frac{8}{16} = \frac{9}{34}, \text{ if the replacement is not made.}$$

Problem 1.150. *A speaks truth 4 out of 5 times. A die is tossed. He reports that there is a six. What is the chance that actually there was six ?*

Solution. Let us define the following events

E_1 : A speaks truth ; E_2 : A tells a lie ; E = A reports a six

From the data given in the problem, we have

$$P(E_1) = \frac{4}{5} \; ; \; P(E_2) = \frac{1}{5} \; ; \; P(A/E_1) = \frac{1}{6} \; ; \; P(A/E_2) = \frac{5}{6}$$

The required probability that actually there was six is

$$P(E_1/E) = \frac{P(E_1) \times P(E/E_1)}{P(E_1) \times P(E/E_1) + P(E_2) \times P(E/E_2)}$$

$$= \frac{\frac{4}{5} \cdot \frac{1}{6}}{\frac{4}{5} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{5}{6}} = \frac{4}{9}.$$

Problem 1.151. *A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner*

(*i*) *There must be one from each category*

(*ii*) *It should have at least one from the purchase department.*

Solution. There are 3 + 4 + 2 + 1 = 10 persons in all and a committee of 4 people can formed cut of them in $^{10}C_4$ ways. Hence exhaustive number of cases is

$$^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4\,!} = 210$$

(*i*) Favourable number of cases for the committee to consist of 4 members, one from each category, is

$$^4C_1 \times {}^3C_1 \times {}^2C_1 \times 1 = 4 \times 3 \times 2 = 24$$

$\therefore$ Required probability $= \frac{24}{210} = \frac{4}{35}$

(*ii*) P[committee has at least one purchase officer]

= 1 – P[committee has no purchase officer]

In order that the committee has no purchase officer, all the 4 members are to be selected from amongst officers of production department, sales department and chartered accountant. *i.e.,* out of 3 + 2 + 1 = 6 members and this can be done in $^6C_4 = \frac{6 \times 5}{1 \times 2} = 15$ ways.

Hence P(committee has no purchase officer) $= \frac{15}{210} = \frac{1}{14}$

$\therefore$ P(committee has at least one purchase officer) $= 1 - \frac{1}{14} = \frac{13}{14}$

Example 1.152. *n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.*

Solution. Since n persons cab be seated in n chairs at a round table in $(n - 1)$! Ways, the exhaustive number of cases = $(n - 1)$! Assuming the two specified persons A and B who sit together as one, we get $(n - 1)$ persons in all, who can be seated at a round table in $(n - 2)$! ways. Further, since A and B can interchange their positions in 2 ! Ways, total number of favourable cases of getting A and B together is $(n - 2)! \times 2!$.

$$\therefore \quad \text{Required probability} = \frac{(n-2)!\times 2!}{(n-1)!} = \frac{2}{n-1}.$$

Problem 1.153. *Four cards are drawn at random from a pack of 5 cards. Find the probability that*

(*i*) *They are a king, a queen, a jack and an ace*

(*ii*) *Two are kings and two are queens*

(*iii*) *Two are black and two are red*

(*iv*) *There are two cards of hearts and two cards of diamods.*

Solution. Four cards can be drawn from a well-shuffled a pack of 52 cards in $^{52}C_4$ ways, which gives the exhaustive number of cases.

(*i*) 1 king can be drawn out of the 4 kings in $^4C_1$ ways.

Similarly, 1 queen, 1 jack and an ace can each be drawn in $^4C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are $^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1$

$$\text{Hence the required probability} = \frac{^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1}{^{52}C_4} = \frac{256}{^{52}C_4}$$

$$(ii)\ \text{Required probability} = \frac{^4C_2 \times {}^4C_2}{^{52}C_4}$$

(*iii*) Since there are 26 black cards and 26 red cards in a pack of cards, the required probability

$$= \frac{^{26}C_2 \times {}^{26}C_2}{^{52}C_4}$$

$$(iv)\ \text{Required probability} = \frac{^{13}C_2 \times {}^{13}C_2}{^{52}C_4}.$$

Problem 1.154. *Twelve balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls.*

Solution. Since each ball can go any one of three boxes, there are 3 ways in which a ball can go to any one of the three boxes. Hence there are 3^{12} ways in which 12 balls can be placed in the three boxes.

Number of ways in which 3 balls out of 12 can go to the first box is ${}^{12}C_3$. Now the remaining 9 balls are to be placed in remaining 2 boxes and this can be done in 2^9 ways.

Hence, the total number of favourable cases = ${}^{12}C_3 \times 2^9$.

$$\therefore \quad \text{Required probability} = \frac{{}^{12}C_3 \times 2^9}{3^{12}}.$$

1.6. RAMSEY NUMBER

Let p and q be 2 positive integers. A positive integer r is said to have the {P, q) – **Ramsey property** if in any group of r people either there is a subgroup of p people known to one another or there is a subgroup of q people not known to one another.

(by Ramsey's theorem all sufficiently large integers r have the (p, q) – Ramsey Property).

The smallest r with the (p, r) – **Ramsey Property** is called the **Ramsey number, R(*p, q*).**

$R(p, q) = R(q, p)$ and $R(p, 1) = 1$, $R(p, 2) = p$

Let k_i ($i = 1, 2, ..., t$) and m be positive integers, with each $k_i \geq m$ and $t \geq 2$.

Let $< C_1, C_2, ... C_t >$ be an ordered partition of the class C of all m-element subsets of an n-element set X. (there are thus $C(n, m)$ elements in C). Then the positive integer n has the generalized $(k_1, k_2, ..., k_t ; m)$ – Ramsey Property if, for some value of i in the range 1 to t, X possesses a k_i – element subset B such that all m-element subsets of B belong to C_i. The smallest such n is the **generalized Ramsey number,** $R(k_1, k_2, ..., k_t ; m)$.

Problem 1.155. *Show that if m and n are integers both greater than 2, then*

$$R(m, n) \leq R(m-1, n) + R(m, n-1).$$

Solution. Let $P \equiv R(m-1, n)$, $q \equiv R(m, n-1)$, and $r \equiv p + q$.

Consider a group $\{1, 2, ..., r\}$ of r people.

Let L be the set of people known to person 1 and M be the set of people not known to person 1.

The 2 sets together have $r - 1$ people, so either L has at least P people or M has at least q people.

(*a*) If L has $P = R(m-1, n)$ people, then, by definition, it contains a subset of $m - 1$ people known to one another or it contains a subset of n people unknown to one another.

In the former case the $m - 1$ people and person 1 constitute m people known to one another.

Thus, in this case, a group of $R(m-1, n) + R(m, n-1)$ people necessarily includes m mutual acquaintances or n mutual strangers.

i.e., $\quad R(m, n) \leq R(m-1, n) + R(m, n-1)$.

(*b*) By the usual symmetry argument the same conclusion follows when M contains q people.

Problem 1.156. *If R(m – 1, n) and R(m, n – 1) are both even the greater than 2, prove that*

$$R(m, n) \leq R(m-1, n) + R(m, n-1) - 1.$$

Solution. Let $P \equiv R(m-1, n)$, $q \equiv R(m, n-1)$, and $r = p + q$.

It suffices to establish that in any group $X = \{1, 2, ..., r - 1\}$ of $r - 1$ people there is either a subgroup of m people who know one another or a subgroup of n people who do not know one another.

Let d_i be the number of people known to person i, for $i = 1, 2, ... r - 1$.

Since knowing is mutual, $d_1 + d_2 + ... + d_{r-1}$ is necessarily even. But $r - 1$ is odd, so d_i is even for atleast 1 i, which we may take to be $i = 1$.

Let L be the set of people known to person 1 and let M be the set of people not known to person 1.

Since there are an even number of people in L, there must be an ever number of people in M as well.

Now either L has atleast $p - 1$ people or M has atleast q people. But $p - 1$ is odd.

So either L has at least p people or M has at least q people.

(*a*) Suppose L has at least P people. Because P = R $(m - 1, n)$, L must contain either $m - 1$ people known to one another or n people not known to one another (in which case the theorem holds.)

In the former case these $m - 1$ people and person 1 will constitute m people known to one another (and the theorem holds.)

(*b*) The case of q or more people in M is handled by symmetry.

Problem 1.157. *Show that if m and n are integers greater than 1, then*

$$R = (m, n) \leq C(m + n - 2, m - 1) \qquad ...(1)$$

Solution. When $m = 2$ or $n = 2$, (1) holds with equality.

The proof is by induction on $k = m + n$.

As we have just seen, the result is true when $k = 4$.

Assume the result true for $k - 1$, then

$$R\,(m - 1, n) \leq C\,(m + n - 3, m - 2) \text{ and}$$

$$R\,(m, n - 1) \leq C\,(m + n - 3, m - 1).$$

Now pascal's identity gives $C\,(m + n - 3, m - 2) + C\,(m + n - 3, m - 1) = X\,(m + n - 2, m - 1)$

So that $R(m - 1, n) + R\,(m, n - 1) \leq C\,(m + n - 2, m - 1)$

But, $R\,(m, n) \leq R\,(m - 1, n) + R\,(m, n - 1)$.

Problem 1.158. *Show that*

(*i*) *R(4, 4) = 18*

(*ii*) *R(4, 3) = 9*

(*iii*) *R(5, 3) = 14*

(*iv*) *R(3, 3) = 6.*

Solution. (*i*) $R(4, 4) \leq R(3, 4) + R(4, 3) = 9 + 9 = 18$

To show that $R(4, 4) > 17$

Consider an arrangement of 17 people about a round table such that each person knows exactly 6 people, the first, second, and fourth persons on one's right and first, and the fourth persons on one's left.

It can be veririfed that in this arrangement there is no subgroup of 4 mutual acquaintances or of 4 mutual strangers.

(*ii*) $R(4, 3) \leq R(3, 3) + R(4, 2) - 1 = 9$

To prove that $R(4, 3) = R(3, 4) > 8$.

We exhibit a group of 8 people which has no subgroup of 3 people known to one another and no subgroup of 4 people not known to one another.

Here is a scenario, 8 people sit about a round table. Each person knows exactly 3 people, the 2 people sitting on either side of him and the person sitting farthest from him.

(*iii*) $R(5, 3) \leq R(4, 3) + R(5, 2) = 9 + 5 = 14$

To see that $R(5, 3) = R(3, 5) > 13$.

Consider a group of 13 people sitting at a round table such that each person knows only the fifth person on his right and the fifth person on his left.

In such a situation there is no subgroup of 3 mutual acquaintances and no subgroup of 5 mutual strangers.

(*iv*) $R(3, 3) \leq 6$.

To show that $R(3, 3) > 5$.

It is enough to consider a seating arrangement of 5 people about a round table in which each person knows only the 2 people on either side.

In such a situation there is no set of 3 mutual acquaintances and no set of 3 people known to one another.

Problem 1.159. *If n points are located in general position in the plane, and if every quadrilateral formed from these n points is convex, then the n points are the vertices of a convex n-gon.*

Solution. Suppose the n points do not form a convex n-gon. Consider the smallest convex polygon that contains the n points. At least one of the n points (say, the point P) is in the interior of this polygon.

Let Q be one of the vertices of the polygon.

Divide the polygon into triangles by drawing line segments joining Q to every vertex of the polygon.

The point P then will be in the interior of one of these triangles, which contradicts the convexity hypothesis.

Problem 1.160. *Show that in any group of 10 people there is always (a) a subgroup of 3 mutual strangers or a subgroup of 4 mutual acquaintances and (b) a subgroup of 3 mutual acquaintances or a subgroup of 4 mutual strangers.*

Solution. (*a*) Let A be 1 of the 10 poeple, the remaining 9 people can be assigned to 2 rooms, those who are known to A are in room Y and those who are not known to A are in room Z.

Either room Y has at least 6 people or room Z has at least 4 people.

(*i*) Suppose room Y has atleast 6 people, then, there is either a subgroup of 3 mutual acquaintances or a subgroup of 3 mutual strangers in this room.

In the former case, A and these 3 people constitute 4 mutual acquaintances.

(*ii*) Suppose room Z has atleast 4 people.

Either these 4 people know one another or at least 2 of them, B and C, do not know each other.

In the former case we have a sub group of 4 mutual acquaintances.

In the later case A, B and C constitute 3 mutual strangers.

(*b*) In the previous scenario, let people who are strangers become acquaintances, and let people who are acquaintances pretend they are strangers. The situation is symmetric.

Problem 1.161. *Let A be any $n \times n$ matrix. Matrix P is an $m \times m$ Principal submatrix of A if P is obtained from A by removing any n-m rows and the same n-m columns. Show that for every positive integer m, there exists a positive integer n such that every $n \times n$ binary matrix A has an $m \times m$ Principal submatrix P in one of the following four categories :*

(*i*) *P is diagonal* (*ii*) *Ever non diagonal entry of P is 1.*

(*iii*) *P is lower triangular and every element in the lower triangle is 1*

(*iv*) *P is upper triangular and every element in the upper triangle is 1.*

Solution. Let n be any positive integer greater than R(m, m, m, m ; 2) and let A = $[a_{ij}]$ be any $n \times n$ binary matrix, the rows of which constitute the set X = $\{r_1, r_2, ..., r_n\}$.

The class C of all 2-element subsets of X is partitioned into 4 classes, as follows :

$$C_1 = \{\{r_i, r_j\} : a_{ji} = 0, a_{ij} = 0\}$$
$$C_2 = \{\{r_i, r_j\} : a_{ji} = 1, a_{ij} = 1\}$$
$$C_3 = \{\{r_i, r_j\} : a_{ji} = 0, a_{ij} = 1\}$$
$$C_4 = \{\{r_i, r_j\} : a_{ji} = 1, a_{ij} = 0\}.$$

Since $n \geq$ R(m, m, m, m ; 2), there exists a subset X′ of X with m elements (rows) such that all 2-elements subsets of X′ are contained in one of these 4 classes.

This implies the existence of an $m \times m$ principal submatrix in one of the categories (*i*) through (*iv*).

Problem 1.162. *An arithmetic progression of length n is a sequence of the form < a, a + d, a + 2d, ..., a + (n – 1)d >*

Show that in any partition of X = {1, 2, ..., 9} into 2 subsets, at least 1 of sets contains an arithmetic progression of length 3.

Solution. Suppose that the theorem is false.

Let X be partitioned into P and Q, and let 5 be an element of P.

Obviously both 1 and 9 [$d = 4$] cannot be in P, so that there are 3 cases to consider.

Case 1. 1 is in P and 9 is in Q.

Since 1 and 5 are in P, 3 is in Q. Since 3 and 9 are in Q, 6 is in P. Since 5 and 6 are in P, 4 is in Q.

Since 3 and 4 are in Q, 2 is in P. Since 5 and 6 are in P, 7 is in Q. Since 7 and 9 are in Q, 8 is in P.

But then P contains the arithmetic Progression 2, 5, 8, *a* contradiction.

Case 2. 9 is in P and 1 is in Q. Set X is invariant when each element is replaced by its tens-complement.

Under this transformation the persent case becomes case 1, which has already been disposed of.

Case 3. 1 and 9 are in Q. The number 7 is either in P or in Q suppose it is in P. Since 5 and 7 are in P, both 3 and 6 are in Q. That means Q has the arithmetic progression 3, 5, 9.

On the otherhand, if 7 is in Q, then 8 is in P.

Since 1 and 7 are in Q, 4 is in P. Since 4 and 5 are in P, 3 is in Q. Since 1 and 3 are in Q, 2 is in P.

Then P has the arithmetic progression 2, 5, 8.

Problem 1.163. *Show that in any group of 20 people there will always be either a subgroup of 4 mutual acquaintances or a subgroup of 4 mutual strangers.*

Solution. Suppose A is one of these 20 people.

People known to A are in room Y and people not known to A are in room Z.

Either room Y has atleast 10 people or room Z has atleast 10 people.

(*i*) If Y has atleast 10 people, then there is either a subgroup of 3 mutual acquaintances or a subgroup of 4 mutual strangers, as asserted, in this room.

In the former case A and these mutual acquaintances will form a subgroup of 4 mutual acquaintances.

(*ii*) Inter change "acqaintances" and "strangers" in (*i*).

1.7 THE CATALAN NUMBERS

A point in the cartesian plane whose coordinates are integers is called a **lattice point.** Consider a path from the origin to the lattice point A(*m, n*), where *m* and *n* are non negative, that

(*i*) starts from the origin

(*ii*) is always parallel to the *x*-axis or the *y*-axis

(*iii*) makes turns only at a lattice point, either along the positive *x*-axis or along the positive *y*-axis,

(*iv*) terminates at A.

A typical path is a sequence of $m + n$ unit steps, *m* of them horizontal and *n* of them vertical. The number of paths is $C(m + n, m) = C(m + n, n)$, the number of ways of reserving positions in the sequence for one or the other kind of step.

A path from P_0 to P_m in the cartesian plane is a sequence $<P_0, P_1, ..., P_m>$ of lattice points, $P_i(x_i, y_i)$, such that for each $i = 0, 1, ..., m - 1$, $x_{i+1} = x_i + 1$, $y_{i+1} = y_i$ or $x_{i+1} = x_i$, $y_{i+1} = y_i + 1$.

This path is **good** if $y_i < x_i$ $(i = 0, 1, ... m)$, otherwise it is **bad.**

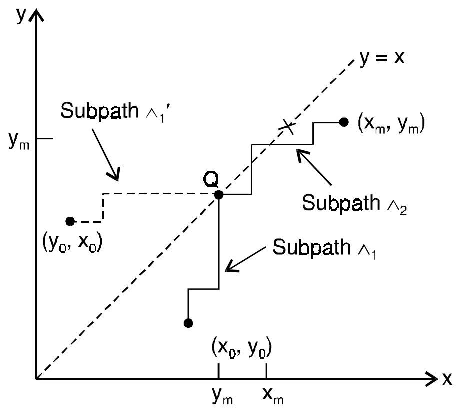


Fig. 1.6.

Set $m = x_m - x_0$ and $n = y_m - y_0$ to obtain the required number as $C(x_m - x_0 + y_m - y_0, x_m - x_0)$.

A **good path** is one that lies entirely below the 45° line (see Fig. 1.6). Thus the conditions $y_0 < x_0$ and $y_m < x_n$ are necessary for a good path, to which may be adjoined $x_0 \le x_m$ and $y_0 \le y_m$ (the *x* and *y* coordinates can never decrease along the path). Under these 4 conditions all paths will be good, unless it is possible for a path to intersect the 45° line at some ordinate less than or equal to y_m, *i.e.,* unless $x_0 \le y_m$. The desired criterion is $y_0 < x_0 < y_m < x_m$.

Fig. 1.6 shows a **bad path** from (x_0, y_0) to (x_m, y_m), it first intersects the line $y = x$ in the lattice point Q. If subpath $\wedge_1$, from (x_0, y_0) to Q, is reflected in the 45° line, $\wedge_1' + \wedge_2$ is a path from (y_0, x_0) to (x_m, y_m).

Any path from (y_0, x_0) to (x_m, y_m) defines by partial reflection a **bad path** from (x_0, y_0) to (x_m, y_m). There are $C(x_m - y_0 + y_m - x_0, x_m - y_0)$ **bad paths,** $C(x_m - x_0 + y_m - y_0, x_m - x_0) - C(x_m - x_0 + y_m - y_0, x_m - y_0)$ **good paths** from (x_0, y_0) to (x_m, y_m).

The n^{th} **catalan number**, C_n, is defined as the number of good paths from (1, 0) to $(n, n-1)$, then

$$C_n = \frac{1}{n} C(2n-2, n-1).$$

[We have $C_n = C(2n-2, n-1) - C(2n-2, n)$

$$= C(2n-2, n-1) \left[1 - \frac{n-1}{n}\right]$$

$$= \frac{1}{n} C(2n-2, n-1)]$$

Problem 1.164. *Find the number of sequences of the form* $\langle u_1\, u_2 \ldots\ldots u_{2n} \rangle$ *such that*

(*i*) u_i *is either* -1 *or* $+1$, *for every i,*

(*ii*) $u_1 + u_2 + \ldots\ldots + u_k \geq 0$, *for* $1 \leq k \leq 2n - 1$, *and*

(*iii*) $u_1 + u_2 + \ldots\ldots + u_{2n} = 0$.

Solution. Consider a path from (0, 0) to (n, n) as traced by a particle which makes unit steps in the x and y directions.

Let the particles location after i steps be (x_i, y_i) and define $u_i \equiv (x_i - x_{i-1}) - (y_i - y_{i-1})$.

Then, if the particle never rise above the line $y = x$, the integers u_i $(i = 1, 2, \ldots\ldots, 2n)$ statisfy (*i*), (*ii*), (*iii*) above.

Conversely, every sequence $\langle u_i \rangle$ that obeys (*i*), (*ii*) and (*iii*) defines a path from (0, 0) to (n, n) that never rises above $y = x$.

Hence, the number of such sequences is C_{n+1}.

Problem 1.165. (The Ballot Problem)

Suppose P and Q are 2 candidates for a public office who secured p votes and q votes, respectively. If $p > q$, *find the probability that P stayed ahead of Q throughout the counting of votes.*

Solution. In the cartesian plane let x and y, respectively, denote the votes accumulated by P and Q at any stage.

Every path from (0, 0) to (p, q) represents a possible history of the voting, and conversely.

Thus, the number of ways the voting could have gone is $C(p + q, p)$, out of which P leads continually in $C(p + q - 1, p - 1) - C(p + q - 1, p)$.

This is the number of good paths from (1, 0) to (p, q).

The desired probability is therefore

$$\frac{C(p+q-1,\, p-1) - C(p+q-1,\, p)}{C(p+q,\, p)} = \frac{p-q}{p+q}.$$

Problem 1.166. *Find the number of paths from (0, 0) to (n, n) such that*

(*a*) *either x > y at all interior lattice points or y > x at all interior lattice points ; and*

(*b*) *$y \le x$ at every lattice point on the path, and*

(*c*) *the path never crosses the line y = x.*

Solution. (*a*) The number of paths of this type will be twice the number of good paths from (1, 0) to (*n, n* – 1), or $2C_n$.

(*b*) Let A be the point (*n*, *n*).

Suppose the origin O(0, 0) is transferred to O′(– 1, 0). The new coordinates are O′(0, 0), O(1, 0), and A(*n* + 1, *n*). The number of good paths from O to A, namely, C_{n+1} is equal to the number of paths from O to A in which $y \le x$ at every lattice point.

(*c*) By reflectional symmetry, the required number is twice the number found in (*b*), or $2C_{n+1}$.

1.8 GROUP

A non empty set G with a binary operation • defined on it constitutes a group (G, *o*) if the following four properties hold.

(*i*) For all *x* and *y* in G, *x o y* is in G. (In multiplicative notation one writes *xy* instead of $x \bullet y$)

(*ii*) There exists an identity element *e* in G such that $x\, o\, e = e\, o\, x = x$ for all *x* in G.

(*iii*) Corresponding to each element *x* in G, there exists an inverse element x^{-1} in G such that $x\, o\, x^{-1} = x^{-1}\, o\, x = e$.

(*iv*) For every *x*, *y* and *z* in G the elements

$x \bullet (y \bullet z)$ and $(x \bullet y) \bullet z$ are identical.

The associativity property (*iv*) allows us to write $x \bullet y \bullet z$ for the triple product. We usually write $a \bullet b$ as *ab* and (G, *o*) as G if there is no risk of ambiguity.

1.8.1 Subgroup

A subset H of G is called a subgroup of (G, *o*), if (H, *o*) is a group.

1.8.2 Finite group

If G is a finite set with $|G| = n$ then (G, *o*) is a finite group of order *n*.

For example, the symmetric difference of sets A and B is defined by $A * B = (A \cup B) - (A \cap B)$ that is, A * B is the set of elements that belong to A or to B but not to both.

1.8.3 Permutation

Suppose that G is a fixed subgroup of the symmetric group of a finite set X and *x* is a given element of X.

Let $Gx \equiv \{g(x) : g \in G\}$

$G_x \equiv \{g \in G : g(x) = x\}$

$F(g) \equiv \{z \in X : g(z) = z\}$.

In words, Gx (the orbit of x with respect to G) is the set of all images of the given element x under the permutations in G ; G_x (the stabilizer of x in G) is the set of all permutations in G that have x as a fixed point ; F(g) (the permutation character of g in X) is the set of all fixed points of a given permutation $g \in$ G.

1.8.4 Permutation Groups and Their Cycle Indices

A permutation of a finite set X is a bijective (one-to-one and onto) mapping from X to X. Suppose f is a permutation of X and x is any element of X. Define recursively, $f'(x) \equiv f(x), f^2(x) \equiv f(f'(x))$, ... $f^i(x) \equiv f(f^{i-1}(x))$, ... since X is finite, there exists a positive integer r such that $f^r(x) = x$.

The sequence $\langle x, f^1(x), f^2(x), ..., f^{r-1}(x) \rangle$ is called a cycle of order (or length) r of the permutation f.

Obviously, every permutation of X can be represented as a composition of k disjoint cycles, where k is atleast 1 and atmost the cardinality of X.

The concept of the cycle representation of a permutation f of X = {1, 2, ... n}. The following algorithm produces this representation :

(*i*) Choose an element i of X (usually $i = 1$). Find the image of i under the mapping f, then the image of the image, then ......, until the image j appears such that $f(j) = i$. Thus the cycle (i ...... j) has been generated.

(*ii*) Choose an element of X not found in any one of the cycles already generated, and use this element as element i in step (*i*), thereby generating a new cycle.

(*iii*) Repeat step (*ii*) until X has been exhausted.

The cycle representation of a permutation is unique upto the order of the cycles in the composition and upto the choice, within each cycle, of the leading element.

For example, Given X = {1, 2, ..., 8} and 1 2 3 4 5 6 7 8 $\xrightarrow{f}$ 3 2 5 1 4 8 6 7

Starting with 1 : $f(1) = 3, f(3) = 5, f(5) = 4, f(4) = 1$.

Thus we have a cycle of length 4 which may be denoted by (1 3 5 4) [or (3 5 4 1) or (5 4 1 3) or (4 1 3 5)]

Starting with 2 : $f(2) = 2$. We have the cycle (2) of length 1.

Starting with 6 : $f(6) = 8, f(8) = 7, f(7) = 6$. Now we have a cycle of length 3 which may be denoted by (6 8 7) [or (8 7 6) or (7 6 8)].

The sum of the lengths has reached 4 + 1 + 3 = 8 = | X | which means we are finished : the cycle representation of f is (1 3 5 4) (2) (6 8 7) (or ......).

1.8.5 Weight

Let the cycle representation of f, a permutation of an n set, consist of a_1 cycles of length 1, a_2 cycles of length 2, ... a_i cycles of length i, ... Then the type of f is the vector $[a_1\ a_2\ ......\ a_n]$, and the weight of the type is the positive integer $W = 1^{a_1}\ 2^{a_2}\ ......\ n^{a_n}$.

For example, the permutation of above example, has 1 cycle of length 1, 1 cycle of length 3, and 1 cycle of length 4. The type of this permutation is [1 0 1 1 0 0 0 0]. The weight of this type is $1'\ 3'\ 4' = 12$.

1.8.6 Cycle index

Let G denote a group, of order m, of permutations of an n-set and length $g \in$ G be of type $[a_1, a_2, ..., a_n]$. The cycle index of g is the monic multinomial

$$Z(g\,;x_1,x_2,......,x_n) \equiv x_1^{a_1}\,x_2^{a_2}\,......\,x_x^{a_n}$$ and the cycle index of G is the multinomial.

$$Z(G\,;x_1,x_2,......,x_n) \equiv \frac{1}{m}\sum_{g \in G} Z\,(g\,;x_1,x_2,......,x_n)$$

For example, suppose the 4 vertices of a square are labeled 1, 2, 3 and 4, clockwise. A clockwise rotation through an angle of 0°, 90°, 180° or 270° takes the square into itself. Thus there are 4 circular or cyclic symmetries. In addition, there are 4 dihedral symmetries that are obtained by reflection of the square in the 2 diagonals and in the 2 lines bisecting opposite sides.

Conversely, the symmetries of the square compose a subgroup G of order 8 of S_4; the elements of G are as follows :

(*i*) The permutation induced by rotating the square clockwise through 0° is $g_1 = e = (1)\,(2)\,(3)\,(4)$, with cycle index $x_1^{\,4}$.

(*ii*) The permutation induced by rotating the square clockwise through 90° is $g_2 = (1\ 2\ 3\ 4)$ with cycle index $x_4^{\,1}$.

(*iii*) The permutation induced by rotating the square clockwise through 180° is $g_3 = (1\ 3)\,(2\ 4)$, with cycle index $x_2^{\,2}$.

(*iv*) The permutation induced by rotating the square clockwise through 270° is $g_4 = (1\ 4\ 3\ 2)$, with cycle index $x_4^{\,1}$.

(*v*) The permutation induced by reflection in the line joining the midpoints of $\overline{12}$ and $\overline{34}$ is $g_5 = (1\ 2)\,(3\ 4)$, with cycle index $x_2^{\,2}$.

(*vi*) The permutation induced by reflection in the line joining the midpoints of $\overline{14}$ and $\overline{23}$ is $g_6 = (1\ 4)\,(2\ 3)$ with cycle index $x_2^{\,2}$.

(*vii*) The permutation induced by reflection in the diagonal joining corners 2 and 4 is $g_7 = (2)\,(4)\,(1\ 3)$, with cycle index $x_1^{\,2}\,x_2^{\,1}$.

(*viii*) The permutation induced by reflection in the diagonal joining corners 1 and 3 is $g_8 = (1)\,(3)\,(2\ 4)$, with cycle index $x_1^{\,2}\,x_2^{\,1}$.

The cycle index of G is therefore

$$Z\,(G\,;x_1,x_2,x_3,x_4) = \frac{1}{8}\,(x_1^{\,4} + 2x_1^{\,2}\,x_2 + 3x_2^{\,2} + 2x_4).$$

1.8.7 Coloring and equivalent w.r.t. group of permutation

A function *f* from a finite set X to a finite set of colors Y is called a coloring of X. Two colorings *f* and ϕ in the set C of all colorings of X are said to be equivalent (indistinguishable) with respect to a group G of permutations of X if there exists a permuation π in G such that $f(x) = \phi(\pi(x))$ for all x in X.

In other words, if we attach names to the elements of X, so that G may be considered a group of 'renamings', then we do not distinguish between 2 colorings of X that become identical under some renaming in G.

Clearly, the relation of indistinguishability is reflexive, symmetric, and transitive, *i.e.,* an equivalence relation.

1.8.8 Pattern

The equivalence classes into which C partitioned by the indistinguishability relation are called the patterns in C (with respect to the group G).

For example, If G = $\{e\}$ then any 2 colorings are distinguishable, so that the number of patterns is the number of colorings. Because

$$Z(G\ ;\ x_1, x_2, \ldots\ x_n) = Z(e\ ;\ x_1, x_2, \ldots\ x_n) = x_1^{\ n}.$$

1.8.9 Pattern Inventory

Let the weight function w map Y into a set of r colors, $\{w(y_1), w(y_2), \ldots\ w(y_r)\}$. The pattern inventory (of C) with respect to G is the multinomial.

PI (G ; $w(y_1), w(y_2), \ldots\ w(y_r)$)

$$\equiv \sum_{\substack{n_1 + n_2 + \ldots + n_r = n \\ n_i \geq 0}} \tau(n_1, n_2, \ldots, n_r)\,[w(y_1)]^{n_1}\,[w(y_2)]^{n_2} \ldots\ldots [w(y_r)]^{n_r}$$

The coefficient $\tau(n_1, n_2, \ldots, n_r)$ gives the number of distinguishable (with respect to G) colorings (= numebr of patterns) that assign color $w(y_1)$ to n_1 elements of X ; color $w(y_2)$ to n_2 elements ; ......, color $w(y_r)$ to n_r elements. The summation is over the sizes of the color classes into which X is divided, the sum consists of $C(n + r - 1, r - 1)$ terms.

1.8.10 Isomorphic group

Groups (G, o) and (G′, $o′$) are isomorphic (identical in structure) if there exists a one-to-one correspondence f between G and G′ such that $f(x\ o\ y) = f(x)\ o'\ f(y)$, for all x and y in G.

1.8.11 Cyclic group

If x is an element in a group (G, o), we write $x\ o\ x$ as x^2, $x\ o\ x^2 = x^2\ o\ x$ as x^3, and so on.

Similary, $x^{-1}\ o\ x^{-1}$ is written as x^{-2}, $x^{-1}\ o\ x^{-2}$ as x^{-3} etc.

Thus the k^{th} power, x^k, of the element x is well defined when k is any non-zero integer, we make the natural definition $x^{\circ} \equiv e$. The group G is said to be a cyclic group if it contains an element x such that every element of G is a power of x. In this case we say that G is generated by x, and we write G = $\langle x \rangle$. If x generates G and if the powers of x are all distinct, G is an infinite cyclic group.

1.8.12 Abelian group

A group (G, o) is abelian if $x\ o\ y = y\ o\ x$ for every x and y in G.

1.8.13 Order of an element

If x is an element of (G, o) and if there exists a positive integer m such that x^m is the identity element e in G, then x is said to be of finite order. If x is of finite order, the smallest positive integer k such that $x^k = e$ is the order of x in G.

1.8.14 Direct product

If G and G′ are two groups, the direct product of G and G′ is the set of all ordered pairs

$$G \times G' = \{g, g'\} : g \in G, g' \in G'\}$$

endowed with the binary operation defined by

$$(g_1, g_1')\,(g_2, g_2') = (g_1 g_2, g_1' g_2').$$

1.8.15 Left and right coset

If H is a subgroup of G and x is an element of G, the set $x\text{H} \equiv \{xh : h \in \text{H}\}$ is called the left coset of H with respect to x, the right coset of H with respect to x is $\text{H}x \equiv \{hx : h \in \text{H}\}$.

1.8.16 Conjugate (Permutation)

Two permutations f and g of X are said to be conjugate if there exists a permutation h of X such that $hf = gh$.

1.8.17 Regular Icosahedron

A regular polytope (a solid in which all faces are congurent polygons and each vertex is incident with the same number of faces) with 12 vertices, 20 faces (congruent equilateral triangles) and 30 edges is called a regular icosahedron.

Theorem 1.20. *Suppose that a finite set X possesses exactly k distinct orbits with respect to a group G of permutations of X. Then*

(*i*) *For every* $x \in X$, $|Gx|\,|G_x| = |G|$

(*ii*) $\sum_{x \in X} |G_x| = k\,|G|$

(*iii*) $\sum_{x \in X} |G_x| = \sum_{g \in G} |F(g)|$

Proof. (*i*) Let g and h be in G_x. Then $g(h(x)) = g(x) = x$ which implies gh is in G_x.

Therefore G_x is a subgroup of G.

To produce a bijection between Gx and the set L of distinct left cosets of G_x.

Let $u \in \text{G}x$, *i.e.,* $u = g(x)$ for some $g \in \text{G}$.

Consider the mapping $u \to g\,\text{G}_x$ from Gx to L

(1) The mapping is onto. In fact, if $l\,\text{G}_x \in \text{L}$, we have l being a permutation of X, $l(x) = y(y \in \text{X})$. This means that $y \in \text{G}x$ and $y \to l\,\text{G}_x$.

(2) The mapping is one-one. Let u and v belong to Gx : $u = g(x)$ and $v = h(x)$, for $g, h \in \text{G}$.

Suppose that in L, $g\,\text{G}_x = h\,\text{G}_x$. Then $h^{-1}g \in \text{G}_x$ which implies $h^{-1}(g(x)) = x$ or $g(x) = h(x)$ or $u = v$.

Thus our mapping is the desired bijection.

(*ii*) There exists elements $x_1, x_2, ..., x_k$ such that $\{\text{G}x_1, \text{G}x_2, ..., \text{G}x_k\}$ is a partition of X. This let us write $\sum_{x \in \text{X}} |\text{G}_x| = \sum_{i=1}^{k} \sum_{x \in \text{G}x_i} |\text{G}_x|$

But, $|\text{G}_x|$ has the constant value $|\text{G}_{x_i}|$ over G_{x_i} (since $x_i \in \text{G}_{x_i}$)

Hence, $\sum_{x \in \text{X}} |\text{G}_x| = \sum_{i=1}^{k} |\text{G}_{x_i}||\text{G}_{x_i}| = \sum_{i=1}^{k} |\text{G}| = k|\text{G}|$

(*iii*) In the sum $\sum_{g \in \text{G}} |\text{F}(g)|$ the count of any $x \in \text{X}$ is $|\text{G}_x|$

Therefore $\sum_{g \in \text{G}} |\text{F}(g)| = \sum_{x \in \text{X}} |\text{G}_x|$.

Theorem 1.20(*a*). *Burnside-Frobenius Theorem*

$$\sum_{g \in G} |F(g)| = k|G|$$

1.8.18 Theorem 21. *Pólya's First Enumeration Theorem*

Let C be the set of all functions (colorings) from an n-set X to an r set Y ($n \geq 2$). Let G be a group of permutations of X, with cycle index Z (G ; x_1, x_2, ..., x_n). Then the number of patterns in C with respect to G is Z (G ; r, r, ..., r).

Proof. The patterns in C with respect to G (a permutation group on X) are the distinct orbits in C with respect to G, and these are the distinct orbits in C′ with respect to G′ (a permutation group on C). Their number is given by the Burnside-Frobenius theorem as

$$k = \frac{1}{|G'|} \sum_{\pi' \in G'} |F(\pi')| \qquad ...(1)$$

where $\qquad F(\pi') = \{f \in C : \pi'(f) = f\}$

Now, because $\pi'(f) = f$ if and only $f(\pi(x)) = f(x)$ for all $x \in X$ and because $|G'| = |G|$, one can convert (1) back to X and G :

$$k = \frac{1}{|G|} \sum_{\pi \in G} |\{f \in C : f(\pi(x)) = f(x) \text{ for all } x \in X\}| \qquad ...(2)$$

Now, if $f(\pi(x)) = f(x)$ and if $(x_1\ x_2\ ......\ x_j)$ is a cycle of π, $f(x_1) = f(x_2) = ...\ f(x_j)$ that is, f is constant over each cycle of π.

Conversely, if f is constant over each cycle of π and if $(xx_t\ ......\ x_u)$ is the cycle involving the arbitrary element $x \in X$, $f(\pi(x)) = f(x_t) = f(x)$.

It follows that the summa in the right hand side of (2) is just the number of ways of coloring X with $r \geq 2$ colors so that elements in the same cycle of the permutation π are given the same color.

If Π is of type $[a_1\ a_2\ ......\ a_n]$, this number of ways is $r^{a_1 + a_2 + ... + a_n}$;

Equation (2) becomes

$$k = \frac{1}{|G|} \sum_{\pi \in G} r^{a_1 + a_2 + ... + a_n} \equiv \frac{1}{|G|} \sum_{\pi \in G} Z(\pi ; r, r, ..., r) \equiv Z(G ; r, r, ..., r) .$$

1.8.19 Theorem 1.22. Pólya's second Enumeration Theorem

The pattern inventory, PI (G ; $w(y_1)$,$w(y_2)$, ..., $w(y_r)$), is the value of the cycle index, Z(G ; x_1, x_2, ..., x_n), at $x_i = [w(y_1)]^i + [w(y_2)]^i + ... + w[w(y_r)]^i$ (i = 1, 2, ... n).

Proof. We note that, the weights function $w(f)$ has the required constancy property needed for an application of the weighted Burnside-Frobenius theorem to the orbits in C with respect to the permutation group G′.

Now, we have $k = \dfrac{1}{|G'|} \sum_{\pi' \in G'} |F(\pi')|$

$$\text{PI}\ (G\ ;\ w(y_1),\ w(y_2),\ \ldots\ldots,\ w(y_r)) = \Sigma w\ (C_i) = \frac{1}{|G'|}\sum_{\pi' \in G'} w(\pi') \qquad \ldots(1)$$

Where $w(\pi') = \sum_{f \in F(\pi')} w(f)$

Convert back to X and G,

$$\text{PI} = \frac{1}{|G|}\sum_{\pi \in G}\left\{\sum_{\substack{f \in C : f(\pi(x)) = f(x) \\ (all\ x)}} [w(f(x_1))][w(f(x_2))]\ldots\ldots[wf(x_n))]\right\} \qquad \ldots(2)$$

The inner summation in (2) may be taken over all functions $f(x)$ that are constant over each cycle of π.

Let π be of type $[a_1\ a_2\ \ldots\ldots\ a_n]$ and define a horrendacs multinomial in the $w(y_i)$ as

$$\Omega \equiv \overbrace{[w(y_1) + w(y_2) + \ldots\ldots + w(y_r)]\ldots\ldots[w(y_1) + w(y_2) + \ldots\ldots + w(y_r)]}^{a_1 \text{ factors}}$$

$$\text{X}\ \overbrace{[w(y_1)^2 + w(y_2)^2 + \ldots\ldots + w(y_r)^2]\ldots\ldots[w(y_1)^2 + w(y_2)^2 + \ldots\ldots + w(y_r)^2]}^{a_2 \text{ factors}}$$

$$\text{X}\ \overbrace{[w(y_1)^3 + w(y_2)^3 + \ldots\ldots + w(y_r)^3]\ldots\ldots[w(y_1)^3 + w(y_2)^3 + \ldots\ldots + w(y_r)^3]}^{a_3 \text{ factors}}$$

X

....

$$\text{X}\ \overbrace{[w(y_1)^n + w(y_2)^n + \ldots\ldots + w(y_r)^n]\ldots\ldots[w(y_1)^n + w(y_2)^n + \ldots\ldots + w(y_r)^n]}^{a_n \text{ factors}}$$

The expansion of Ω consists of $r^{a_1 + a_2 + \ldots\ldots\ a_n}$ terms, which number is also the number of functions $f(x)$ that are constant over each cycle of π. The equality is no accident, we now demonstrate that the individual terms in the expansion are precisely the weights w of the individual functions $f(x)$.

Suppose that the cycles in the representation of π are put into one-to-one correspondence with the factors of Ω in the natural way : the one-cycles correspond one-to-one with the first a_1 factors, the 2 cycles, with the next a_2 factors, and so on.

If $f(x)$ maps a given j cycle T into y_v, draw a circle around the quantity $w(y_v)^j = \underset{x \in T}{\pi}\ w(f(x))$.

The expansion term given by the product of all circled quantities (one in each factors of Ω) will equal $\underset{U}{\pi}\left[\underset{x \in U}{\pi}\ w(f(x))\right]$ in which U runs through all cycles of π.

But these cycles effect a partition of X, so our expansion term is just $\underset{x \in X}{\pi} w(f(x)) = W(f)$.

We have just proved that the inner sum in (2) has the value Ω. But, by construction. $\Omega = Z(\pi\ ;\ x_1, x_2, \ldots\ldots, x_n)|_{x_j} = w(y_1)^j + w(y_2)^j + \ldots\ldots + w(y_r)^j\ (j = 1, 2, \ldots\ldots n)$.

1.8.20 Theorem 1.23. *Lagrange's theorem*

The order of a finite group is divisible by the order of any subgroup.

Proof. Let the group be of order n and let a given subgroup, of order S, have r distinct left cosets.

Let x be any element of G. Then x is an element of xH, since $x = xe$ and e is in H.

Thus every element of G is in at least 1 left coset of H. Two distinct left cosets have no elements in common.

Thus the left cosets of H make up a partition of G.

Let $H = \{h_1, h_2, h_3, \ldots\ldots h_k\}$ and let x be any element of G. Then $xH = \{xh_1, xh_2, \ldots\ldots xh_k\}$.

The elements of xH must be distinct, for $xh_i = xh_j$ would imply $h_i = h_j$. Hence $|xH| = k$.

Therefore $rS = n$.

1.8.21 Theorem 1.24. *Characterization theorem for cyclic groups.*

If G is a group of order $n \geq 2$, the following are equivalent :

(i) G is a cyclic group

(ii) For each divisor d of n, the cardinality of $\{x \in G : x^d = e\}$ is d.

(iii) For each divisor d of n, the cardinality of {x ∈ G : the order of x is d} is $\phi(d)$.

Proof. $(i) \Rightarrow (ii)$

Suppose G is generated by x. Let $n = dk$ and consider the collection $Y \equiv \{x^\circ, x^k, x^{2k}, x^{3k}, \ldots, x^{(d-1)k}\}$.

The d elements in this collection are distinct (because x is of order n). The typical element x^{ik} of Y satisfies $(x^{ik})^d = (x^{dk})^i = (x^n)^i = e^i = e$.

Thus Y is a subset of $\{x \in G : x^d = e\}$.

Conversely, let y be any element of G such that $y^d = e$.

Since x is a generator of G, there exists an integer $0 \leq m \leq n - 1$ such that $y = x^m$.

Therefore $x^{md} = e$.

But x is of order n ; so that, for some integer r,

$$md = rn = rdk \text{ or } m = rk.$$

Thus $y = x^{rk}$, with $0 \leq r \leq d - 1$ (because $r/d = m/n$) which means that $y \in Y$.

Consequently, $\{x \in G : x^d = e\}$ and Y are identical sets. So that $|\{x \in G : x^d = e\}| = |Y| = d$.

$(ii) \Rightarrow (iii)$

Let y be an element of G, of order C. Then $y^d = e$ if and only if $\frac{c}{d}$ (c divides d).

Consequently, $\{x \in G : x^d = e\}$ may be partitioned in such manner that the i^{th} cell consists of all elements of G whose order equals the i^{th} divisor of d. Define $f(c)$ to be the number of elements of order c, and specialize d to a divisor of n. Then by (*ii*),

$$\sum_{\frac{c}{d}} f(c) = d$$

The möbius formula yields $f(d) = \sum_{\frac{c}{d}} \mu(c)\left(\frac{d}{c}\right) = \phi(d)$

(*iii*) ⇒ (*i*)

By (*iii*), with $d = n$, there exist $\phi(n) \geq 1$ elements of order n in G. Hence $G = C_n$.

Problem 1.167. *A group (G, o) is abelian if x o y = y o x for every x and y in G. Show that every cyclic group is abelian.*

Solution. Let x and y be 2 elements in a cyclic group G generated by g. Because $x = g^m$ and $y = g^n$ for some integers m and n.

$$x \ o \ y = g^m \ o \ g^n = g^{m+n} = g^{n+m} = g^n \ o \ g^m = y \ o \ x.$$

Problem 1.168. *Prove that, in any group (a) the identity element is unique ; and (b) the inverse of any element is unique.*

Solution. (*a*) Suppose there existed two identities, e and f.

Then, since e is a right-identity and f is a left identity, $f = f \ o \ e = e$.

(*b*) If element x had 2 inverses, y and z, the associative law would give

$$(y \ o \ x) \ o \ z = y \ o \ (x \ o \ z) \text{ or } e \ o \ z = y \ o \ e \quad \text{or} \quad z = y.$$

Problem 1.169. *Show that the set of all integers under the binary operation of addition is an infinite cyclic group.*

Solution. If z is the set of all integers, $(z, +)$ is a group because all 4 group axioms are satisfied by the structure.

Let $G = < a >$ be an infinite cyclic group.

The mapping $f : Z \to G$ defined by $f(z) = a^z$ is obviously a bijection, and we have

$$f(z + w) = a^{z+w} = a^z \ o \ a^w = f(z) \ o \ f(w)$$

Therefore the 2 groups are isomorphic, which means that $(z, +)$ is the infinite cyclic group $< 1 >$.

Problem 1.170. *Show that, under an isomorphism,*

(a) the identity elements of G and G′ correspond

(b) if u and v are inverses in G, then f (u) and f (v) are inverses in G′ and

(c) give an example of 2 isomorphic groups of order n.

Solution. (*a*) In G, $x \ o \ e = e \ o \ x = x$;

Whence $f(x \ o \ e) = f(e \ o \ x) = f(x)$ or $f(x) \ 0' \ f(e) = f(e) \ 0' \ f(x) = f(x)$

which shows that the idenity in G′ is $e' = f(e)$.

(*b*) From $u \ o \ v = v \ o \ u = e$ and part (*a*).

$$f(u)\ o'\ f(v) = f(v)\ o'\ f(v) = e'.$$

(*c*) One such pair is composed of the group of rotational symmetries of a regular *n*-gon and the group ({0, 1, 2, ..., $n - 1$}, +), where the operation + is addition modulo *n*.

Problem 1.171. *Show that if H (| x | = k) is a finite subgroup of G, then every left (right) coset of H has cardinality k.*

Solution. Let H = $\{h_1, h_2, h_3, ..., h_k\}$ and let *x* be any element of G. Then *x*H = $\{xh_1, xh_2, ..., xh_k\}$.

The elements of *x*H must be distinct, for $xh_i = xh_j$ would imply $h_i = h_j$.

Hence | *x*H | = *k*.

Problem 1.172. *Show that if H is a subgraph of G and if x and y are in G, then either xH ∩ yH is empty or xH = yH.*

Solution. If *x*H ∩ *y*H is not empty, there exists an element *z* which is in *x*H and also in *y*H.

Hence there exist *h* and *h*′ in H such that

$z = xh = yh'$, which in turn implies $y^{-1}x = h'h^1$ is in H, which gives *x*H = *y*H.

Problem 1.173. *Show that a subset H of a finite group is a subgroup if and only if H is closed with respect to multiplication.*

Solution. Let *x* and *y* be 2 elements in H, and let the order of *y* be *m*. Then $y^m = e$ implies $y^{m-1} = y^{-1}$ and by hypothesis, y^{m-1} is in H.

Thus *x* and y^{-1} are in H.

Problem 1.174. *Show that the class of distinct left cosets of a subgroup H of a group G constitutes a partition of the group.*

Solution. Let *x* be any element of G. Then *x* is an element of *x*H, since $x = xe$ and *e* is in H.

Thus every element of G is in at least 1 left coset of H.

Two distinct left cosets have no elements in common.

Thus the left cosets of H make up a partition of G.

Problem 1.175. *Prove that a subgroup of a cyclic group is cyclic.*

Solution. Let G′ be a subgroup of G = <*x*>.

Every element in G′ is of the form x^k, let *m* be the smallest positive *k* for which x^k is in G′.

Now, for any integer *k*, the division algorithm gives $k = qm + r$, where $0 \le r < m$.

Hence, $x^k = x^{qm+r} = (x^m)^q\ x^r = ux^x$ in which $u \in$ G′ because $x^m \in$ G′ and G′ is closed under multiplication. It follows that u^{-1} also belongs to G′, so that, if $x^k \in$ G, $x^r = u^{-1}\ x^k \in$ G′.

If *r* were positive, this would violate the minimality of *m*.

Therefore, $r = 0$ and each x^k in G′ may be written as $(x^m)^q$; *i.e.,* G′ = < x^m >.

Problem 1.176. *If G is a cyclic group of order* n*, find the number of distinct generators of G.*

Solution. Suppose G = <*x*> = $\{e, x^1, x^2, ..., x^{n-1}\}$.

Let *m* be any positive integer less than, and relatively prime to, *n*, consider the cyclic group

G′ = <x^m> = $\{e, x^{1m}, x^{2m}, ...\ x^{(n-1)m}\}$.

To establish that G′ = G it suffices to show that the elements of G′ are distinct.

Suppose, on the contrary, that for some $0 \le b < a \le n - 1$ we had $x^{am} = x^{bm}$. Then necessarily

$(a - b)m = cn$ for some integer c.

But, m and n being relatively prime, this equation would require that n divide $a - b$, an impossibility since $0 < a - b < n$.

If, on the other hand, $m = rp$ and $n = sp$ $(r < s < n)$ then $(s^m)^s = (x^n)^r = e$ so that the order of x^m, and consequently the order of $<x^m>$ is smaller than n.

The conclusion is that G has just as many generators as there are positive integers less than, and relatively prime to n, *i.e.*, $\phi(n)$ generators.

Problem 1.177. *If G and G′ are two groups, the direct product of G and G′ is the set of all ordered pairs,* $G \times G' = \{(g, g') : g \in G, g' \in G'\}$ *endowed with the binary operation defined by* $(g_1, g_1')(g_2 . g_2') = (g_1g_2, g_1' g_2')$ *show that* $G \times G'$ *is a group.*

Solution. By definition, the product of 2 elements in $G \times G'$ is in $G \times G'$.

Further, $(g, g')(e, e') = (ge, g'e') = (g, g') = (eg, e'g') = (e, e')(g, g')$.

Thus (e, e') is the identity in the direct product.

Also, $(g, g')(g^{-1}, g'^{-1}) = (g^{-1}, g'^{-1})(g, g') = (e, e')$.

So each element in the direct product has an inverse element. The associativity rule is obviously satisfied in the product structure. Thus $G \times G'$ is a group.

1.8.22 Theorem 1.25. *The Burnside-Frobenius Theorem with weights*

Suppose that $X_1, X_2, ..., X_k$ *are the distinct orbits in the set* $X = \{x_1, x_2, ..., x_n\}$ *with respect to the permutation group* $G = \{g_1, g_2, ... g_m\}$. *On X define a weight function w(x)-weights may be numbers or algebraic symbols, with the property that whenever* x_r *and* x_s *are in the same orbit,* $w(x_r) = w(x_s)$. *Use the following recipe to induce a weight function on G :*

$$W(g_i) = \sum_{x \in G(g_i)} w(x) \qquad (i = 1, 2, ..., m)$$

that is, the weight of permutation in G is the total weight of its fixed points in X, then $\sum_{i=1}^{m} w(g_i) = \left(\sum_{P=1}^{k} w_P\right)$ *m in which* w_P *(P = 1, 2, ..., k) is the unique value assumed by w(x) over* X_p.

Proof. Let t be an element of X_P, so that $X_P = Gt$ and $w(t) = w_P$.

By definition of the stabilizer, t contributes its weight to exactly $|G_t|$ summands on the left side of (1)

$$k = \frac{1}{|G|}\sum_{\pi \in G} [\{f \in C : f(\pi(x)) = f(x) \text{ for all } x \in X\}] \qquad ...(1)$$

its contribution is thus $|G_t|\, w_P$. Because $|G_x|$ is constant over X_P, any other element of X_P makes the same contribution.

Consequently, the net contribution of X_P to the left side is

$$|G_t| \times |G_t|\, w_P = mw_P$$

As this weight is precisely reflected in the right side (1) the equation is proved.

1.8.23 Theorem 1.26. *Cauchy's Formula*

The number of permutations of X = {1, 2, ..., n} that are of type [$a_1\, a_2 \ldots\ldots a_n$] is $\dfrac{n!}{Wa_1!\, a_2! \ldots\ldots a_n!}$

where $W = 1^{a_1}\, 2^{a_2} \ldots\ldots a^{a_n}$ *is the weight of the type.*

Proof. The number of ways of partitioning X into a_1 cells of cardinality 1, a_2 cells of cardinality 2, ... a_n of cardinality n, is given by

$$N = \frac{n!}{[a_1!\,(1!)^{a_1}]\,[a_2!\,(2!)^{a_2}] \ldots\ldots [a_n!\,(n!)^{a_n}]}$$

But a cell is not the same as a cycle.

Infact, a cell with q elements gives rise to $(q - 1)!$ distinct cycles of length $q - 1$ for each circular permutation of the elements. Hence the desired number is

$$N[(1-1)!]^{a_1}\,[(2-1)!]^{a_2} \ldots\ldots [(n-1)!]^{a_n} = \frac{n!}{[a_1!\,1^{a_1}]\,[a_2!\,2^{a_2}] \ldots\ldots [a_n!\,n^{a_n}]}$$

$$= \frac{n!}{Wa_1!\,a_2! \ldots\ldots a_n!}.$$

1.8.24 Theorem 1.27. *Cayley's Theorem*

Every finite group is isomorphic to a group of permutations.

Proof. The idea behind the proof is very simple, because each element of a group G has its inverse, the rows of the multiplication table for G must be distinct permutations of G. Thus, given the finite group (G, o), where G = ($g_1, g_2, \ldots\ldots, g_m$}, define m distinct permutations of G by

$$\pi_1(g) = g_1\, o\, g \quad \pi_2(g) = g_2\, o\, g \ldots\ldots \pi_m(g) = g_m\, o\, g$$

Consider the group (G′, o'), where G′ = {$\pi_1, \pi_2, \ldots\ldots, \pi_m$} and where o' denotes multiplication of permutations as defined. The mapping f : G → G′ defined by

$$f(g_i) = \pi_i \ (i = 1, 2, \ldots\ldots, m)$$

is obviously a bijection. Indeed, it is an isomorphism, for, if $g_i\, o\, g_j = g_k$, then for each $g \in$ G,

$$\pi_k(g) = g_k\, o\, g = (g_i\, o\, g_j)\, o\, g = g_i\, o\, (g_j\, o\, g)$$

$$= g_i\, o\, \pi_j(g) = \pi_i(\pi_j(g)) = (\pi_i\, o'\, \pi_j)(g)$$

i.e., $\pi_i\, o'\, \pi_j = \pi_k$ (f preserves group multiplication)

Problem 1.178. *Show that there are precisely 17, 824 distinguishable (under rotations) vertex colorings of the regular dodecahedron using 1 or 2 colors.*

Solution. The regular icosahedron will have as its geometric dual a solid with 20 vertices and 12 faces, each of which is a regular pentagon ; this is the regular dodecahedron.

Therefore, the cycle index as

$$Z(G\,;\, x_1, x_2, \ldots\ldots, x_{20}) = \frac{1}{60}\,(x_1^{20} + 15x_2^{10} + 20x_1^2x_3^6 + 24x_5^4)$$

The number of vertex colorings is then

$$Z(G\ ;\ 2, 2, ..., 2) = \frac{1}{60}(1, 069, 440) = 17{,}824.$$

Problem 1.179. *Find the number of ways, under the rotational group, of coloring the vertices and faces of a regular octahedron so that 4 vertices are red, 2 vertices are blue, 4 faces are green, and 4 faces are yellow.*

Solution. Because the vertex coloring and the face coloring are independent, we may treat them separately and then use the product rule.

The cycle index of the group of vertex permutation is

$$\frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_2^3 + 6x_1^2x_4 + 8x_3^2)$$

Therefore, the pattern inventory for red (R) and blue (B) is

$$\frac{1}{24}[(R + B)^6 + 3(R + B)^2 (R^2 + B^2)^2 + 6(R^2 + B^2)^3 + 6(R + B)^2 (R^4 + B^4) + 8(R^3 + B^3)^2]$$

The coefficient R^4B^2 in the pattern inventory is

$$\frac{1}{24}\left[\frac{6!}{4!2!} + 3(3) + 6\left(\frac{3!}{2!1!}\right) + 0\right] = 2$$

The cycle index of the group of face permutation is

$$\frac{1}{24}(x_1^8 + 9x_2^4 + 8x_1^2x_3^2 + 6x_4^2)$$

The pattern inventory for green (G) and yellow (Y) is

$$\frac{1}{24}[(G + Y)^8 + 9(G^2 + Y^2)^4 + 8(G + Y)^2 (G^3 + Y^3)^2 + 6(G^4 + Y^4)]$$

The coefficient of G^4Y^4 in the pattern inventory is

$$\frac{1}{24}\left[\frac{8!}{4!4!} + 9\left(\frac{4!}{2!2!}\right) + 8(4) + 6(2)\right] = 7$$

Thus there are (2)(7) = 14 ways of coloring.

Problem 1.180. *Find the number of distinguishable necklaces consisting of 7 stones, of which 2 stones are red, 3 stones are blue, 2 stones are green, when (a) only rotational symmetrices (of a regular polygon with 7 vertices) are considered ; and (b) both rotational and reflectional symmetrices are considered.*

Solution. (*a*) The group here is cyclic and of prime order

$$Z\ (C_7\ ;\ x_1, x_2, ..., x_7) = \frac{1}{7}(x_1^7 + 6x_7)$$

$$\text{PI}(C_7 ; R, B, G) = \frac{1}{7}\ [(R + B + G)^7 + 6(R^7B^7 + G^7)]$$

The number we seek τ (2, 3, 2) will be $\frac{1}{7}$ times the coefficient of $R^2B^3G^2$ in $(R + B + G)^7$.

The multinomial theorem gives

$$\tau(2, 3, 2) = \frac{1}{7}\frac{7\,!}{2\,!3\,!2\,!} = 30.$$

(*b*) The group is the dihedral group H_{14}.

$$\text{PI}(H_{14} ; R, B, G) = \frac{1}{2}\ \text{PI}(C_7 ; R, B, G) + (R + B + G)(R^2 + B^2 + G^2)^3$$

using the result of (*a*), we have

$$\tau(2, 3, 2) = \frac{1}{2}\ (30) + \frac{1}{2}\left(0 + \frac{3\,!}{1\,!1\,!1\,!} + 0\right) = 18.$$

Problem 1.181. *Let X = {1, 2, 3, 4}, Y = {y_1, y_2} ; w(y_1) = R, w(y_2) = B ; and G = {(1) (2) (3) (4), (1 2) (3 4), (1 3) (2 4), (1 4) (2 3)}.*

Find the pattern inventory for the set C of all functions from X to Y.

Solution. The cycle index is $\frac{1}{4}\ (x_1{}^4 + 3x_2{}^2)$.

By pólya's first theorem, with $r = |\,Y\,| = 2$, the number of pattern in C is

$$k = \frac{1}{4}\ (2^4 + 3\ .\ 2^2) = 7.$$

To visualize these 7 patterns, it is helpful to have a concrete model of X and G.

Fortunately, we have several available. If X is identified with the vertex set of the square, then G will be the subgroup $\{g_1, g_3, g_5, g_6\}$ of the full symmetry group D_8.

Now, there are 5 possible values of the assignment vector (n_1, n_2) : (0, 4) (1, 3) (2, 2) (3, 1) (4, 0)

Obviously, (0, 4) and (4, 0) each determine a single pattern (there's only one way to paint all vertices the same color) ; the respective weights of these patterns are $w(C_1) = B^4$ and $w(C_2) = R^4$.

Similarly, (1, 3) and (3, 1) generate 1 pattern apiece (there is a reflection or rotation in G that will give the odd-colored vertex any desired location) ; $w(C_3) = RB^3$, $w(C_4) = R^3B$.

By elimination (2, 2) must give rise to 7 – 4 = 3 patterns, with

$w(C_5) = w(C_6) = w(C_7) = R^2B^2$ (as shown figure below)

Thus, $\text{PI}(G ; R, B) = \sum_{i=1}^{7} w(C_i)\ R^4 + R^3B + 3R^2B^2 + RB^3 + B^4.$

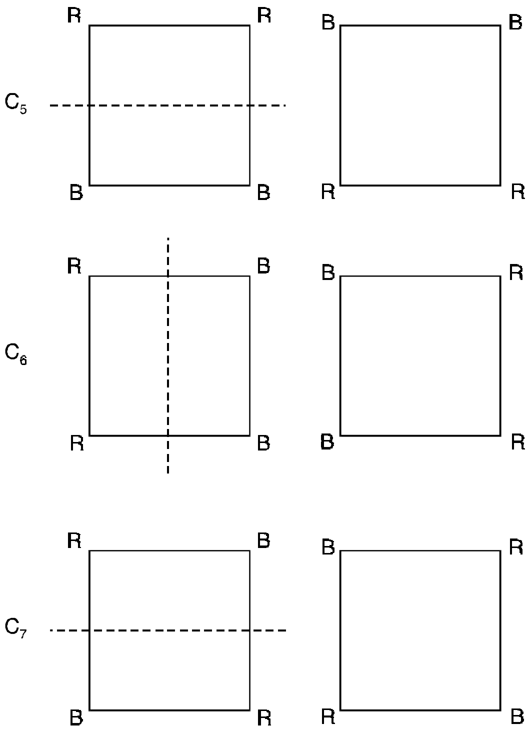


Fig. 1.

Problem 1.182. *Let C be the (finite) set of all functions f from a finite set X to a finite set Y, and let G be a group of permutation of X. For each π in G, define a mapping π′ from C to C by π′(f (x)) = f (π(x)) (for each x ∈ X and each f ∈ C)*

Prove that (a) π′ is a permutation of C, and

(b) G′ = {π′ : π ∈ G} is a group.

Solution. (*a*) If $\pi'(f_1) = \pi'(f_2)$, then $f_1(\pi(x)) = f_2(\pi(x))$ for every $x \in X$, which implies that $f_1(t) = f_2(t)$ for every $t \in X$.

So $f_1 = f_2$ (π' is injective). Also, π' is surjective, infact, for any $f \in C$,

$$f(x) = f(\pi(\pi^{-1}(x))) = \pi'(f(\pi^{-1}(x))) \equiv \pi'((f\pi^{-1})(x))$$

Hence, as a bijection, π' is a permutation of C.

(*b*) To show that G′ is closed with respect to multiplication (composition).

Let π_1 and π_2 in G respectively determine π_1' and π_2' in G′.

Our assertion is that $\pi_1\pi_2$ in G determines $\pi_1'\pi_2'$ in G.

i.e., $(\pi_1\pi_2)' = \pi_1'\pi_2'$.

$$(\pi_1\pi_2)'(f(x)) = f(((\pi_1\pi_2)(x)) = f((\pi_1(\pi_2(x)))$$

$$= \pi_1'(f(\pi_2(x))) = \pi_1'(\pi_2' f(x))) = (\pi_1'\pi_2')(f(x)).$$

Problem 1.183. *Let G be a group of permutations of* $X = \{x_1, x_2, \ldots\ldots x_n\}$ *and let C be the set of all functions from X to* $Y = \{y_1, y_2, \ldots\ldots y_n\}$. *If w(y) is a given weight function on Y, we induce a weight function w(f) on C by the formula*

$$w(f) = [w(f(x_1))]\,[w(f(x_2))] \ldots\ldots [w(f(x_n))].$$

(*a*) *If f and* ϕ *in C are equivalent with respect to G, prove that* $w(f) = w(\phi)$.

(*b*) *Denote by* $C_1, C_2 \ldots\ldots C_k$ *the distinct patterns in C ; let* $w(C_i)$ $(i = 1, 2, \ldots\ldots, k)$ *stand for the constant value of w over* C_i. *Show that the pattern inventory of C can be expressed as*

$$PI\,(G\,;\, w(y_1), w(y_2), \ldots\ldots, w(y_r)) = \sum_{i=1}^{k} w(C_i)\,.$$

Solution. (*a*) Since *f* and ϕ are equivalent, there exists a permutation π of *x* such that $f(x) = \phi(\pi(x))$ for all *x* in X.

Therefore
$$\begin{aligned} w(f) &= [w(f(x_1))]\,[w(f(x_2))] \ldots\ldots [w(f(x_n))] \\ &= [w(\phi(\pi(x_1)))]\,[w(\phi(\pi(x_2)))] \ldots\ldots [w(\phi(\pi(x_n)))] \\ &= [w(\phi(x_1'))]\,[w(\phi(x_2'))] \ldots\ldots [w\phi(x_n'))] = w(\phi). \end{aligned}$$

(*b*) In a permutation of colored objects, the numbers or red objects or green objects, etc. clearly do not change. It follows that all colorings *f* making up a given pattern C_i in C are characterized by the same "assignment vector" $(n_1, n_1, \ldots\ldots, n_r)$.

This means that any $f \in C_i$ maps n_1 elements of X into y_1, n_2 elements into y_2, ..., n_r elements into y_r, so that the weight of C_i is

$$w(C_i) = w(f) = [w(y_1)]^{n_1}\,[w(y_2)]^{n_2} \ldots\ldots [w(y_r)]^{n_r}$$

Now, in the definition of the pattern inventory, the coefficient $\tau\,(n_1, n_2, \ldots\ldots, n_r)$ is defined to be the number of pattern answering to the vector $(n_1, n_2, \ldots\ldots, n_r)$.

Hence we can write.

$$\sum_{i=1}^{k} w(C_i) \equiv \text{total weight of the } k \text{ patterns}$$

$$= \sum_{(n_1, n_2, \ldots\ldots n_r)}^{k} [\text{total weight of patterns answering to } (n_1, n_2, \ldots\ldots, n_r)]$$

$$= \sum_{(n_1, n_2, \ldots\ldots n_r)} \tau\,(n_1, n_2, \ldots\ldots n_r)\,[w(y_1)]^{n_1}\,[w(y_2)]^{n_2} \ldots\ldots [w(y_r)]^{n_r}$$

$$\equiv PI(G\,;\, w(y_1), w(y_2), \ldots\ldots w(y_r)).$$

Problem 1.184. *(a) How may one define a cycle index for an orbitrary finite group ?*

(b) Illustrate the procedure for the group of subsets of X = {a, b} under the symmetric difference.

Solution. (*a*) Take the cycle index of the permutation group G′ to be the cycle index of the abstract group G.

(*b*) The multiplication table for G = {ϕ, {*a*}, {*b*}, X} is

*	ϕ	{*a*}	{*b*}	X	
ϕ	ϕ	{*a*}	{*b*}	X	$\rightarrow \pi_1$: (*) ({*a*})({*b*})(X)
{*a*}	{*a*}	ϕ	X	{*b*}	$\rightarrow \pi_2$: (ϕ{*a*}) ({*b*}X)
{*b*}	{*b*}	X	ϕ	{*a*}	$\rightarrow \pi_3$: (ϕ{*b*})({*a*}X)
X	X	{*b*}	{*a*}	ϕ	$\rightarrow \pi_4$: (ϕX)({*a*}{*b*})

To the right of each row is shown the cycle representation of the permutation is G′ generated by that row.

These 4 permutations have the respective cycle indices

$x_1^4, x_2^2, x_2^2, x_2^2$ hence

$$Z = (G\ ;\ x_1, x_2, x_3, x_4) = Z(G'\ :\ x_1, x_2, x_3, x_4) = \frac{1}{4}\ (x_1^4 + 3x_2^2).$$

Problem 1.185. *Obtain the cycle index of the group of permutations of the 6 edges induced by the rotational symmetrices of the regular tetrahedron.*

Solution. The tetrahedron is ABCD (vertices), with edges AB, AC, AD,BC, BD, and CD marked as 1, 2, 3, 4, 5 and 6 respectively. The 12 rotational symmetries listed the following edges permutations.

(*i*) *e* = (1)(2)(3)(4)(5)(6)

(*ii*) (1 2 3)(4 6 5) and (1 3 2)(4 5 6) about vertex A, and a similar pair for each of the other 3 vertices.

(*iii*) (1)(6)(2 5)(3 4) and 2 similar permutations.

Hence $$Z(G\ ;\ x_1, x_2, \ldots\ldots. x_6) = \frac{1}{12}\,(x_1^6 + 3x_1^2x_2^2 + 8x_3^2).$$

Problem 1.186. *In a military mess the food trays are rectangular and divided into 4 equal rectangular compartments. Find the number of distinguishable ways of filling a tray with 4 foods if the long dimension must be parallel to the table edge.*

Solution. Label the corners of the tray 1, 2, 3 and 4 (clockwise), where $\overline{12}$ is a longer side. The symmetry group G of the rectangle is composed of the following permutations :

(1) (2) (3) (4)	[the zero rotation]
(1 3) (2 4)	[180° rotation]
(1 2) (3 4)	[reflection in perpendicular axis]
(1 4) (2 3)	[reflection in parallel axis]

Hence, $Z(G\ ;\ x_1, x_2, x_3, x_4) = \frac{1}{4}\,(x_1^4 + 3x_2^2)$ and

$$Z(G\ ;\ 4, 4, 4, 4) = \frac{1}{4}\,(4^4 + 3.4^2) = 76 \text{ ways}.$$

Problem 1.187. *Two permutations f and g of X are said to be conjugate if there exists a permutation h of X such that hf = gh. Show that 2 permutations are conjugate if and only if they are of the same type.*

Solution. If f and g are conjugate permutations, there exists a permutation h such that $g = hfh^{-1}$.

Suppose $C = (x_1 x_2 \ldots\ldots x_r)$ is a cycle of f, of length r.

Then $f(x) = x_2 \quad f(x_2) = x_3 \ldots\ldots f(x_r) = x_1$

Let $h(x_i) = y_i$ for each i. Then

$$g(y_i) = h(f(h^{-1}(y_i)) = h(f(x_i)) = h(x_{i+1}) = y_{i+1}$$

in which the subscript is to be evaluated modulo r.

Thus every cycle of length r corresponds to a cycle of g of length r, and vice versa. So f and g are of the same type.

On the other hand, assume that f and g are of the same type, and let $C = (x_1 x_2 \ldots\ldots x_r)$ be a cycle of f.

Then g has a cycle of the form $C' = (y_1 y_2 \ldots\ldots y_r)$.

Define $h(x_i) = y_i$ over C and similarly over every other cycle of f, this makes h a bijection from X to X, or a permutation of X. We have

$$h(f(x_i)) = h(x_{i+1}) = y_{i+1} = g(y_i) = g(h(x_i))$$

so f and g are conjugate.

Problem 1.188. *Find the number of ways of coloring the corners of a regular pentagon using 3 colors if indistinguishability is with respect to the subgroup of rotational symmetries.*

Solution. There are $3^5 = 243$ ways of coloring the 5 corners if rotational symmetries are ignored, thus we have a set C of 243 elements.

The group G′ of rotational symmetries has 5 elements.

Let g_1' be the identity, then $F(g_1')$ has 243 elements.

The other rotations will preserve a color configuration if and only if it involves a single color.

This means that $F(g_2')$, $F(g_3')$, $F(g_4')$, $F(g_5')$ have 3 elements each.

Thus, the number of colorings is $\left(\frac{1}{5}\right)(243 + 12) = 51$.

Problem 1.189. *Show that if G is a finite group of order n, then all elements of G are of finite order and no order exceeds n.*

Solution. If $x \in G$, the elements in $\{x^k : k = 0, 1, \ldots..\}$ cannot be all distinct, since G is finite.

Hence there must exist integers p and q, where $p > q \geq 0$, such that $x^p = x^q$, which implies $x^{p-q} = e$. So, x is of finite order, say, k.

Because x generates a subgroup of order k, $k \leq n$.

Problem 1.190. *A finite cyclic group of order m is denoted by C_m. Show that if m and n are relatively prime, the direct product $C_m \times C_n$ is a cyclic group.*

Solution. As the direct product is a finite group of order mn, it suffices to prove that it contains an element of order mn.

Let C_m be generated by x and C_n be generated by y, and let k be the order of the element $Z = (x, y)$ of the direct product.

Then $Z^k = (e, e') = (x^k, y^k)$, which implies that k is a multiple of m and a multiple of n.

Since k is the smallest positive integer P such that $Z^P = (e, e')$, it follows that k is the least common multiple of m and n.

But, if m and n are relatively prime, their least common multiple is their product.

Thus the order of Z is mn.

Problem 1.191. *If $f = (1\ 2\ .....\ n)$ is a permutation of $X = \{1, 2, ...\ n\}$ then the cyclic group (of permutations) $G = < f >$ is of order n. Prove that in the cycle representation of any element of G all cycles are of same length.*

Solution. The type of $f^{\circ} = e$ is $[n\ 0\ 0\ 0\ .....\ 0]$.

Let $m = m(i)$ be the length of the shortest cycle in the cycle representation of f^i $(1 \le i \le n - 1)$ and let x be an element in some cycle of f^i of length m. Then $f^{im}(x) = (f^i)^m(x) = x$.

Now, if $y \in$ X, both x and y belong to the same cycle of the permutation $f = (1\ 2\ ......\ n)$. This implies that there exists r such that $f^r(x) = y$.

Consequently, $(f^i)^m(y) = f^{im} f^r(x) = f^r f^{im}(x) = f^r(x) = y$. So the element y belongs to a cycle in f^i whose length divides m. But m is the length of the shortest cycle in f^i. Thus, every cycle in f^i is of length $m(i)$, which common length must be a divisor of n.

Problem 1.192. *If $X = \{1, 2, 3, 4\}$ and $G = \{g_1, g_2, g_3, g_4\}$ is a group of permutations of X, where*

$$g_1 = (1)(2)(3)(4) \qquad g_3 = (1)(2)(3\ 4)$$
$$g_2 = (1\ 2)(3)(4) \qquad g_4 = (1\ 2)(3\ 4)$$

find the cycle index of G.

Solution. Since X has 4 elements, the index has 4 variable x_i, where $i = 1, 2, 3, 4$. The element g_1 has 4 cycles of length 1 ; so its contribution is x_1^4.

Both g_2 and g_3 have 2 cycles of length 1 and 1 cycle of length 2, their contribution is $2x_1^2x_2$.

The contribution of g_4 is x_2^2. Thus the cycle index of the group is $(|\,G\,| = 4)$

$$Z = (G\ ;\ x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + 2x_1^2x_2 + x_2^2).$$

Problem 1.193. *The length of a stick is n feet. The individual feet are marked consecutively 1, 2, 3, ... n. The only symmetries are rotations about the center through 0° and 180°. Obtain the cycle index of this permutation group.*

Solution. Here $G = \{e, g\}$

(i) If $n = 2k$, $g = (\overline{1\ 2k})\ (\overline{2\ 2k-1})\ (\overline{3\ 2k-2})\ ......\ (\overline{k\ k+1})$

and so the cycle index of $\frac{1}{2}\ (x_1^{2k} + x_2^{k})$.

(ii) If $n = 2k + 1$, $g = (\overline{1\ 2k+1})\ (\overline{2\ 2k})\ ......\ (\overline{k\ k+2})\ (k + 1)$

and so the cycle index is $\frac{1}{2}(x_1^{2k+1} + x_1x_2^{k})$.

1.9 GENERATING FUNCTION

A generating function can be defined as "Given a numeric sequence $a = a_0, a_1, a_2, ..., a_n$ ...... the series.

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$

is called the **generating function** of the sequence".

Suppose $h = h_0, h_1, h_2, ..., h_n$, ... is a sequence of numbers. We write it as an infinite sequence, but we mean to include finite sequences also as well.

If $g(x) = h_0 + h_1x + h_2x^2 + ... + h_nx^n + ...$

$$= \sum_{n=0}^{\infty} h_n x^n$$

$h_0, h_1, h_2, ..., h_n$ is finite sequence, we make it of finite length, by setting $h_r = 0$ for $r > n$.

Thus, the function generates the sequence as its sequence of coefficients. If the sequence is finite then there is an m for which $h_r = 0$ for $r > m$.

In this case $g(x)$ is an ordinary polynomial in x of degree m. The function $g(x) = (1 + x)^m$ generates the binomial coefficients $h_r = C(m, r)$.

Therefore a generating function in which coefficients of x^n are sequence terms of the sequence h, is **called the Binomial generating function** of the sequence h.

The binomial coefficient $C(m, r)$ gives us the total number of combinations of r selections from m objects.

We know that $P(m, r) = r ! * C(m, r)$. This means if $C(m, r)$ is coefficient of x^r then $P(m, r)$ is coefficient of $\frac{x^r}{r!}$.

Let $h(x)$ is a generating function for a sequence a given in series form as below

$$h(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + ...$$

$$= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \text{ this series is exponential.}$$

The generating function $h(x)$ defined as above for a sequence $a = a_0, a_1, a_2$, ... is called **Exponential generating function.**

Problem 1.194. *Find a generating function for a_r = the number of non negative integral solutions of $e_1 + e_2 + e_3 + e_4 + e_5 = r$.*

where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd, and $1 \le e_5 \le 9$.

Solution. Let $A_1(x) = A_2(x) = 1 + x + x^2 + x^3$

$A_3(x) = A_4(x) = x^2 + x^3 + x^4 + x^5 + x^6$

and $A_5(x) = x + x^3 + x^5 + x^7 + x^9$.

Thus, the generating function we want is

$$A_1(x)\ A_2(x)\ A_3(x)\ A_4(x)\ A_5(x)$$
$$= (1 + x + x^2 + x^3)^2\ (x^2 + x^3 + x^4 + x^5 + x^6)^2\ (x + x^3 + x^5 + x^7 + x^9).$$

Problem 1.195. *Find a generating function for a_r = the number of non negative integral solutions to $e_1 + e_2 + ... + e_n = r$, where $0 \le e_i \le 1$.*

Solution. Let $A_i(x) = 1 + x$ for each $i = 1, 2, ..., n$.

Thus, the generating function we want is

$$A_1(x)\ A_2(x)\ ......\ A_n(x) = (1 + x)^n.$$

The binomial theorem gives all the coefficients and thus we know the number of solutions to be above equation is $C(n, r)$.

Problem 1.196. *Find a generating function for a_r = the number of non negative integral solutions to $e_1 + e_2 + ... + e_n = r$, where $0 \le e_i$ for each i.*

Solution. Here since there is no upper bound constraint on the e_i's.

Let $A_1(x)\ A_2(x)\ ......\ A_n(x) = (1 + x + x^2 + ... + x^k\ ......)^n$. We know that $\sum_{r=0}^{\infty} C(n-1+r, r)x^r$ must be another expressions for this same generating function

that is, $\left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{r=0}^{\infty} C(n-1+r, r)x^r$.

In particular, $\left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{r=0}^{\infty} (r+1)x^r$ and $\left(\sum_{k=0}^{\infty} x^k\right)^3 = \sum_{r=0}^{\infty} \frac{(r+2)(r+1)}{2} x^2$

Since for $n = 2$, $C(n - 1 + r, r) = r + 1$ and
for $n = 3$, $C(n - 1 + r, r) = (r + 2)(r + 1)/2$.

Problem 1.197. *Find a generating function for a_r = the number of ways of distributing r similar balls into n numbered boxes where each box is non empty.*

Solution. First we model this problem as an integral solution of an equation problem, namely, we are to count the number of integral solutions to $e_1 + e_2 + ... + e_n = r$, where each $e_i \ge 1$.

Then, in turn, we build the generating function

$$(x + x^2 + ...)^n = \left(\sum_{r=1}^{\infty} x^r\right)^n, \text{ must equal } \sum_{r=n}^{\infty} C(r-1, n-1)x^r.$$

Problem 1.198. *Find a generating function for a_r = the number of ways the sum r can be obtained when :*

(*a*) *2 distinguishable dice are tossed.*

(*b*) *2 distinguishable dice are tossed and the first shows an even number and the second shows an odd number.*

(*c*) *10 distinguishable dice are tossed and 6 specified dice show an even number and the remaining 4 show an odd number.*

Solution. In (*a*) We are to count the number of integral solutions of $e_1 + e_2 = r$, where $1 \le e_i \le 6$. Then a_r is the coefficient of x^r in the generating function

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

In (*b*) We are looking for the coefficient of x^r in

$(x^2 + x^4 + x^6)(x + x^3 + x^5)$ since $1 \le e_1 \le 6$ and e_1 is even while $1 \le e_2 \le 6$ and e_2 is odd.

Likewise, the generating function called for in (C) is $(x^2 + x^4 + x^6)^6\,(x + x^3 + x^5)^4$.

Problem 1.199. *Find a generating function to count the number of integral solutions to* $e_1 + e_2 + e_3 = 10$ *if for each i,* $0 \le e_i$.

Solution. Here we can take two approaches. Of course we are looking for the coefficient of x^{10} in $(1 + x + x^2 + x^3 + ...)^3$.

But since the equation is a model for the distribution of 10 similar balls into 3 boxes we see that each $e_i \le 10$ for we cannot place more than 10 balls in each box.

Thus we could also interpret the problem as one where we are to find the coefficient of x^{10} in

$$(1 + x + x^2 + ... + x^{10})^3.$$

Problem 1.200. *Use generating functions to find the number of ways to select r objects of n different kinds if we must select atleast one object of each kind.*

Solution. Since we need to select atleast one object of each kind, each of the *n* kinds of objects contributes the factor $(x + x^2 + x^3 + ...)$ to the generating function G(x) for the sequence $\{a_r\}$, where a_r is the number of ways to select *r* objects of *n* different kinds if we need atleast one object of each kind.

Hence, $G(x) = (x + x^2 + x^3 + ...)^n = x^n(1 + x + x^2 + ...)^n = \dfrac{x^n}{(1-x)^n}$

Using the extended Binomial theorem, we have

$$G(x) = \frac{x^n}{(1-x)^n} = x^n \,.\, (1 - x)^{-n}$$

$$= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r = x^n \sum_{r=0}^{\infty} (-1)^r \; C(n + r - 1, r)\, (-1)^r\, x^r$$

$$= \sum_{r=0}^{\infty} C(n + r - 1,\ r)\, x^{n+r}$$

$$= \sum_{t=n}^{\infty} C(t - 1, t - n)\, x^t \;=\; \sum_{r=n}^{\infty} C(r - 1, r - n)\, x^r\ .$$

We have shifted the summation in the next-to-last equality by setting $t = n + r$ so that $t = n$ when $r = 0$ and $n + r - 1 = t - 1$, and then we replaced *t* by *r* as the index of summation in the last equality to our original notation.

Hence, there are C(r – 1, r – n) ways to select r objects of n different kinds if we must select at least one object of each kind.

Problem 1.201. *Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.*

(For example, there are two ways to pay for an item that costs \$3 when the order in which the tokens are inserted does not matter ; inserting three \$1 token or one \$1 token and a \$2 token. When the order matters, there are three ways ; inserting three \$1 tokens, inserting a \$1 token and then a \$2 token, or inserting a \$2 token and then a \$1 token).

Solution. Consider the case when the order in which the tokens are inserted does not matter.

Here, all we care about is the number of each token used to produce a total of r dollars.

Since we can use any number of \$1 tokens, any number of \$2 tokens, and any number of \$5 tokens, the answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + x^6 + ...)\ (1 + x^5 + x^{10} + x^{15} + ...).$$

The first factor in this product represents the \$1 tokens used, the second the \$2 tokens used, and the third the \$5 tokens used.

For example, the number of ways to pay for an item costing \$7 using \$1, \$2 and \$5 tokens is given by the coefficient of x^7 in this expansion, which equals 6.

When the order in which the tokens are inserted matters, the number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in $(x + x^2 + x^5)^n$.

Since each of the r tokens may be a \$1 token, a \$2 token, or a \$5 token.

Since any number of tokens may be inserted, the number of ways to produce r dollars using \$1, \$2, or \$5 tokens, when the order in which the tokens are inserted matters, is the coefficient of x^r in

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + ...$$

$$= \frac{1}{1-(x+x^2+x^5)} = \frac{1}{1-x-x^2-x^5}$$

where we have added the number of ways to insert 0 tokens, 1 token, 2 tokens, 3 tokens, and so on, and where we have used the identity $\frac{1}{(1-x)} = 1 + x + x^2 + ...$, with x replaced with $x + x^2 + x^5$.

For example, the number of ways to pay for an item costing \$7 using \$1, \$2, and \$5 tokens, when the order in which the tokens are used matters, is the coefficient of x^7 in this expansion, which equals 26.

To see that this coefficient equals 26 requires the addition of the coefficients of x^7 in the expansions $(x + x^2 + x^5)^k$ for $2 \le k \le 7$.

This can be done by hand with considerable computation, or a computer algebra system can be used.

Problem 1.202. *In how many different ways can eight identical cookies be distributed among three distinct children to each child receives at least two cookies and no more than four cookies ?*

Solution. Since each child receives at least two but no more than four cookies, for each child there is a factor equal to $(x^2 + x^3 + x^4)$ in the generating function for the sequence $\{C_n\}$, where C_n is the number of ways to distribute n cookies.

Since there are three children, this generating function is

$$(x^2 + x^3 + x^4)^3.$$

We need the coefficient of x^8 in this product. The reason is that the x^8 terms in the expansion correspond to the ways that three terms can be selected, with one from each factor, that have exponents adding upto 8.

Furthermore, the exponents of the term from the first, second and third factors are the numbers of cookies the first, second and third children receive, respectively. Computation shows that this coefficient equals 6.

Hence, there are six ways to distribute the cookies so that each child receives at least two, but no more than four, cookies.

Problem 1.203. *Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, and e_3 are non negative integers with $2 \le e \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.*

Solution. The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7).$$

This follows since we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where the exponents e_1, e_2 and e_3 satisfy the equation $e_1 + e_2 + e_3 = 17$ and the given constraints. It is not hard to see that the coefficient of x^{17} in this product is 3. Hence there are three solutions.

Problem 1.204. *How many integer solutions are there for the equation $c_1 + c_2 + c_3 + c_4 = 25$ if $0 \le c_i$ for all $1 \le i \le 4$?*

Solution. For each child the possibilities can be described by the polynomial

$$1 + x + x^2 + x^3 + \ldots\ldots + x^{25}$$

Then the answer to this problem is the coefficient of x^{25} in the generating function

$$f(x) = (1 + x + x^2 + \ldots\ldots + x^{25})^4.$$

The answer can also be obtained as the coefficient of x^{25} in the generating function

$$g(x) = (1 + x + x^2 + x^3 + \ldots\ldots + x^{25} + x^{26} + \ldots\ldots\ )^4$$

if we rephrase the question in terms of distributing, from a large (or unlimited) number of pennies, 25 pennies among four children.

Note that the terms x^k, for all $k \ge 26$.

Problem 1.205. *If there is an unlimited number (or at least 24 of each colour) of red, green white, and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even number of white beans and at least six black ones ?*

Solution. The polynomials associated with the jelly bean colours are as follows :

red (green) : $1 + x + x^2 + \ldots\ldots + x^{24}$, where the leading 1 is for $1x^\circ$, because one possibility for the red (and green) jelly beans is that none of that colour is selected.

White : $(1 + x^2 + x^4 + x^6 + \ldots\ldots + x^{24})$

Black : $(x^6 + x^7 + x^8 + \ldots\ldots + x^{24})$.

So the answer to the problem is the coefficient of x^{24} in the generating function

$$f(x) = (1 + x + x^2 + \ldots\ldots + x^{24})^2\ (1 + x^2 + x^4 + \ldots\ldots + x^{24})\ (x^6 + x^7 + \ldots\ldots\ x^{24})$$

One such selection is five red, three green, eight white and eight black jelly beans.

This arises from x^5 in the first factor, x^3 in the second factor, and x^8 in the last two factors.

Problem 1.206. *While shopping one saturday, Mildred buys 12 oranges for her children, Grace, Mary, and Frank. In how many ways can she distribute the oranges so that Grace gets atleast four, and Mary and Frank get at least two, but Frank gets no more than five ?*

Table 1 lists all the possible distributions

G	*M*	*F*	*G*	*M*	*F*
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	2	3
5	3	4	7	3	2
5	4	4	8	2	2
5	5	2			

Solution. We see that we have all the integer solutions to the equation $c_1 + c_2 + c_3 = 12$ where $4 \le c_1$, $2 \le c_2$ and $2 \le c_3 \le 5$.

Considering the first two cases in this table, we find the solutions 4 + 3 + 5 = 12 and 4 + 4 + 4 = 12. When multiplying polynomials we add the powers of the variable, and here, when we multiply the three polynomials

$$(x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5)$$

two of the ways to obtain x^{12} are as follows :

1. From the product $x^4x^3x^5$, where x^4 is taken from $(x^4 + x^5 + x^6 + x^7 + x^8)$, x^3 from $(x^2 + x^3 + x^4 + x^5 + x^6)$, and x^5 from $(x^2 + x^3 + x^4 + x^5)$.

2. From the product $x^4x^4x^4$ where the first x^4 is found in the first polynomial the second x^4 in the second polynomial, and the third x^4 in the third polynomial.

1.9.1. Partitions of Integers

In number theory, we are confronted with partitioning a positive integer n into positive summands and seeking the number of such partitions, without regard to order. This number is denoted by P(n).

For example,

P(1) = 1 : 1

P(2) = 2 : 2 = 1 + 1

P(3) = 3 : 3 = 2 + 1 = 1 + 1 + 1

P(4) = 5 : 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1

P(5) = 7 : 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1

= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.

We should like to obtain P(n) for a given n without having to list all the partitions. We need a tool to keep track of the numbers of 1's, 2's, ... n's that are used to summands for n.

If $n \in Z^+$, the number of 1's we can use is 0 or 1 or 2 or ......

The power series $1 + x + x^2 + x^3 + x^4 + ...$ keeps account of this for us.

In like manner, $1 + x^2 + x^4 + x^6 + ...$ keeps track of the number of 2's in the partition of n, while $1 + x^3 + x^6 + x^9 + ...$ accounts for the number of 3's.

Therefore, in order to determine P(10), for instance, we want the coefficient of x^{10} in $f(x) = (1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + x^9 + ...) ...... (1 + x^{10} + x^{20} + ...)$ or in

$$g(x) = (1 + x + x^2 + x^3 + ... + x^{10})(1 + x^2 + x^4 + ... + x^{10})$$
$$(1 + x^3 + x^6 + x^9 + ...) ......(1 + x^{10}).$$

$f(x)$ can be written in the more compact form

$$f(x) = \frac{1}{1-x}\,\frac{1}{(1-x^2)}\,\frac{1}{(1-x^3)}\;......\;\frac{1}{(1-x^{10})} = \prod_{i=1}^{10}\frac{1}{(1-x^i)}$$

If this product is extended beyond $i = 10$, we get $P(x) = \prod_{i=1}^{\infty}\left[\frac{1}{(1-x^i)}\right]$, which generates the sequence P(0), P(1), P(2), P(3) ......, where we define P(0) = 1.

It is impossible to actually calculate the infinite number of terms in the product P(x).

If we consider only $\prod_{i=1}^{r}\left[\frac{1}{(1-x^i)}\right]$ for some fixed r, then the coefficient of x^n here is the number of partitions of n into summands that do not exceed r.

The difficulty in calculating P(n) from P(x) for large values of n, the idea of the generating function will be useful in studying certain kinds of partitions.

Generating functions play an essential role in the theory of partitions. Recall that a partition of a positive integer r is a collection of positive integers with sum r.

$P(r) \equiv$ number of distinct partitions of r

$P_n(r) \equiv$ number of partitions of r into parts at most equal to n

$\equiv$ the number of solutions in non negative integers of $1u_1 + 2u_2 + ... + nu_n = r$

$q_n(r) \equiv$ number of partitions of r into at most n parts

$\equiv$ number of distributions of r identical objects (1 s) among n identical places, empty places being permitted.

1.9.2. The Exponential Generating Function

For each $n \in Z^+$, $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$.

So $(1 + x)^n$ is the (ordinary) generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, ... \binom{n}{n}$, 0, 0, ...

Also $\binom{n}{r}$ = C(*n, r*) when we wanted to emphasize that $\binom{n}{r}$ represented the number of combinations of *n* objcets taken *r* at a time, with $0 \le r \le n$.

Consequently, $(1 + x)^n$ generates the sequence

$$C(n, 0), C(n, 1), C(n, 2), ... C(n, n), 0, 0, ...$$

Now for all $0 \le r \le n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where P(*n, r*) denotes the number of permutations of *n* objects taken *r* at a time.

So $\quad (1 + x)^n = C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + ... + C(n, n)x^n$

$$= P(n, 0) + P(n, 1)x + P(n, 2)\frac{x^2}{2!} + P(n, 3)\frac{x^3}{3!} + ... + P(n, n)\frac{x^n}{n!}.$$

Hence, if in $(1 + x)^n$ we consider the coefficient of $\frac{x^r}{r!}$, with $0 \le r \le n$.

For a sequence a_0, a_1, a_2, a_3, ... of real numbers,

$$f(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + ...$$

$$= \sum_{i=0}^{\infty} a_n \frac{x^n}{n!}$$

is called the **exponential generating** function for the given sequence.

1.9.3. The Summation Operator

This section introduces a technique that helps us go from the (ordinary) generating function for the sequence a_0, a_1, a_2, ... to the generating function for the sequence $a_0, a_0 + a_1, a_0 + a_1 + a_2$, ...

For $f(x) = a_0 + a_1x + a_2x^2 + ...$

Consider the function $\frac{f(x)}{(1-x)}$

$$\frac{f(x)}{(1-x)} = [a_0 + a_1x + a_2x^2 + ...][1 + x + x^2 + x^3 + ...]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + ...$$

So $\frac{f(x)}{(1-x)}$ generates the sequence of sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3$, ...

This is the convolution of the sequence a_0, a_1, a_2, ... and the sequence b_0, b_1, b_2, ... where $b_n = 1$ for all $n \in$ N.

Problem 1.207. *Find the generating function for the sequence a = 1, 1, 1, 1, ...*

Solution. The given sequence : $a = 1, 1, 1, 1, \ldots\ldots$

Here, the general term $a_n = 1$.

Let $f(x)$ be the binomial generating function for the given sequence, then $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (1)\, x^n \qquad (\because \quad a_n = 1)$$

$$= 1 + x + x^2 + x^3 + \ldots\ldots$$

$$= \frac{1}{1-x} \qquad (\because \quad \text{the series is geometric with } x \text{ as common ratio})$$

$\therefore \quad f(x) = \dfrac{1}{1-x}$ is the binomial generating function for the sequence $a = 1, 1, 1, 1, \ldots\ldots$

Let $g(x)$ be the exponential generating function for the same sequence, then $g(x)$ can be written as

$$g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (1) \cdot \frac{x^n}{n!} \qquad (\because \quad a_n = 1)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

$$= e^x, \qquad (\because \quad \text{the series is exponential with } x \text{ as base})$$

$\therefore \quad g(x) = e^x$ is the exponential generating function for the sequence $a = 1, 1, 1, 1, \ldots\ldots$.

Problem 1.208. *Find the generating function for the sequence*

b = 1, 3, 9, ..., 3^n, ...

Solution. The given sequence : $b = 1, 3, 9, \ldots\ldots, 3^n, \ldots\ldots$

Here, the general term $b_n = 3^n$

Let $f(x)$ be the binomial generating function for the given sequence, then $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n \qquad (\because \quad b_n = 3^n)$$

$$= 1 + 3x + (3x)^2 + (3x)^3 + \ldots\ldots$$

$$= \frac{1}{1-3x} \qquad (\because \quad \text{the series is geometric with } 3x \text{ as common ratio})$$

$\therefore \quad f(x) = \dfrac{1}{1-3x}$ is the binomial generating function for the sequence $b = 1, 3, 3^2, 3^3, \ldots\ldots 3^n$, ...

Let $g(x)$ be the exponential generating function for the same sequence, then $g(x)$ can be written as

$$g(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 3^n \cdot \frac{x^n}{n!} \qquad (\because \quad b_n = 3^n)$$

$$= 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\ldots$$

$$= e^{3x} \qquad (\because \quad \text{the series is exponential with } 3x \text{ as base})$$

$\therefore \quad g(x) = e^{3x}$ is the exponential generating function for the sequence $b = 1, 3, 3^2, \ldots\ldots$

Problem 1.209. *Find the binomial generating function for the sequence*

$a = 1, 2, 3, \ldots\ldots, r, \ldots\ldots$

Solution. The given sequence : $a = 1, 2, 3, \ldots\ldots, r, \ldots\ldots$.

Here, the general term $a_r = r$

Let $f(x)$ be the binomial generating function of the given sequence, then

$f(x)$ can be written as

$$f(x) = \sum_{r=0}^{\infty} a_r\, x^r = \sum_{r=0}^{\infty} r \cdot x^r \qquad (\because \quad a_r = r)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \qquad \ldots(1)$$

The equation (1) is arithmetic-geometric in which first factor is following arithmetic progression and second following geometric.

We find the sum of the series by multiplying equation (1) with x and then by subtracting the result from equation (1).

Thus, we get

$$(1 - x)\, f(x) = 1 + x + x^2 + x^3 + \ldots\ldots$$

$$\Rightarrow \qquad f(x) = \frac{1}{(1-x)^2}$$

this is the required binomial generating function.

Problem 1.210. *Find the discrete numeric function (sequence) whose exponential generating function is given by $2e^x$.*

Solution. The given generating function can be written in series form as

$$2e^x = 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\right]$$

Clearly, the coefficient of $\frac{x^n}{n!}$ for $n \geq 0$, are 2, 2, 2, ...

$\therefore$ If the discrete numeric function is a, then

$$a = 2, 2, 2, 2, ...\ .$$

Problem 1.211. *Find the exponential generating function for sequence*

$$t = {}^nP_0, {}^nP_1, {}^nP_2, ...\ {}^nP_n.$$

Solution. The given sequence is finite. All terms, in this sequence for $m > n$ are zero.

Let $h(x)$ be the exponential generating function for the given sequence, then

$h(x)$ can be written as

$$h(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} = \sum_{i=0}^{\infty} {}^n\text{P}_i \frac{x^i}{i!} \qquad (\because \quad a_i = {}^n\text{P}_i)$$

$$= {}^n\text{P}_0 + {}^n\text{P}_1\, x + {}^n\text{P}_2 \frac{x^2}{2!} + ... + {}^n\text{P}_n \frac{x^n}{n!}$$

$$= {}^n\text{C}_0 + {}^n\text{C}_1 x + {}^n\text{C}_2\, x^2 + ... + {}^n\text{C}_n\, x^n \qquad (\because \quad {}^n\text{P}_r = r\,!\,.\, {}^n\text{C}_r)$$

$$= (1 + x)^n$$

$\therefore$ The exponential generating function for the sequence

$t = {}^n\text{P}_0, {}^n\text{P}_1, {}^n\text{P}_2, ..., {}^n\text{P}_n$ is given by

$h(x) = (1 + x)^n$.

Theorem 1.28. *Let $f(x)$ is the generating function for a and $g(x)$ is the generating function for b, then $f(x) + g(x)$ is the generating function for $a + b$.*

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$

be the generating functions for a and b,

then

$$f(x) + g(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$$

$$= \sum_{n=0}^{\infty} (a_n + b_n)\, x^n$$

$\therefore$ The sequence generated by $f(x) + g(x)$ is

$$(a_0 + b_0), (a_1 + b_1), (a_2 + b_2), ...$$

These terms are precisely the sum of the corresponding terms from a and b.

Thus, $f(x) + g(x)$ is generating function for the sum of the sequence $a + b$.

Note 1. If $f(x)$ and $g(x)$ are generating functions for sequences a and b respectively then $f(x) * g(x)$ is not the generating function for $a * b$.

2. Let $f(x)$ and $g(x)$ are generating functions for finite sequences a and b respspcetively. The convolution of two sequences a and b is the sequence c defined as

$$C_n = \sum_{r=0}^{n} a_r \, b_{n-r} \text{ , then}$$

$f(x) \cdot g(x)$ is the generating function for convolution of a and b.

3. For $k > 0$, the function $g(x) = \dfrac{1}{(1-x)^k}$ generates the sequence $a = \{C(n + k - 1, n)/n = 0, 1, 2, 3, ...\}$.

Thus the nth coefficient is the number of ways to select n objects of k times.

$$\frac{1}{(1-x)^k} = \left[\frac{1}{(1-x)}\right]^k = (1 + x + x^2 + x^3 + ...)^k$$

If we carry out this multiplication k times, then x^n appear as many times as there are non negative integer solutions to the equation

$$x_1 + x_2 + x_3 + ... + x_k = n$$

We know that, this number is $C(n + k - 1, n)$.

Problem 1.212. *For the following two sequences a and b, whose general terms are given, find a + b and ab.*

$$a_r = \begin{cases} 0 & \text{for } \; 0 \le r \le 2 \\ 2^{-r} + 5 & \text{for } \; r \ge 3 \end{cases} \quad \textit{and}$$

$$b_n = \begin{cases} 3 - 2^r & \text{for } \; 0 \le r \le 1 \\ r + 2 & \text{for } \; r \ge 2 \end{cases}$$

Solution. Let C be the sequence for $a + b$,

at $r = 2$, the general definitions of a and b cannot be simply added.

For $r \ge 3$ and $0 \le r \le 1$, bottom and top definitions will be added, but at $r = 2$ the value will be determined and will be placed in the definition.

Let $d = ab$.

In this case, we shall find the sequence formula for d by multiplying the definitions of a and b and setting the range for sequence position r accordingly.

$$C_r = a_r + b_r = \begin{cases} 3 - 2^r & \text{for } \; 0 \le r \le 1 \\ 4 & \text{for } \; r = 2 \\ 2^{-r} + r + 7 & \text{for } \; r \ge 3 \end{cases}$$

and $$C_r = a_r * b_r = \begin{cases} 0 & \text{for } \; 0 \le r \le 2 \\ r2^{-r} + 2^{-r+1} + 5r + 10 & \text{for } \; r \ge 3 \end{cases}.$$

Problem 1.213. *Find the number of positive integral solution to the equation $x + y + z = 10$.*

Solution. Here x, y and z are positive integers satisfying the given equations.

To find the number of possible solutions to the above equation.

Here x belongs to {1, 2, 3, 4, ...} and so are y and z.

If different possible positive integers for x represented by the powers of x then the series for x can be written as

$$x + x^2 + x^3 + \ldots\ldots$$

This problem is reduced to selection of 10 objects of 3 kinds.

The generating function $f(x)$ for this problem is then written as

$$f(x) = (x + x^2 + x^3 + \ldots\ldots)^3$$

$$= \frac{x^3}{(1-x)^3}$$

The required result is given by the coefficient of x^{10} in $f(x)$.

This is equal to the coefficient of x^7 in $(1 - x)^{-3}$.

$$\Rightarrow \quad {}^{7+3-1}C_7 = {}^9C_7 \qquad (\because \quad n = 7, k = 3)$$

$$= \frac{9 * 8}{2} = 36.$$

Problem 1.214. *Find the generating function for the sequence 1, a, a^2, ..., where a is a fixed constant.*

Solution. Let $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \ldots\ldots$

So, $G(x) - 1 = ax + a^2 x^2 + a^3x^3 + \ldots\ldots.$

or $$\frac{G(x)-1}{ax} = 1 + ax + a^2x^2 + \ldots\ldots$$

or $$\frac{G(x)-1}{ax} = G(x)$$

$$\Rightarrow \quad G(x) = \frac{1}{1-ax}$$

The required generating function is therefore, $\dfrac{1}{1-ax}$.

Problem 1.215. *Find the generating function of a sequence $\{a_k\}$ if $a_k = 2 + 3k$.*

Solution. The generating function of a sequence whose general term is 2 is $F(x) = \dfrac{2}{1-x}$

The generating of a sequence whose general term is $3k$ is $G(x) = \dfrac{3x}{(1-x)^2}$.

Hence the required generating function is

$$F(x) + G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2}.$$

Problem 1.216. *Find the sequences corresponding to the ordinary generating functions* (*a*) $(3 + x)^3$ *and* (*b*) $3x^3 + e^{2x}$.

Solution. (*a*) $(3 + x)^3 = 27 + 27x + 9x^2 + x^3$

The sequence is (27, 27, 9, 1, 0, 0, 0, ...)

(*b*) $3x^3 + e^{2x} = 1 + 2x + \frac{2^2}{2!}x^2 + \left(3 + \frac{2^3}{3!}\right)x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + ...$

The sequence is $\left(1, 2, \frac{2^2}{2!}, \frac{2^3}{3!} + 3, \frac{2^4}{4!}, ...\right)$.

Problem 1.217. *Find a closed form for the generating function for each of the following sequence*

(*a*) *0, 0, 1, 1, 1, ...*

(*b*) *1, 1, 0, 1, 1, 1, 1, 1, ...*

(*c*) *1, 0, – 1, 0, 1, 0, – 1, 0, 1, ...*

(*d*) *C(8, 0), C(8, 1), C(8, 2),......, C(8, 8), 0, 0, ...*

(*e*) *3, – 3, 3, – 3, 3, – 3, ...*

Solution. (*a*) We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... \infty$$

$$= \sum_{n=0}^{\infty} x^n .$$

So, the generating function of 1, 1, 1, ..., is $\frac{1}{1-x}$.

Now $\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2}$.

Hence $\frac{x^2}{1-x}$ is the generating function 0, 0, 1, 1, 1, ...

(*b*) Here $\frac{1}{1-x} - x^2 = 1 + x + x^3 + ... \infty = \sum_{\substack{n=0 \\ n \neq 2}}^{\infty} x^n$

So, the generating function of 1, 1, 0, 1, 1, 1, ... is $\frac{1}{1-x} - x^2$.

(*c*) We know that $\frac{1}{1+x^2} = (1 + x^2)^{-1}$

$$= 1 - x^2 + x^4 - x^6 + x^8 \text{ ...... } \infty$$
$$= 1 + 0.x + (-1)x^2 + 0.x^3 + 1.x^4 + 0.x^5 + (-1) \,.\, x^6 + \text{......}$$

So the generating function of 1, 0, – 1, 0, 1, 0, – 1, ... is $\frac{1}{1+x^2}$.

(*d*) We know that

$$(1 + x)^8 = C(8, 0)x^0 + C(8, 1)x + \text{......} + C(8, 8)x^8 + 0 + 0 + \text{......}$$
$$= \sum_{n=0}^{\infty} C(8, n)x^n \,.$$

So, the generating function of C(8, 0), C(8, 1), ..., C(8, 8), 0, 0, ...

(*e*) We have

$$\frac{3}{1-x} = 3(1 - x)^{-1} = 3(1 - x + x^2 - x^3 + \text{......})$$
$$= 3 + (-3)x + 3x^2 + (-3)x^3 + \text{......}$$
$$= \sum_{n=0}^{\infty} (-3)^n x^n \,.$$

Hence, the required generating function is $\frac{3}{1+x}$.

Theorem 1.29. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k \quad \text{and}$$

$$f(x)\, g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j\, b_{k-j} \right) x^k \,.$$

Theorem 1.30. *(The Extended Binomial Theorem)*

Let x be a real number with $|\, x \,| < 1$ and let u be a real number. Then $(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.

Table. Some useful Generating functions :

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$ $= 1 + C(n, 1)x + C(n, 2)x^2 + \ldots\ldots + x^n$	$C(n, k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$ $= 1 + C(n, 1)ax + C(n, 2)a^2x^2 + \ldots\ldots + a^n x^n$	$C(n, k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$ $= 1 + C(n, 1)x^r + C(n, 2)x^{2r} + \ldots\ldots + x^{rn}$	$C(n, k/r)$ if r/k, 0, 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots\ldots + x^n$	1 if $k \le n$, 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots\ldots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \ldots\ldots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \ldots\ldots$	1 if r/k, 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \ldots\ldots$	$k + 1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n, 1)x + C(n+1, 2)x^2 + \ldots\ldots$	$C(n + k - 1, k) = C(n + k - 1, n - 1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n, 1)x + C(n+1, 2)x^2 - \ldots\ldots$	$(-1)^k C(n + k - 1, k)$ $= (-1)^k C(n + k - 1, n - 1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$ $= 1 + C(n, 1)ax + C(n+1, 2)a^2x^2 + \ldots\ldots$	$C(n + k - 1, k)a^k$ $= C(n + k - 1, n - 1)a^k$

(*Contd.*)

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$	$\frac{1}{k!}$
$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$	$\frac{(-1)^{k+1}}{k}$

Problem 1.218. *What is the generating function for the sequence*

1, 1, 1, 1, 1, 1 ?

Solution. The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

We have $\frac{(x^6 - 1)}{(x-1)} = 1 + x + x^2 + x^3 + x^4 + x^5$.

Consequently, $G(x) = \frac{(x^6-1)}{(x-1)}$ is the generating function of the sequence 1, 1, 1, 1, 1, 1.

Problem 1.219. *Let m be a positive integer. Let $a_k = C(m, k)$, for k = 0, 1, 2, ..., m. What is the generating function for the sequence a_0, a_1, ..., a_m ?*

Solution. The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + ... + C(m, n)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$.

Problem 1.220. *Let $f(x) = \frac{1}{(1-x)^2}$, find the coefficients a_0, a_1, a_2, ... in the expansion*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k .$$

Solution. We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

Hence, we have

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1)x^k .$$

Problem 1.221. *Find the values of the extended binomial coefficients*

$$\binom{-2}{3} \text{ and } \binom{1/2}{3}.$$

Solution. Taking $u = -2$ and $k = 3$ in

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots\ldots(u-k+1)/k\,! & \text{if} \quad k > 0 \\ 1 & \text{if} \quad k = 0 \end{cases}$$

which gives, $\binom{-2}{3} = \dfrac{(-2)(-3)(-4)}{3\,!} = -4$

Similarly, taking $u = \dfrac{1}{2}$ and $k = 3$ gives us

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3\,!}$$

$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\Big/6 = \frac{1}{16}.$$

Problem 1.222. *Find the generating functions for $(1 + x)^{-n}$ and $(1 - x)^{-n}$, where n is a positive integer, using the extended Binomial theorem.*

Solution. By the extended Binomial theorem, it follows that

$$(1 + x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

Using $\binom{-n}{r} = (-1)^r\, C(n + r - 1, r)$, which provides a simple formula for $\binom{-n}{k}$, we obtain

$$(1 + x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \;\; C(n + k - 1, k)x^k.$$

Replacing x by $-x$, we find that

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} C(n + k - 1, k)x^k.$$

Problem 1.223. *Use generating functions to find the number of k-combinations of a set with n elements. (Assume that the Binomial theorem has already been established).*

Solution. Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

$$f(x) = \sum_{k=0}^{n} a_k x^k \;.$$

Here $f(x)$ is the generating function for $\{a_k\}$, where a_k represents the number of k-combinations of a set with n elements.

Hence, $f(x) = (1 + x)^n$.

But by Binomial Theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where $$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence, C(*n, k*), the number of *k*-combinations of a set with *n* elements, is $\dfrac{n!}{k!(n-k)!}$.

Problem 1.224. *Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.*

Solution. Let G(x) be the generating function for the sequence $\{a_r\}$, where a_r equals the number of r-combinations of a set with n elements with repetitions allowed.

That is, $$G(x) = \sum_{r=0}^{\infty} a_r\, x^r.$$

Since we can select any number of a particular member of the set with n elements when we form an r-combinations with repetition allowed, each of the n elements contributes $(1 + x + x^2 + x^3 + ...)$ to a product expansion for G(x).

Each element contributes this factor since it may be selected zero times, one time, two times, three times, and so on, when an r-combinations is formed (with a total of r elements selected).

Since there are n elements in the set and each contributes this same factor to G(x), we have

$$G(x) = (1 + x + x^2 + ...)^n$$

As long as $|\, x \,| < 1$, we have $1 + x + x^2 + ... = \dfrac{1}{1-x}$

So, $G(x) = \dfrac{1}{(1-x)^n} = (1-x)^{-n}$

Applying the extended Binomial Theorem, it follows that

$$(1-x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r$$

The number of r-combinations of a set with n elements with repetitions allowed, when r is a positive integer, is the coefficient a_r of x^r in this sum.

Consequently, we find that a_r equals

$$\binom{-n}{r}(-1)^r = (-1)^r\, C(n + r - 1, r)\, .\, (-1)^r$$

$$= C(n + r - 1, r).$$

Problem 1.225. *A box contains many identical red, blue, white, and green marbles. Find the ordinary generating function corresponding to the problem of finding the number of ways of choosing r marble from the box such that the sample does not have more than 2 red, more than 3 blue, more than 4 white, and more than 5 green.*

Solution. The generating function is

$$(1 + x + x^2)(1 + x + x^2 + x^3)(1 + x + ... + x^4)(1 + x + ... + x^5)$$
$$= (1 - x^3)\,(1 - x^4)\,(1 - x^5)\,(1 - x^6)\,(1 - x)^{-4}.$$

Problem 1.226. (Uniqueness of Base-b Representation)

If b is an integer greater than 1, prove by means of a generating function that an arbitrary positive integer r can be written as

$$r = r_0 b^0 + r_1 b^1 + r_2 b^2 + ...,\ 0 \le r_i \le b - 1 \qquad ...(1)$$

in 1 and only 1 way.

Solution. The generating function for the number of solution vectors (r_0, r_1, r_2, ...) of (1) is obviously

$$[1 + x^{1(1)} + x^{2(1)} + ... + x^{(b-1)(1)}]\,[1 + x^{1(b)} + x^{2(b)} + ... + x^{(b-1)(b)}]$$
$$\times [1 + x^{1(b^2)} + x^{2(b^2)} + ... + x^{(b-1)(b^2)}]\ ......$$

$$= \frac{1 - x^b}{1 - x}\ \frac{1 - x^{b^2}}{1 - x^b}\ \frac{1 - x^{b^3}}{1 - x^{b^2}}\ ......$$

$$= \frac{1}{1 - x} = 1 + x + x^2 + x^3 + ...$$

Each coefficient in the generating function is 1,

i.e., any r has a unique base-b representation.

Problem 1.227. *Given a positive integer k, use the generating-function method to find the number of solutions in non negative integers of $u_1 + u_2 + ... + u_k = n$,*

(*a*) *the first 2 variables are atmost 2, and*

(*b*) *the sum of the first two variables is atmost 2.*

Solution. (*a*) $f(x) = (1 + x + x^2)^2\,(1 + x + x^2 + ...)^{k-2}$

$$= (1 - x^3)^2\,(1 - x)^{-k}.$$

(*b*) **Case 1.** Both variables are 0, and the sum is 0.

Case 2. One of them is 1 and the order is 0, and the sum is 1.

Case 3. Both are 1 or one of them is 2 and the other is 0, and the sum is 2.

These cases occur in 1, 2, and 3 ways, respectively.

So the generating function is

$$(1 + 2x + 3x^2)(1 + x + x^2 + ...)^{k-2} = (1 + 2x + 3x^2)(1 - x)^{-k+2}.$$

Problem 1.228. *Find an ordinary generating function that solves the problem of finding the number of positive 5-digit integers with digit sum r.*

Solution. The leading digit is atleast 1 and most 9 ; the other 4 digits are non negative and atmost 9.

Hence, the generating function is

$$(x + x^2 + ... + x^9)\ (1 + x + x^2 + ... + x^9)^4 = x(1 - x^9)(1 - x^{10})(1 - x)^{-2}.$$

Problem 1.229. *Let $f(x) = (1 + x + ... + x^n)^3$ and $g(x) = (1 + x + ... + x^{n-1})^3$. Use a combinatorial argument to show that the coefficient of x^{2n+1} in $f(x)$ is equal to the coefficient of x^{2n-2} in $g(x)$.*

Solution. Consider the equation $a + b + c = 2n + 1$, where the 3 variables are non negative integers at most equal to n.

The number of solutions is the coefficient of x^{2n+1} in $f(x)$.

But no variable in this equation is 0, for then 1 of the remaining 2 variables would have to exceed n.

So the number of solutions is also equal to the coefficient of x^{2n+1} in

$$(x + x^2 + ... + x^n)^3 = x^3\ g(x)$$

which is the coefficient of x^{2n-2} in $g(x)$.

Problem 1.230. *Find the coefficient of x^{27} in*

(a) $(x^4 + x^5 + x^6 + ...)^5$ and (b) $(x^4 + 2x^5 + 3x^6 + ...)^5$.

Solution. (*a*) Since $(x^4 + x^5 + ...)^5 = x^{20}\ (1 - x)^{-5}$, what is required is the coefficient of x^7 in $(1 - x)^{-5}$.

This is C(11, 4).

(*b*) Since $(x^4 + 2x^5 + 3x^6 + ...)^5 = x^{20}\ [(1 - x)^{-2}]^5 = x^{20}(1 - x)^{-10}$ we require the coefficient of x^7 in $(1 - x)^{-10}$, which is C(16, 9).

Problem 1.231. *Find the sequences corresponding to the ordinary generating functions*

(a) $(3 + x)^3$ (b) $3x^3 + e^{2x}$ and (c) $2x^2(1 - x)^{-1}$.

Solution. (*a*) $(3 + x)^3 = 27 + 27x + 9x^2 + x^3$

The sequence is $< 27, 27, 9, 1, 0, 0, 0, ... >$.

(*b*) $3x^3 + e^{2x} = 1 + 2x + \dfrac{2^2}{2!}x^2 + \left(3 + \dfrac{2^3}{3!}\right)x^3 + \dfrac{2^4}{4!}x^4 + \dfrac{2^5}{5!}x^5 + ...$

The sequence is $\left\langle 1, 2, \dfrac{2^2}{2!}, \dfrac{2^3}{3!} + 3, \dfrac{2^4}{4!} \right\rangle$.

(*c*) $2x^2(1 - x)^{-1} = 2x^2\ (1 + x + x^2 + x^3 + ...)$

The sequence is $< 0, 0, 2, 2, 2, ...>$.

Problem 1.232. *Using the generating function of $\sum_{k=0}^{n} C(n, k)x^k = (1 + x)^n$, establish Pascal's identity $C(n + 1, r) = C(n, r) + C(n, r - 1)$.*

Solution. The coefficient of x^r in $(1 + x)^{n+1}$ is C(n + 1, r).

But $(1 + x)^{n+1} = (1 + x)^n$, and the coefficient of x^r in the right-hand side is C(n, r) + C(n, r – 1).

Problem 1.233. *Find the ordinary generating function of the sequence*

$$<C(r + n - 1, n - 1)>_{r \geq 0}.$$

(*a*) *by a combinatorial argument, and*

(*b*) *by differentiation of the infinite geometric series.*

Solution. (*a*) From $x_1 + x_2 + ... + x_m = n$, it is known that $C(r + n - 1, n - 1)$ counts the non negative integral solutions of $u_1 + u_2 + ... + u_n = r$.

Therefore, one can write

$$\frac{1}{(1-x)^n} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) ...... \left(\frac{1}{1-x}\right)$$

$$= (1 + x + x^2 + ... + x^{u_1} + ...)(1 + x + x^2 + ... + x^{u_2} + ...) \\ ...... (1 + x + x^2 + ... + x^{u_n} + ...)$$

$$= \sum_{r=0}^{\infty}\left(\sum_{\substack{u_1 + u_2 + ... + u_n = r \\ u_i \geq 0}} 1\right)x^r$$

$$= \sum_{r=0}^{\infty} C(r + n - 1, n - 1)x^r$$

(*b*) Differentiate $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$n - 1$ times to obtain.

$$\frac{(n-1)!}{(1-x)^n} = \sum_{k=n-1}^{\infty} k(k-1) \; ...... \; (k - n + 2)x^{k-n+1}$$

$$= \sum_{r=0}^{\infty} (r + n - 1)(r + n - 2) \; .... \; (r + 1)x^r$$

$$= \sum_{r=0}^{\infty} \frac{(r+n-1)!}{r} x^r$$

Division of both sides by $(n - 1)$! reproduces the result of (*a*).

Problem 1.234. *Use the generating function method*

(*a*) *to count the distinct binary solutions of* $u_1 + u_2 + ... + u_n = r$

(*b*) *to establish the pigeonhole principle.*

Solution. (*a*) The generating function (on *r*) is

$$(1 + x)^n = \sum_{r=0}^{n} C(n, r)x^r$$

Thus these $C(n, r)$ solutions ($= 0$ for $r > n$)

(*b*) The function $(1 + x)^n$ is also the generating function corresponding to the problem of distributing r identical pigeons among n distinct pigeonholes so that each hole receives fewer than 2 pigeons.

The coefficient of x^{n+1} in the generating function is zero.

Hence, when $n + 1$ pigeons are distributed, some hole receives at least 2 pigeons.

Problem 1.235. *If throwing a die 5 times constitutes a trial, with the 5 throws considered distinguishable, find the number of trials that produce a total of 12 or fewer dots.*

Solution. Let $a_r \equiv$ number of trials that produce r dots

$A_r \equiv$ number of trials that produce atmost r dots

Clearly, $\langle a_r \rangle$, has the generating function

$$(x^1 + x^2 + \ldots\ldots + x^6)^5 = \left(\frac{x - x^7}{1 - x}\right)^5$$

Hence $\langle A_r \rangle$ has the generating function

$$(1 - x)^{-1}\left(\frac{x - x^7}{1 - x}\right)^5 = (x - x^7)^5 (1 - x)^{-6}$$

$$= (x^5 - 5x^{11} + 10x^{17} - \ldots\ldots) \sum_{r=0}^{\infty} C(r + 5, 5)x^r$$

The coefficient of x^{12} on the right is

$$A_{12} = 1.C(12, 5) - 5 \,.\, C(6, 5) = 762.$$

Problem 1.236. *Prove, for all r and n, $P_n(r) = q_n(r)$, without drawing the star diagram.*

Solution. The system $1u_1 + 2u_2 + 3u_3 + \ldots\ldots + nu_n = r$ (u_i a non negative integer)

Which by definition has precisely $P_n(r)$ solutions, is taken by the substitution

$$\begin{aligned} u_n &= w_1 \\ u_{n-1} &= w_2 - w_1 \\ u_{n-2} &= w_3 - w_2 \\ &\ldots\ldots\ldots\ldots \\ u_1 &= w_n - w_{n-1} \end{aligned} \quad \ldots(1)$$

into the system

$$w_1 + w_2 + w_3 + \ldots\ldots + w_n = r$$

$$w_1 \le w_2 \le w_3 \le \ldots\ldots \le w_n \quad (w_i \text{ a non negative integer})$$

This later system has precisely $q_n(r)$ solutions.

But the mapping (1) is obviously bijective, so that

$$P_n(r) = q_n(r).$$

1.9.4. Problem 1.237. (Euler's Theorem)

Let P(r, ODD) be the number of ways of partitioning r into (possibly repeated) odd parts. Show that, for every r, P(r, ODD) = $P^{\#}$ (r).

Solution. The ordinary generating function of < P(r, ODD)> is given by

$$\frac{1}{(1-x^2)(1-x^3)(1-x^5).......} = \left(\frac{1-x^2}{1-x^1}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)......$$

$$= (1 + x^1)(1 + x^2)(1 + x^3)(1 + x^4) ......$$

$$= \text{generating function of} < P^{\#}(r) >$$

$$= \sum_{r=0}^{\infty} P^{\#}(r)\, x^r.$$

Problem 1.238. *Find the exponential generating function of <a_r>, where a_r is the number of r sequences of the set E = {e_1, e_2, ..., e_n }.*

Solution. Looking at the product

$$G(x) \equiv \left(1+x+\frac{x^2}{2!}+......+\frac{x^{i_1}}{i_1!}+......\right)\left(1+x+\frac{x^2}{2!}+......+\frac{x^{i_2}}{i_2!}+......\right)$$

$$......\left(1+x+\frac{x^2}{2!}+......+\frac{x^{i_n}}{i_n!}+......\right)$$

One sees that the r sample of E consisting of i_1e_1's, i_2e_2's, ..., i_ne_n's where i_1, i_2, ..., i_n are non negative integers with sum r-contributes

$$\frac{r!}{i_1!\,i_2!......i_n!} = P(r\,;\,i_1, i_2, ...\, i_n)$$

to the coefficeint of $\frac{x^r}{r!}$ in G(x).

The number of r sequences of E generated by permutation of the given r sample.

Hence, the total coefficient of $\frac{x^r}{r!}$ is just a_r.

This is, G(x) = $(e^x)^n$ is the desired exponential generating function.

Problem 1.239. *If a leading digit of O is permitted, find the numbers of r-digit binary numbers that can be formed using*

(a) *an even number of 0^s and an even number 1 s,*

(b) *an odd number of 0^s and an odd number of 1 s.*

Solution. Here we are counting r sequences of the set {0, 1} that obey certain restrictions.

(a) The exponential generating function is

$$F_e(x) = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+......\right)^2 = \left(\frac{e^x+e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^{2x} + e^{-2x} + 2)$$

The coefficient of $\frac{x^r}{r!}$ in $F_e(x)$ is 2^{r-1} if r is even, and 0 if r is odd.

$$(b)\ F_0(x) = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= F_e(x) - 1$$

Thus the answer is the same as in (*a*).

Problem 1.240. *The exponential generating function for the Bell numbers was found in*

$$B_n = n!\sum_{{}^1P_1 + {}^2P_2 + ... + {}^nP_n = n} \frac{1}{[P_1!(1!)P_1][P_2!(2!)^{P_2}]......[P_n!(n!)^{P_n}]}$$

and $B_n = \left.\frac{d^n}{dx^n} e^{e^x - 1}\right|_{x=0}$ *to be* $e^{e^x - 1}$.

Check this result by use of the exponential generating function for the stirling numbers of the second kind.

Solution. By the definition of the two kinds of numbers

$$B_n = \sum_{m=0}^{\infty} S(n, m),\ B_0 = S(0, 0) = 1$$

If $n \geq 1$, only n of the summands are non zero.

Because the exponential generating of a sum is the sum of the generating functions and

$$(e^x - 1)^m = \left(x + \frac{x^2}{2!} + ... + \frac{x^{n_1}}{n_1!} + ...\right)\left(x + \frac{x^2}{2!} + ... + \frac{x^{n_2}}{n_2!} + ...\right)$$

$$x + \frac{x^2}{2!} + ... + \frac{x^{nm}}{n_m!} + ...$$

gives :

Exponential generating function of

$$\langle B_n \rangle = \sum_{m=0}^{\infty} \frac{1}{m!}(e^x - 1)^m = e^{x^x - 1}.$$

Problem 1.241. *Find the number of r-letter sequences that can be formed using the letters P, Q, R and S such that in each sequence there are an odd number of P's and an even number of Q's.*

Solution. The answer is the coefficient of $\dfrac{x^r}{r!}$ in

$$\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\ldots\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\right)(e^x)(e^x)$$

$$=\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right)e^{2x} = \frac{1}{4}(e^{4x}-1)$$

This coefficient is 4^{r-1}.

Problem 1.242. *The sequence of Bernoulli numbers, $<b_n>_{n\geq 0}$, has the exponential generating function* $\dfrac{x}{e^x-1}$.

Show that (a) $b_3 = b_5 = b_7 = \ldots\ldots = 0$ and

(b) $b_n = \displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\,m!\,S(n,m)$.

Solution. (*a*) $b_1\dfrac{x^1}{1!}+b_3\dfrac{x^3}{3!}+b_5\dfrac{x^5}{5!}+\ldots\ldots = \dfrac{1}{2}\left(\dfrac{x}{e^x-1}-\dfrac{-x}{e^{-x}-1}\right) = -\dfrac{1}{2}x$

whence $b_1 = -\dfrac{1}{2}$ and $b_3 = b_5 = \ldots\ldots = 0$.

(*b*) Two sequences are identical if and only if they have the same exponential generating function.

The exponential generating function of the numbers

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\,m!\,S(n,m)$$

is given by

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}(e^x-1)^m = \frac{1}{1-e^x}\sum_{m=0}^{\infty}\frac{(1-e^x)^{m+1}}{m+1}$$

$$= \frac{1}{1-e^x}\int_0^{1-e^x}\frac{1}{1-u}\,du$$

$$= \frac{1}{1-e^x}(-x) = \frac{x}{e^x-1}.$$

Problem 1.243. *Find the number of ways of distributing 10 distinguishable books among 4 distinguishable shelves so that each shelf gets atleast 2 and atmost 7 books.*

Solution. This is a problem in restricted sequences.

Here we want to count the 10-sequences of a 4 set (the i^{th} term of a sequence is the shelf to which the i^{th} book is assigned) that can exist under the given limitations.

The appropriate exponential generating function is

$$G(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^7}{7!}\right)^4$$

$$= x^8 \left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\ldots\right)^4$$

and the answer is the coefficient of $\frac{x^{10}}{10!}$ in G(x), which is $\frac{10!}{16}$ = 226,800.

1.9.5. Problem 1.244. (Dobinski's Equality)

$$\textit{Prove that } B_n = \begin{cases} 1 & n = 0 \\ e^{-1} \sum_{k=1}^{\infty} \frac{k^n}{k!} & n = 1, 2, 3, \ldots\ldots \end{cases}$$

Solution. The exponential generating function of the number on the right of the asserted equality is

$$1 + e^{-1} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{k^n}{k!}\right)\frac{x^n}{n!} = 1 + e^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=1}^{\infty} \frac{(kx)^n}{n!}$$

$$= 1 + e^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} [(e^x)^k - 1]$$

$$= 1 + e^{-1} (e^{e^x} - e) = e^{e^x} - 1.$$

which is just the exponential generating function of $\langle B_n \rangle$.

Problem 1.245. *Derive the linear recursion relation for the Bell numbers from the exponential generating function*

$$\left(B_n = \sum_{k=0}^{n-1} C(n-1, k) B_k\right).$$

Solution. Differentiation of $e^{e^x - 1} = \sum_{r=0}^{\infty} B_r \frac{x^r}{r!}$

gives $(e^{e^x - 1})(e^x) = \sum_{r=0}^{\infty} B_r \frac{x^{r-1}}{(r-1)!}$

On the right the coefficient of $\dfrac{x^r}{r!}$ is B_{r+1}, on the left it is

$$\sum_{i=0}^{r} C(r, i) B_i$$

Therefore, $B_{r+1} = \sum_{i=0}^{r} C(r, i) B_i$ or $B_r = \sum_{i=0}^{r-1} C(r-1, i) B_i$.

Problem 1.246. *Obtain a linear recurrence relation for the Bernoulli numbers.*

Solution. We have $x = (e^x - 1) \sum_{n=0}^{\infty} b_n \dfrac{x^n}{n!}$

The left-hand side is the exponential generating function of < 0, 1, 0, 0, 0, ...>. Since $e^x - 1$ generates <0, 1, 1, 1, ...>.

The right hand side generates $\left\langle \sum_{i=1}^{n} C(n, i) b_{n-i} \right\rangle$

Consequently, $b_0 = 1$ and for $n \geq 2$

$$\sum_{i=1}^{n} C(n, i) b_{n-i} = 0 \text{ or } b_{n-1} = -\frac{1}{n} \sum_{i=2}^{n} C(n, i) b_{n-i} .$$

Problem 1.247. *For every positive integer n let $S_0(n) = n$ and*

$$S_m(n) = 1^m + 2^m + 3^m + \ldots\ldots + (n-1)^m$$

where m is a positive integer. Obtain the exponential generating function of the sequence

$$\langle S_m(n) \rangle_{m \geq 0}.$$

Solution. By linearity,

generating function of $\langle S_m (k + 1)\rangle_{m \geq 0}$ – generating function of $\langle S_m (k) \rangle_{m \geq 0}$ = generating function of $\langle S_m(k + 1) - S_m(k)\rangle_{m \geq 0}$

= generating function of $\langle k^m \rangle_{m \geq 0} = e^{kx} = (e^x)$

This simple recurrence relation is solved by summation

$[S_m(0) \equiv 0]$;

generating function of $\langle S_m (n)\rangle_{m \geq 0} = \sum_{k=0}^{n-1} (e^x)^k = \dfrac{e^{nx} - 1}{e^x - 1}$.

Problem 1.248. *The Bernoulli polynomial of degree m is defined by*

$$B_m(t) = \sum_{i=0}^{m} C(m, i) b_i\, t^{m-i}$$

(*a*) *Show that* $S_{m-1}(n) = [B_m(n) - B_m(0)]/m$.

(*b*) *Obtain the exponential generating function (in the variable x) of the sequence* $\langle B_m(t)\rangle_{m \geq 0}$.

Solution. (*a*) We have $S_m(n) = \dfrac{1}{m+1} \displaystyle\sum_{i=0}^{m} C(m+1, i) b_i n^{m+1-i}$

$$S_{m-1}(n) = \frac{1}{m}\left[\sum_{i=0}^{m} C(m, i) b_i n^{m-1} - C(m, m) b_m \; n^{m-m}\right]$$

$$= \frac{1}{m}[B_m(n) - B_m(0)].$$

(*b*) The sequences $\langle b_m\rangle_{m \geq 0}$ and $\langle t^m\rangle_{m \geq 0}$ have respective exponential generating functions

$$\frac{x}{e^x - 1} \text{ and } e^{tx}.$$

Then by the binomial convolution

$$\left\langle \sum_{i=0}^{m} C(m, i) b_i t^{m-i} \right\rangle_{m \geq 0} = \langle B_m(t)\rangle_{m \geq 0}.$$

must have the product $\dfrac{xe^{tx}}{(e^x - 1)}$ as its exponential generating function.

Problem 1.249. *Give the ordinary generating function of the sequence* $\langle n(3 + 5n)\rangle$.

Solution. We have $n(3 + 5n) = 3n + 5n^2$

By $g(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, the respective generating functions of $\langle n\rangle$ and $\langle n^2\rangle$ are

$$\frac{x}{(1-x)^2} \text{ and } \frac{x(1+x)}{(1-x)^3}.$$

Hence, the answer is

$$\frac{3x}{(1-x)^2} + \frac{5x(1+x)}{(1-x)^3} = \frac{8x + 2x^2}{(1-x)^3}.$$

Problem 1.250. (Restricted Partitions)

Given a collection k of n distinct positive integers $\alpha_1 < \alpha_2 < \ldots\ldots < \alpha_n$, *and an arbitrary positive integer r, let* $f_n(r) \equiv$ *number of partitions of r into parts selected (with replacement) from k.*

Determine the ordinary generating function of $\langle f_n(0) \equiv 1, f_n(1), \ldots\rangle$.

Solution. Here we want to count the solutions in non negative integers of

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n = r$$

and so we write
$$\prod_{i=1}^{n} \frac{1}{1-x^{\alpha_i}} = [1 + x^{\alpha_1} + (x^{\alpha_1})^2 + ... + (x^{\alpha_1})^{u_1} + ...]$$
$$\times [1 + x^{\alpha_2} + (x^{\alpha_2})^2 + ... + (x^{\alpha_2})^{u_2} + ...] \times ......$$
$$...... \times [1 + x^{\alpha_n} + (x^{\alpha_n})^2 + ... + (x^{\alpha_n})^{u_n} + ...]$$
$$= \sum_{r=0}^{\infty} f_n(r)\, x^r .$$

Problem 1.251. *In an experiment, 4 differently coloured dice are thrown simultaneously, and the numbers are added. Find the numbers of distinct experiments such that (a) the total is 18 and (b) the total is 18 and the green die shows an even number.*

Solution. (*a*) The answer is the coefficient of x^{18} in the generating function
$$(x^1 + x^2 + ... + x^6)^4 = (x - x^7)^4\,(1 - x^{-4})$$
$$= (x^4 - 4x^{10} + 6x^{16} - ......) \sum_{r=0}^{\infty} C(r+3, 3)x^r$$

which is seen to be
$$1 . C(17, 3) - 4 . C(11, 3) + 6 . C(5, 3) = 80.$$

(*b*) Now the generating function is
$$(x^2 + x^4 + x^6)(x^1 + x^2 + ... + x^6)^3$$
$$= (x^2 + x^4 + x^6)(x^3 - 3x^9 + 3x^{15} - x^{21}) \sum_{r=0}^{\infty} C(r+2, 2)x$$
$$= (x^5 + x^7 + x^9 - 3x^{11} - 3x^{13} - 3x^{15} + 3x^{17} + ...) \sum_{r=0}^{\infty} C(r+2, 2)x$$

in which the coefficient of x^{18} is
$$1.C(15, 2) + 1.C(13, 2) + 1.C(11, 2) - 3.C(9, 2) - 3.C(7, 2)$$
$$- 3.C(5, 2) + 3.C(3, 2) \text{ or } 46.$$

Problem 1.252. *Find the number of ways of forming a committee of 9 people drawn from 3 different parties so that no party has an absolute majority in the committee.*

Solution. If any party is excluded, one of the other parties will have an absolute majority. So there must be atleast 1 person from each party. And no party can have more than 4 representatives in the committee.

Thus, the generating function is
$$f(x) = (x + x^2 + x^3 + x^4)^3$$
$$= (x^3 - 3x^7 + 3x^{11} - x^{15}) \sum_{r=0}^{\infty} C(r+2, 2)x^r$$

The answer is the coefficient of x^9 in $f(x)$, which is

$$1.C(8, 2) - 3.C(4, 2) = 10.$$

Problem 1.253. *Establish the upper bound $P(r) < \exp. (3\sqrt{r})$.*

Solution. If $g(x)$ is the generating function of <P(r)> then for any r and all $0 < x < 1$,

where
$$g(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)......}$$

$$P(r)x^r < g(x) \text{ or } \log P(r) < \log g(x) - r \log x$$

From the well-known expansion

$$\log \frac{1}{1-u} = u + \frac{u^2}{2} + \frac{u^3}{2} + ... (0 \le u < 1)$$

it follows that

$$\log g(x) = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + ...\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + ...\right)$$

$$+ \left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + ...\right) + ...$$

$$= (x + x^2 + x^3 + ...) + \frac{1}{2}(x^2 + x^4 + x^6 + ...) + \frac{1}{3}(x^3 + x^6 + x^9 + ...)$$

$$= \frac{x}{1-x} + \frac{1}{2}\frac{x^2}{1-x^2} + \frac{1}{3}\frac{x^3}{1-x^3} + ...$$

Now, for $0 < x < 1$ and $k = 1, 2, 3, ...$

$$\frac{x^k}{1-x^k} = \frac{x}{1-x}\ \frac{x^{k-1}}{1+x+x^2+......+x^{k-1}} < \frac{x}{1-x}\ \frac{x^{k-1}}{x^{k-1}+x^{k-1}+......+x^{k-1}}$$

$$= \frac{x}{1-x}\frac{1}{k}$$

Whence $\log g(x) < \dfrac{x}{1-x}\left[1 + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{3}\right)^2 + ...\right]$

$$= \frac{x}{1-x}\ \frac{\pi^2}{6}$$

Further, $\quad -\log x = \log \frac{1}{x} = \int_1^{1/x} \frac{dt}{t} < \int_1^{1/x} dt = \frac{1-x}{x}.$

Therefore we have

$$\log P(r) < \frac{\pi^2}{6}\frac{x}{1-x} + r\frac{1-x}{x} \quad \text{...(1)}$$

Using standard calculus to minimize the right-hand side of (1) over $0 < x < 1$, one obtains

$$\log P(r) < \frac{2\pi}{\sqrt{6}}\sqrt{r} < 3\sqrt{r}$$

and the proof is complete.

Problem 1.254. *Establish the lower bound $P(r) \geq 2^q$ for $r \geq 2$, where $q \equiv \lfloor \sqrt{r} \rfloor$.*

Solution. The bound may be established by inspection for $q = 1, 2$; so assume $q \geq 3$.

It is asserted that each non empty subset S of

$X = \{1, 2, 3, ..., q\}$ generates a partition of r.

In fact, if $\sigma(S) \equiv$ Sum of the elements of S

then $$\sigma(S) \leq \sigma(X) = \frac{q^2+q}{2} \leq \frac{r+\sqrt{r}}{2} < \frac{r+r}{2} = r.$$

So that $S \cup \{r - \sigma(S)\}$ is the desired partition of r.

Furthermore, distinct subsets generate distinct partitions.

To see that this is so, let

$$S_1 \cup \{r - \sigma(S_1)\} \text{ and } S_2 \cup \{r - \sigma(S_2)\} \quad \text{...(1)}$$

be the partitions generated by the distinct k subsets $(1 \leq k \leq q-1)S_1$ and S_2.

For $i = 1, 2,$

we have $$r - \sigma(S_i) \geq q^2 - [\sigma(X) - 1] = q^2 - \left[\frac{q^2+q-2}{2}\right]$$

$$= \frac{q^2-q+2}{2} = q + \frac{(q-2)(q-1)}{2}$$

Consequently, for $q \geq 3$, $\quad r - \sigma(S_i) > q$ ...(2)

If the 2 partitions (1) coincided, and if $\sigma(S_1) = \sigma(S_2)$, then S_1 must coincide with S_2, which is contrary to the hypothesis.

On the otherhand, if the 2 partitions (1) coincided, and if $\sigma(S_1) \neq \sigma(S_2)$, then $r - \sigma(S_1)$ would have to be an element of S_2, which is ruled out by (2).

The conclusion is that the number of partitions of r must exceed the number of non null subsets of X :

$$P(r) > 2^q - 1 \geq 2^q.$$

Problem 1.255. *Define*

$q^{\#}(r, E)$ = *number of partitions of r into an even number of unequal parts*

$q^{\#}(r, 0)$ = *number of partitions of r into an odd number of unequal parts*

Prove that $(1 - x)(1 - x^2)(1 - x^3) \ldots\ldots = \sum_{r=0}^{\infty} [q^{\#}(r, E) - q^{\#}(r, 0)]\, x^r$.

Solution. Because $(1 - x)(1 - x^2)(1 - x^3) \ldots\ldots$

$$= [1 + (-1)x][1 + (-1)x^2][1 + (-1)x^3] \ldots\ldots$$

any partition of r into an even number, e, of unequal parts will contribute $(-1)^e = +1$ to the coefficient of x^r in the infinite product.

Analogously, any partition of r into an odd number, 0, of unequal parts will contribute $(-1)^0$ $= -1$.

Therefore, the coefficient of x^r is

$$q^{\#}(r, \mathrm{E})(+1) + q^{\#}(r, 0)(-1) = q^{\#}(r, \mathrm{E}) - q^{\#}(r, 0) \text{ as asserted.}$$

1.9.6. Problem 1.256. (Euler's First Identity)

Derive $(1 + x^1)(1 + x^3)(1 + x^5) \ldots\ldots = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(1 - x^2)(1 - x^4)(1 - x^6) \ldots\ldots (1 - x^{2k})}$

(an empty product equals unity).

Solution. The left hand side is the ordinary generating function of $< \mathrm{P}^{\#}(r, \mathrm{ODD})>$.

In view of $\mathrm{P}^{\#}(r, \mathrm{ODD}) = \sum_{k=1}^{\lfloor \sqrt{r} \rfloor} \mathrm{P}_{2k}(r - k^2, \mathrm{EVEN})$

$$= \sum_{k=1}^{\lfloor \sqrt{r} \rfloor} q_k(r - k^2, \mathrm{EVEN}).$$

it is enough to prove that the k^{th} summand ($k = 0, 1, 2, \ldots\ldots$) on the right is the generating function of $< \mathrm{P}_{2k}(r - k^2, \mathrm{EVEN})>_{r \geq 0}$. But this is obvious, for

$$\frac{x^{k^2}}{(1 - x^2)(1 - x^4) \ldots\ldots (1 - x^{2k})} = x^{k^2}(1 + x^2 + x^4 + \ldots\ldots)(1 + x^4 + x^8 + \ldots\ldots)$$

$$\ldots\ldots(1 + x^k + x^{4k} + \ldots\ldots)$$

$$= x^{k^2} \sum_{s=0}^{\infty} \mathrm{P}_{2k}(\mathrm{S}, \mathrm{EVEN})\, x^s = \sum_{s=0}^{\infty} \mathrm{P}_{2k}(\mathrm{S}, \mathrm{EVEN})\, x^{s+k^2}$$

$$= \sum_{r=0}^{\infty} \mathrm{P}_{2k}(r - k^2, \mathrm{EVEN})\, x^r$$

1.9.7. Problem 1.257. (Euler's Second Identity)

Show that $(1 + x^2)(1 + x^4)(1 + x^6) \ldots\ldots = \sum_{k=0}^{\infty} \frac{x^{k(k+1)}}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots(1-x^{2k})}$.

Solution. Suppose that we are given a partition of r into k distinct even parts.

Then subtraction of 1 from each part yields a unique partition of $r - k$ into k distinct odd parts.

Conversely, addition of 1 ...... yields a unique ......

By this one-to-one correspondence, and by the result

$$P^{\#}(r, \text{ODD}) = \sum_{k=1}^{\lfloor\sqrt{r}\rfloor} P_{2k}\,(r - k^2, \text{EVEN}) = \sum_{k=1}^{\lfloor\sqrt{r}\rfloor} q_k\,(r - k^2, \text{EVEN})$$

$$\begin{pmatrix}\text{No. of partitions of } r \text{ into} \\ k \text{ distinct even parts}\end{pmatrix} = \begin{pmatrix}\text{No. of partitions of } r - k \text{ into} \\ k \text{ distinct odd parts}\end{pmatrix}$$

$$= P_{2k}((r-k) - k^2, \text{EVEN})$$
$$= P_{2k}\,(r - k(k+1), \text{EVEN})$$

Therefore, the proof of Euler's second identity reduces to establishing that the ordinary generating function of

$$\langle P_{2k}(r - k(k+1), \text{EVEN})\rangle_{r \geq 0} \text{ is just}$$

$$\frac{x^{k(k+1)}}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots(1-x^{2k})}$$

This may be carried out by inspection.

Problem 1.258. *The number of partitions of r into n distinct (unequal) parts is denoted by* $q^{\#}(r, n)$*. Prove that*

$$q^{\#}(r, n) = q(r - C(n, 2), n)$$

Solution. By definition the system

$$u_1 + u_2 + \ldots\ldots + u_n = r$$
$$0 < u_1 < u_2 < \ldots\ldots < u_n \qquad \ldots(1)$$

has precisely $q^{\#}(r, n)$ solutions in integers u_i. Under the bijective transformation

$$u_1 = w_1$$
$$u_2 = w_2 + 1$$
$$u_3 = w_3 + 2$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$u_n = w_n + (n - 1)$$

(1) goes over into

$$w_1 + w_2 + \ldots\ldots + w_n = r - C(n, 2) \qquad \ldots(2)$$
$$0 < w_1 \leq w_2 \leq w_3 \leq \ldots\ldots \leq w_n$$

But, again by definitions, (2) has exactly

$q(r - C(n, 2), n)$ solutions in integers w_i.

Problem 1.259. *Show that*

$$P^{\#}(r, ODD) = \sum_{k=1}^{\lfloor\sqrt{r}\rfloor} P_{2k}\,(r - k^2, EVEN) = \sum_{k=1}^{\lfloor\sqrt{r}\rfloor} q_k\,(r - k^2, EVEN)\,.$$

Solution. As in Fig. 1.7, represent a partition of r into distinct odd parts by nested elbows.

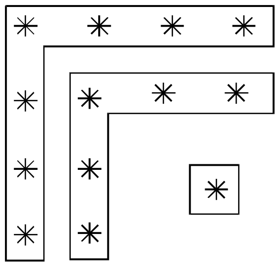


Fig. 1.7

Let k be the number of parts. Then k is the largest integer such that the diagram contains a $k \times k$ square (called the DURFEE square) having as one corner the asterisk in the first row and first column.

Clearly, $1 \le k \le \lfloor\sqrt{r}\rfloor$, in Fig. 1.7, which diagrams

$$23 = 11 + 9 + 3, \quad \text{one has } k = 3.$$

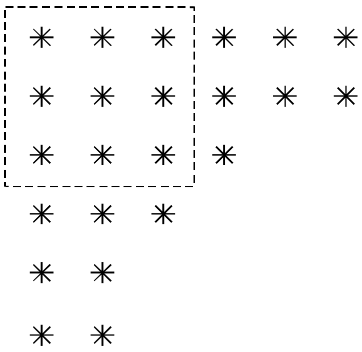


Fig. 1.8

The remaining $r - k^2$ asterisks can be assembled into a partition of $r - k^2$ with all parts even, in two different ways :

(*i*) If the i^{th} part in the partition is the total number of asterisks in the $(k + i)^{\text{th}}$ row and the $(k + i)^{\text{th}}$ column, then all parts are less than or equal to $2k$.

(*ii*) If the j^{th} part in the partition is the number of asterisks in the j^{th} elbow which lie outside the square, then there are at most k parts.

Since the geometric argument is reversible, we conclude that the number of partitions of r into k distinct odd parts is given by either $P_{2k}(r - k^2, \text{EVEN})$ or $q_k(r - k^2, \text{EVEN})$.

Hence, a summation on k yields the required results. Note that, as usual, the sums may be extended from $k = 0$ to $k = \infty$, since only null summands are thereby introduced.

Problem 1.260. *Let $P^{\#}(r)$ be the number of partitions of r into unequal parts. Obtain the ordinary generating function of $<P^{\#}(r)>$.*

Solution. A given positive integer i appears either 0 times or 1 time among the parts of r ;

Hence, $(1 + x^1)(1 + x^2)(1 + x^3) \ldots\ldots (1 + x^i) = \sum_{r=0}^{\infty} P^{\#}(r)x^r$

It is evident that for a particular value of r say, $r = s$ on the first s factors of the infinite product need be retained.

Problem 1.261. *Let $P(r, n) \equiv$ number of partitions of r with largest part n*

$q(r, n)$ = number of partitions of r into exactly n parts.

(*a*) *Prove that, for all r and n, $P(r, n) = q(r, n)$*

(*b*) *Determine the ordinary generating function (on r) of either sequence.*

Solution. (*a*) $P^*(r, n) = P_n(r) - P_{n-1}(r)$

$= q_n(r) - q_{n-1}(r) = q(r, n).$

(*b*) The ordinary generating function of $<P_n(r)> = <q_n(r)>$ is

$$g_n(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)}$$

$$f_n(x) \equiv \sum_{r=0}^{\infty} P(r, n)x^r = \sum_{r=0}^{\infty} P_n(r)x^r - \sum_{r=0}^{\infty} P_{n-1}(r)x^r$$

$$= g_n(x) - g_{n-1}(x) = x^n g_n(x).$$

Problem 1.262. *Establish the recurrence relation*

$$q_n(r) = q_{n-1}(r) + q_n(r - n)$$

(also satisfied by $P_n(r)$)

(*a*) *by solving a distribution problem, and*

(*b*) *by use of theorem ; the ordinary generating function of*

$$<P_n(r)> = <q_n(r)> \text{ is } g_n(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)} \qquad \ldots(1)$$

Solution. (*a*) Imagine you are given a heap of r identical 1 s and a row of n identical boxes.

Partitioning r into exactly n parts, which, by definition, can be accomplished in $q_n(r) - q_{n-1}(r)$ ways is tantamount to first putting *a*1 into each box (1 way) and then arbitrarily distributing the remaining $(r - n)$ 1 s among the n boxes [$q_n(r - n)$ ways].

Thus, by the product rule

$$q_n(r) - q_{n-1}(r) = 1 \,.\, q_n(r - n).$$

(*b*) From (1), $(1 - x^n)\, g_n(x) = g_{n-1}(x)$

Equating coefficients of x^r,

$$q_n(r) - q_n(r-n) = q_{n-1}(r).$$

Problem 1.263. *Find the generating function for $P_d(n)$, the number of partitions of a positive integer n into distinct summands.*

Solution. Let us consider the 11 partitions of 6 :

1. 1 + 1 + 1 + 1 + 1 + 1	2. 1 + 1 + 1 + 1 + 2	
3. 1 + 1 + 1 + 3	4. 1 + 1 + 4	
5. 1 + 1 + 2 + 2	6. 1 + 5	
7. 1 + 2 + 3	8. 2 + 2 + 2	
9. 2 + 4	10. 3 + 3	11. 6

Partitions (6), (7), (9), and (11) have distinct summands so $P_d(6) = 4$.

In calculating $P_d(n)$, for each $k \in Z^+$ there are two choices : either k is not used as one of the summands of n, or it is. This can be accounted for by the polynomial $1 + x^k$, and consequently, the generating function for these partitions is

$$P_d(x) = (1 + x)(1 + x^2)(1 + x^3) \ldots\ldots = \prod_{i=1}^{\infty} (1 + x^i)$$

For each $n \in Z^+$, $P_d(n)$ is the coefficient of x^n in $(1 + x)(1 + x^2) \ldots\ldots (1 + x^n)$.

When $n = 6$, the coefficient of x^6 in $(1 + x)(1 + x^2) \ldots\ldots (1 + x^6)$ is 4.

Problem 1.264. *Find the generating function for the number of ways an advertising agent can purchase n minutes ($n \in Z^+$) of air time if time slots for commercials come in block of 30, 60 or 120 seconds.*

Solution. Let 30 seconds represent one time unit. Then the answer is the number of integer solutions to the equation $a + 2b + 4c = 2n$ with $0 \le a, b, c$.

The associated generating function is

$$f(x) = (1 + x + x^2 + \ldots\ldots)(1 + x^2 + x^4 + \ldots\ldots)(1 + x^4 + x^8 + \ldots\ldots)$$

$$= \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^4}\,.$$

and the coefficient of x^{2n} is the number of partitions of $2n$ into 1's, 2's and 4's.

Problem 1.265. *A ship carries 48 flags, 12 each of the colors red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.*

(*a*) *How many of these signals use an even number of blue flags and an odd number of black flags ?*

(*b*) *How many of the signals have atleast three white flags or no white flags at all ?*

Solution. (*a*) The exponential generating function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right)$$

Consider all such signals made up of n flags, where $n \geq 1$.

The last two factors in $f(x)$ restrict the signals to an even number of blue flags and an odd number of black flags, respectively.

Since $$f(x) = (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{1}{4}\right)(e^{2x})(e^{2x} - e^{-2x})$$

$$= \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right)\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}$$

The co-efficient of $\frac{x^{12}}{12!}$ in $f(x)$ yields $\left(\frac{1}{4}\right)(4^{12}) = 4^{11}$ signals made up of 12 flags with an even number of blue flags and an odd number of black flags.

(*b*) The exponential generating function

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)(e^x)^2$$

$$= e^{3x}\left(e^x - x - \frac{x^2}{2!}\right)$$

$$= e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2 e^{3x}$$

$$= \sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$$

Here the factor $\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\right) = e^x - x - \frac{x^2}{2!}$ in $g(x)$ restricts the signals to those that contain three or more of the 12 white flags, or none at all.

The answer for the number of signals sought here is the coefficient of $\frac{x^{12}}{12!}$ in $g(x)$.

We consider each summand, we find :

(i) $\sum_{i=0}^{\infty} \frac{(4x)^i}{i!}$, here we have the term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$. So the coefficient of $\frac{x^{12}}{12!}$ is 4^{12}.

(ii) $x\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$. Now we see that in order to get $\frac{x^{12}}{12!}$ we need to consider the term

$x\left[\frac{(3x)^{11}}{11!}\right] = 3^{11}\left(\frac{x^{12}}{11!}\right) = (12)(3^{11})\left(\frac{x^{12}}{12!}\right)$ and here the coefficient of $\frac{x^{12}}{12!}$ is $(12)(3^{11})$ and

(iii) $\left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i}\right)$, for this summand we observe that $\left(\frac{x^2}{2}\right)\left[\frac{(3x)^{10}}{10!}\right] = \left(\frac{1}{2}\right)(3^{10})\left(\frac{x^{12}}{10!}\right)$

$= \left(\frac{1}{2}\right)(12)(11)(3^{10})\left(\frac{x^{12}}{12!}\right)$, where the coefficient of $\frac{x^{12}}{12!}$ is $\left(\frac{1}{2}\right)(12)(11)(3^{10})$.

Consequently, the number of 12 flags signals with atleast three white flags, or none at all is

$$4^{12} - 12(3^{11}) - \left(\frac{1}{2}\right)(12)(11)(3^{10}) = 10{,}754{,}218.$$

Problem 1.266. *Find a formula to express* $0^2 + 1^2 + 2^2 + \ldots\ldots + n^2$ *as a function of n.*

Solution. We start with $g(x) = \frac{1}{(1-x)} = 1 + x + x^2 + \ldots\ldots$

Then $(-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} = \frac{dg(x)}{de} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

So $\frac{x}{(1-x)^2}$ is the generating function for 0, 1, 2, 3, 4, ...

Repeating this technique, we find that

$$x\frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = \frac{x(1+x)}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + \ldots\ldots$$

So $\dfrac{x(1+x)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2, \ldots\ldots$

$$\frac{x(1+x)}{(1-x)^3} \cdot \frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$$

is the generating function for $0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, 0^2 + 1^2 + 2^2 + 3^2, \ldots\ldots$

Hence the coefficient of x^n in $\dfrac{x(1+x)}{(1-x)^4}$ is $\sum_{i=0}^{n} i^2$. But the coefficient of x^n in $\dfrac{x(1+x)}{(1-x)^4}$ can also be calculated as follows :

$$\frac{x(1+x)}{(1-x)^4} = (x + x^2)(1-x)^{-4} = (x + x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots\ldots\right]$$

So the coefficient of x^n is

$$\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$$

$$= (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2}$$

$$= \binom{n+2}{n-1} + \binom{n+1}{n-2} = \frac{(n+2)\,!}{3\,!\,(n-1)\,!} + \frac{(n+1)\,!}{3\,!\,(n-2)\,!}$$

$$= \frac{1}{6}\,[(n+2)(n+1)(n) + (n+1)(n)(n-1)]$$

$$= \frac{1}{6}\,(n)(n+1)\,[(n+2) + (n-1)]$$

$$= (n)(n+1)(2n+1)/6.$$

Problem 1.267. *A company hires 11 new employees, each of whom is to be assigned to one of four sub divisions. Each sub-division will get atleast one new employee. In how many ways can these assignments be made ?*

Solution. Calling the subdivisions A, B, C and D, we can equivalently count the number of 11-letter sequences in which there is atleast one occurrence of each of the letters A, B, C and D.

The exponential generating function for these arrangements is

$$f(x) = \left(x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots\ldots\right)^4$$

$$= (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1$$

The answer then is the coefficient of $\frac{x^{11}}{11!}$ in $f(x)$:

$$4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11})$$

$$= \sum_{i=0}^{4} (-1)^i \binom{4}{i} (4-i)^{11} .$$

1.9.8. Calculational Techniques

Definition and examples

The most important concept we introduce is that of division of formal **power series.** First let us discuss the meaning of $\frac{1}{A(x)}$. If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal power series then A(x) is said to have a multiplicative inverse if there is a formal power series $B(x) = \sum_{k=0}^{\infty} b_k k x^k$ such that A(x) B(x) = 1.

In particular, if A(x) has a **multiplicative inverse**, then we see that $a_0 b_0 = 1$, so that a_0 must be non zero.

The converse is also true. Infact, if $a_0 \neq 0$, then we can determine the coefficients of B(x) by writing down the coefficients of successive powers of x in A(x) B(x) from the definition of product of 2 power series, and then equating these to the coefficients of like powers of x in the power series 1. Therefore we have

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 &= 0 \\ &\ldots\ldots \\ a_0 b_n + a_1 b_{n-1} + \ldots\ldots + a_n b_0 &= 0 \text{ and so on.} \end{aligned}$$

From the first equation, we can solve for $b_0 = \frac{1}{a_0}$, from the second, we find $b_1 = \frac{-a_1 b_0}{a_0} = \frac{-a_1}{a_0^2}$

in the third equation, we get

$$b_2 = \frac{-a_1 b_1 - a_2 b_0}{a_0} = \frac{a_1^2 - a_2 a_0}{a_0^3}$$

from the fourth, we find

$$b_3 = \frac{-a_1 b_2 - a_2 b_1 - a_3 b_0}{a_0}$$

We can substitute into this expression for b_0, b_1 and b_2 to obtain an expression for b_3 involving only the coefficients of A(x). Continuing in this manner, we can solve for each coefficient of B(x).

Thus, we established that a formal power series

$A(x) = \sum_{n=0}^{\infty} a_n x^n$ has a multiplicative inverse if the constant term a_0 is different from zero.

If A(x) and C(x) are power series, we say that A(x) divides C(x) if there is a formal power series D(x) such that C(x) = A(x) D(x), and we write $D(x) = \frac{C(x)}{A(x)}$.

Of course, for arbitrary formal power series, A(x) and C(x), it need not be the case that A(x) divides C(x).

However, if $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is such that $a_0 \neq 0$, then A(x) has a multiplicative inverse B(x) $= \frac{1}{A(x)}$ and then A(x) divides any C(x), let $D(x) = C(x)B(x) = \left(C(x)\frac{1}{A(x)}\right)$.

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $a_0 = 0$, but some coefficient of A(x) is not zero, then let a_k be the first non zero coefficient of A(x), and $A(x) = x^k A_1(x)$, where a_k, the constant term of $A_1(x)$ is non zero.

Then in order for A(x) to divide C(x) it must be true that x^k is also a factor of C(x), that is, $C(x) = x^k C_1(x)$ where $C_1(x)$ is a formal power series. If this is the case, then cancel the common powers of x from both A(x) and C(x) and then we can find $\frac{C(x)}{A(x)} = \frac{C_1(x)}{A_1(x)}$ by using the multiplicative inverse of $A_1(x)$.

1.9.9. Geometric Series

Let us use the multiplicative inverse for A(x) = 1 – x.

Let $B(x) = \frac{1}{A(x)} = \sum_{n=0}^{\infty} b_n x^n$.

Solving successively for b_0, b_1, ..., as above, we see that

$$b_0 = \frac{1}{a_0} = 1$$

$$b_1 = \frac{-a_1 b_0}{a_0} = \frac{-(-1)(1)}{(1)} = 1$$

$$b_2 = \frac{-a_1b_1 - a_2b_0}{a_0} = \frac{-(-1)(1)-(0)(1)}{1} = 1$$

$$b_3 = \frac{-a_1b_2 - a_2b_1 - a_3b_0}{a_0} = 1, \text{ and so on.}$$

We see that each $b_i = 1$ so that we have an expression for the geometric series

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r \qquad ...(1)$$

If we replace in the above expression x by ax where a is a real number, then we see that

$$\frac{1}{1-ax} = \sum_{r=0}^{\infty} a^r x^r \qquad ...(2)$$

the so called geometric series (with common ratio a).

In particular, let $a = -1$, then we get

$$\frac{1}{1+x} = \sum_{r=0}^{\infty} (-1)^r x^r = 1 - x + x^2 - x^3 \qquad ...(3)$$

the so called alternating geometric series.

Likewise,
$$\frac{1}{1+ax} = \sum_{r=0}^{\infty} (-1)^r a^r x^r \qquad ...(4)$$

Suppose that n is a positive integer. If $B_1(x)$, $B_2(x)$, ... $B_n(x)$ are the multiplicative inverse of $A_1(x)$, $A_2(x)$ ...... and $A_n(x)$, respectively, then $B_1(x)$ $B_2(x)$ ...... $B_n(x)$ is the multiplicative inverse of $A_1(x)A_2(x)$ ...... $A_n(x)$, just multiply

$A_1(x)A_2(x)$ ...... $A_n(x)$ by $B_1(x)$ $B_2(x)$ ...... $B_n(x)$ and use the facts that $A_i(x)B_i(x) = 1$ for each i.

In particular, if $B(x)$ is the multiplicative inverse of $A(x)$, then $(B(x))^n$ is the multiplicative inverse of $(A(x))^n$. Let us apply this observation to $A(x) = 1 - x$.

For n a positive integer,

$$\frac{1}{(1-x)^n} = \left(\sum_{k=0}^{\infty} x^k\right) = \sum_{r=0}^{\infty} C(n-1+r, r)x^r \qquad ...(5)$$

The fact that $\sum_{k=0}^{\infty} x^k$ is the multiplicative inverse of $1 - x$.

The equality $\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)x^r$ could also be proved by mathematical induction and use of the identity $C(n-1, 0) + C(n, 1) + C(n+1, 2) + \ldots\ldots + C(n+r-1, r) = C(n+r, r)$

By replacing x by $-x$ in the above we get the following identity :
For n a positive integer,

$$\frac{1}{(1+x)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)(-1)^r x^r \qquad ...(6)$$

Following this pattern, replace x by ax in (5) and (6) to obtain

$$\frac{1}{(1-ax)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)a^r x^r \qquad ...(7)$$

$$\frac{1}{(1+ax)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)(-a)^r x^r \qquad ...(8)$$

Likewise, replace x by x^k in (1) to get for k a positive integer,

$$\frac{1}{1-x^k} = \sum_{r=0}^{\infty} x^{kr} = 1 + x^k + x^{2k} + ... \qquad ...(9)$$

and

$$\frac{1}{1+x^k} = \sum_{r=0}^{\infty} (-1)^r x^{kr} \qquad ...(10)$$

If a is a non zero real number,

$$\frac{1}{a-x} = \frac{1}{a}\left(\frac{1}{1-\frac{x}{a}}\right) = \frac{1}{a} \sum_{r=0}^{\infty} \frac{x^r}{a^r} \qquad ...(11)$$

and

$$\frac{1}{x-a} = -\frac{1}{a-x} = -\frac{1}{a} \sum_{r=0}^{\infty} \frac{x^r}{a^r} \qquad ...(12)$$

If n is a positive integer,

$$1 + x + x^2 + ...+ x^n = \frac{1-x^{n+1}}{1-x} \qquad ...(13)$$

If n is a positive integer,

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$$

$$(1 + x^k)^n = 1 + \binom{n}{1}x^k + \binom{n}{2}x^{2k} + ... + \binom{n}{n}x^{nk}$$

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + (-1)^n \binom{n}{n}x^n$$

$$(1-x^k)^n = 1 - \binom{n}{1}x^k + \binom{n}{2}x^{2k} + \ldots\ldots + (-1)^n \binom{n}{n}x^{nk}.$$

1.9.10. Use of Partial Fraction Decomposition

If A(x) and C(x) are polynomials, we compute $\frac{C(x)}{A(x)}$ by using the above identities and partial fractions.

If A(x) is a product of linear factors,

$A(x) = a_n(x - \alpha_1)^{r_1}(x - \alpha_2)^{r_2} \ldots\ldots (x - \alpha_k)^{r_k}$ and if C(x) is any polynomial of degree less than the degree of A(x), then $\frac{C(x)}{A(x)}$ can be written as the sum of elementary fractions as follows :

$$\frac{C(x)}{A(x)} = \frac{A_{11}}{(x-\alpha_1)^{r_1}} + \frac{A_{12}}{(x-\alpha_1)^{r_1-1}} + \ldots\ldots + \frac{A_1 r_1}{(x-\alpha_1)}$$

$$+ \frac{A_{21}}{(x-\alpha_2)^{r_2}} + \frac{A_{22}}{(x-\alpha_2)^{r_2-1}} + \ldots\ldots + \frac{A_2 r_2}{(x-\alpha_2)} + \ldots\ldots$$

$$+ \frac{Ak_1}{(x-\alpha_k)^{r_k}} + \frac{Ak_2}{(x-\alpha_k)^{r_k-1}} + \ldots\ldots \frac{Ak_{rk}}{(x-\alpha_k)}.$$

To find the numbers A_{11}, ..., A_{krk}, we multiply both sides of the last equation by $(x - \alpha_1)^{r_1}(x - \alpha_2)^{r_2} \ldots\ldots (x - \alpha_k)^{r_k}$ to clear of denominators and then we equate coefficients of the same powers of x. Then the required coefficients can be solved from the resulting system of equations.

Problem 1.268. *Calculate* $B(x) = \sum_{r=0}^{\infty} b_r x^r = \frac{1}{(x^2 - 5x + 6)}$.

Solution. Since $x^2 - 5x + 6 = (x - 3)(x - 2)$.

We see that $\frac{1}{x^2 - 5x + 6} = \frac{A}{x-3} + \frac{B}{x-2}$

Thus, $A(x - 2) + B(x - 3) = 1$.

Let $x = 2$ and we find $B = -1$

Let $x = 3$ and we see that $A = 1$.

Thus $\frac{1}{x^2 - 5x + 6} = \frac{1}{x-3} - \frac{1}{x-2}$

Then we use

$$\frac{1}{a-x} = \frac{1}{a}\left(\frac{1}{1-\frac{x}{a}}\right) = \frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$$

$$\frac{1}{x-a} = -\frac{1}{a-x} = -\frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$$

to see that

$$\frac{1}{x^2-5x+6} = -\frac{1}{3-x} + \frac{1}{2-x}$$

$$= -\frac{1}{3\left(1-\frac{x}{3}\right)} + \frac{1}{2\left(1-\frac{x}{2}\right)}$$

$$= -\frac{1}{3}\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r x^r + \frac{1}{2}\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r x^r$$

$$= \sum_{r=0}^{\infty}\left(-\frac{1}{3^{r+1}} + \frac{1}{2^{r+1}}\right)x^r$$

$$= \mathrm{B}(x)$$

Therefore, for each r, $b_r = -\frac{1}{3^{r+1}} + \frac{1}{2^{r+1}}$

Thus $\frac{x^5}{x^2-5x+6} = x^5\sum_{r=0}^{\infty}\left(-\frac{1}{3^{r+1}} + \frac{1}{2^{r+1}}\right)x^r$

$$= \sum_{r=0}^{\infty}\left(-\frac{1}{3^{r+1}} + \frac{1}{2^{r+1}}\right)x^{r+5}$$

and if we make the substitution $k = r + 5$ we see that

$$\frac{x^5}{x^2-5x+6} = \sum_{k=5}^{\infty}\left(-\frac{1}{3^{k-4}} + \frac{1}{2^{k-4}}\right)x^k = \sum_{k=0}^{\infty} d_k x^k$$

and what this final equality says is that

$$d_0 = d_1 = d_2 = d_3 = d_4 = 0$$

$$d_5 = -\frac{1}{3} + \frac{1}{2}$$

$$d_6 = -\frac{1}{3^2} + \frac{1}{2^2}$$

$$d_k = -\frac{1}{3^{k-4}} + \frac{1}{2^{k-4}} \text{ if } k \geq 5 \text{ and so on.}$$

Problem 1.269. *Compute the coefficients of*

$$\sum_{r=0}^{\infty} d_r x^r = \frac{x^2 - 5x + 3}{x^4 - 5x^2 + 4}.$$

Solution. Since $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4)$

$$= (x-1)(x+1)(x-2)(x+2)$$

We can write

$$\frac{x^2 - 5x + 3}{x^4 - 5x^2 + 4} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$$

Multiplication by $x^4 - 5x^2 + 4$ gives

$$x^2 - 5x + 3 = A(x+1)(x-2)(x+2) + B(x-1)(x-2)(x+2) + C(x-1)(x+1)(x+2) + D(x-1)(x+1)(x-2)$$

Let $x = 1$, then all terms of the right hand side that involve the factor $x - 1$ vanish, and we have $-1 = -6$ A or $A = \frac{1}{6}$.

Similarly putting $x = -1$, $x = 2$, and $x = -2$, we find $B = \frac{3}{2}$, $C = -\frac{1}{4}$, and $D = -\frac{17}{12}$.

Thus, $$\frac{x^2 - 5x + 3}{x^4 - 5x^2 + 4} = \frac{1}{6(x-1)} + \frac{3}{2(x+1)} - \frac{1}{4(x-2)} - \frac{17}{12(x+2)}$$

$$= \frac{1}{2}\left[-\frac{1}{3(1-x)} + \frac{3}{1+x} + \frac{1}{4\left(1-\frac{x}{2}\right)} - \frac{17}{12\left(1+\frac{x}{2}\right)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3}\sum_{r=0}^{\infty} x^r + 3\sum_{r=0}^{\infty}(-1)^r x^r + \frac{1}{4}\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\right)x^r - \frac{17}{12}\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r x^r\right]$$

$$= \frac{1}{2}\sum_{r=0}^{\infty}\left[\left(-\frac{1}{3}\right) + 3(-1)^r + \frac{1}{4}\frac{1}{2^r} - \frac{17}{12}\left(-\frac{1}{2}\right)^r\right]x^r$$

Therefore, $d_r = \frac{1}{2}\left[-\frac{1}{3} + 3(-1)^r + \frac{1}{2^{r+2}} - \frac{17}{3}(-1)^r \frac{1}{2^{r+2}}\right]$

which can be simplified to

$$d_r = \begin{cases} \frac{1}{2}\left[-\frac{1}{3} + 3 + \frac{1}{2^{r+2}}\left(1 - \frac{17}{3}\right)\right] = \frac{1}{3}\left(4 - \frac{14}{2^{r+3}}\right) \text{if } r \text{ is even} \\ \frac{1}{2}\left[-\frac{1}{3} - 3 + \frac{1}{2^{r+2}}\left(1 + \frac{17}{3}\right)\right] = \frac{1}{3}\left(-5 + \frac{5}{2^{r+1}}\right) \text{if } r \text{ is odd} \end{cases}$$

After doing these examples we see that it is desirable to write $\frac{C(x)}{A(x)}$ in the form

$$\frac{B_{11}}{\left[1-\left(\frac{x}{\alpha_1}\right)\right]^{r_1}} + \frac{B_{12}}{\left[1-\left(\frac{x}{\alpha_1}\right)\right]^{r_1-1}} + \ldots\ldots + \frac{B_1 r_1}{\left[1-\left(\frac{x}{\alpha_1}\right)\right]} + \ldots\ldots + \frac{Bk_1}{\left[1-\left(\frac{x}{\alpha_k}\right)\right]^{r_k}}$$

$$+ \frac{Bk_2}{\left[1-\left(\frac{x}{\alpha_k}\right)\right]^{r_k-1}} + \ldots\ldots + \frac{Bk_{rk}}{\left[1-\left(\frac{x}{\alpha_k}\right)\right]}$$

where $A(x) = a_n(x - \alpha_1)^{r_1} (x - \alpha_2)^{r_2} \ldots\ldots (x - \alpha_k)^{r_k}$ and then solve for the constants B_{11}, ..., $B_1 r_1$, ..., B_{k_1} ......, B_{kr_k}, by algebraic techniques. This is desirable because in this form we can readily apply the formulas

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r \text{ through } \frac{1}{(1+ax)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)(-a)^r x^r .$$

Without having to resort to the intermediate step of applying $\frac{1}{a-x} = \frac{1}{a}\left(\frac{1}{1-\frac{x}{a}}\right) = \frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$

and $$\frac{1}{x-a} = -\frac{1}{a-x} = -\frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$$

Problem 1.270. *Find the coefficient of x^{20} in $(x^3 + x^4 + x^5 \ldots\ldots)^5$.*

Solution. Simplify the expression by extracting x^3 from each factor.

Thus, $(x^3 + x^4 + x^5 + \ldots\ldots)^5 = [x^3(1 + x + \ldots\ldots)]^5$

$$= x^{15}\left(\sum_{r=0}^{\infty} x^r\right)^5 = \frac{x^{15}}{(1-x)^5}$$

$$= x^{15} \sum_{r=0}^{\infty} C(5-1+r, r)x^r$$

The coefficient of x^{20} in the original expression becomes the coefficient of x^5 in $\sum_{r=0}^{\infty} C(4+r, r)x^r$.

Thus, the coefficient we seek is when $r = 5$ in the last power series, that is, the coefficient is

$$C(4 + 5, 5) = C(9, 5).$$

Problem 1.271. *Calculate the coefficient of x^{15} in*

$$A(x) = (x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + ... + x^{15}).$$

Soltuion. Note that we can rewrite the expression for A(x) as

$$x^2(1 + x + x^2 + x^3)(x)(1 + x + ... + x^6)(1 + x + ... + x^{15})$$

$$= x^3 \frac{(1-x^4)}{(1-x)} \frac{(1-x^7)}{1-x} \frac{(1-x^{16})}{1-x}$$

$$= x^3 \frac{(1-x^4)(1-x^7)(1-x^{16})}{(1-x)^3}$$

The coefficient of x^{15} in A(x) is the same as the coefficient of x^{12} in $\frac{(1-x^4)(1-x^7)(1-x^{16})}{(1-x)^3}$

$$= (1 - x^4)(1 - x^7)(1 - x^{16}) \left(\sum_{r=0}^{\infty} C(r+2, r)x^r \right)$$

Since the coefficient of x^{12} in a product of several factors can be obtained by taking one term from each factor so that the sum of their exponents equals 12, we see that the term x^{16} in the third factor and all terms of degree greater than 12 in the last factor need not be considered.

Hence we look for the coefficient of x^{12} in

$$(1 - x^4)(1 - x^7) \sum_{r=0}^{12} C(r+2, r)x^r$$

$$= (1 - x^4 - x^7 + x^{11}) \sum_{r=0}^{\infty} C(r+2, r)x^r.$$

Problem 1.272. *Find the number of ways of placing 20 similar balls into 6 numbered boxes so that the first box contains any number of balls between and 5 inclusive and the other 5 boxes must contain 2 or more balls each.*

Solution. The integer solution of an equation model is : count the number of integral solutions to $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = 20$ where $1 \le e_1 \le 5$ and $2 \le e_2, e_3, e_4, e_5, e_6$.

First, we will count the solutions where $1 \le e_1$ and $2 \le e_i$ for $i = 2, 3, 4, 5, 6$.

We do this by placing 1 ball in box number one, 2 balls each in the other 5 boxes, and then counting the number of ways to distribute the remaining 9 balls into 6 boxes with unlimited repetition. There are C(14, 9) ways to do this.

But then we wish to discard the number of solutions for which $6 \le e_1$ and $2 \le e_i$ for $i = 2, 3, 4, 5, 6$. There are C(9, 4) of these.

Hence the total number of solutions subject to the constraints is C(14, 9) – C(9, 4).

Now let us solve this problem with generating function.

The generating function we consider is

$$\begin{aligned}(x + x^2 + x^3 + x^4 + x^5)\,(x^2 + x^3 + \ldots\ldots)^5 &= x(1 + x + x^2 + x^3 + x^4)\,[x^2(1 + x + x^2 + \ldots\ldots)]^5 \\ &= x(1 + x + x^2 + x^3 + x^4)(x^{10})(1 + x + x^2 + \ldots\ldots)^5 \\ &= x^{11}(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + \ldots\ldots)^5\end{aligned}$$

We desire to compute the coefficient of x^{20} in this last product but we need only compute the coefficient of x^9 in

$(1 + x + x^2 + x^3 + x^4)\,(1 + x + x^2 + \ldots\ldots)^5$, which can be rewritten as

$$\left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x}\right)^5 = (1 - x^5)\left(\frac{1}{1-x}\right)^6$$

$$= (1 - x^5)\left(\sum_{r=0}^{\infty} C(r+5, r)x^r\right)$$

Thus, the coefficient of x^9 in this last product is C(14, 9) – C(9, 4).

Problem 1.273. *Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$.*

Solution. Since $\dfrac{1}{x-a} = \left(-\dfrac{1}{a}\right)\left(\dfrac{1}{\left(1-\left(\dfrac{x}{a}\right)\right)}\right)$

$$= \left(-\frac{1}{a}\right)\left[1+\left(\frac{x}{a}\right)+\left(\frac{x}{a}\right)^2+\ldots\ldots\right] \text{ for any } a \neq 0.$$

We could solve this problem by finding the coefficient of x^8 in $\dfrac{1}{[(x-3)(x-2)^2]}$ expressed as

$$\left(-\frac{1}{3}\right)\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+......\right]\left(\frac{1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)\right.$$

$$\left.+\binom{-2}{2}\cdot\left(\frac{-x}{2}\right)^2+......\right]$$

An alternative techniques uses the partial fraction decomposition

$$\frac{1}{(x-3)(x-2)^2}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

This decomposition implies that

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3)$$

or $\quad 0.x^2 + 0.x + 1 = 1 = (A + B)x^2 + (-4A + -5B + C)x + (4A + 6B - 3C)$

By comparing coefficients, we find that $A + B = 0$,

$$-4A - 5B + C = 0 \text{ and } 4A + 6B - 3C = 1,$$

Solving these equations yields $A = 1$, $B = -1$, $C = -1$

Hence, $\dfrac{1}{(x-3)(x-2)^2} = \dfrac{1}{x-3}+\dfrac{1}{x-2}-\dfrac{1}{(x-2)^2}$

$$=\left(\frac{-1}{3}\right)\frac{1}{1-\left(\frac{x}{3}\right)}+\left(\frac{1}{2}\right)\frac{1}{\left(1-\left(\frac{x}{2}\right)\right)}+\left(\frac{-1}{4}\right)\frac{1}{\left(1-\left(\frac{x}{2}\right)\right)^2}$$

$$=\left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i+\left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^i$$

$$+\left(\frac{-1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)+\binom{-2}{2}\left(\frac{-x}{2}\right)^2+......\right]$$

The coefficient of x^8 is $\left(\dfrac{-1}{3}\right)\left(\dfrac{1}{3}\right)^8+\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)^8+\left(\dfrac{-1}{4}\right)\dbinom{-2}{8}\left(\dfrac{-1}{2}\right)^8$

$$=-\left[\left(\frac{1}{3}\right)^9+7\left(\frac{1}{2}\right)^{10}\right].$$

Problem 1.274. *In how many ways can a police contain distribute 24 rifle shells to four police officer so that each officer gets at least three shells, but not more than eight ?*

Solution. The choices for the number of shells each officer receives are given by

$$x^3 + x^4 + ... + x^8.$$

There are four officers, so the resulting generating function is

$$f(x) = (x^3 + x^4 + ... + x^8)^4$$

We seek the coefficient of x^{24} in $f(x)$, with

$$(x^3 + x^4 + ... + x^8)^4 = x^{12}(1 + x + x^2 + ... + x^5)^4$$

$$= x^{12}\left(\frac{(1-x^6)}{(1-x)}\right)^4.$$

The answer is the coefficient of x^{12} in

$$(1 - x^6)^4(1 - x)^{-4} = \left[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\binom{4}{3}x^{18}+x^{24}\right]$$

$$\left[\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+......\right]$$

which is $$\left[\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}\right]$$

$$=\left[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}\right] = 125.$$

Problem 1.275. *Verify that for all $n \in Z^+$,* $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$.

Solution. Since $(1 + x)^{2n} = [(1 + x)^n]^2$, by comparison of coefficients the coefficients of x^n in $(1 + x)^{2n}$, which is $\binom{2n}{n}$, must equal the coefficient of x^n in

$$\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+......+\binom{n}{n}x^n\right]^2, \text{ and this is}$$

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+......+\binom{n}{n}\binom{n}{0}$$

with $$\binom{n}{r}=\binom{n}{n-r}, \text{ for all } 0 \le r \le n.$$

Problem Set 1.1

1. Find a generating function for the sequence $A = \{a_r\}_{r=0}^{\infty}$ where

$$a_r = \begin{cases} 1 & \text{if } 0 \le r \le 2 \\ 3 & \text{if } 3 \le r \le 5 \\ 0 & \text{if } r \ge 6 \end{cases}$$

2. Find the generating function for the number of r-combinations of $\{3.a, 5.b, 2.c\}$

3. Write a generating function for a_n, the number of ways of obtaining the sum n when tossing q distinguishable dice. Then find a_{25}.

4. How many ways are there to paint 20 identical rooms in a hotel with 5 colours if there is only enough blue, pink and green paint to paint 3 rooms ?

5. Write the generating function for the sequence $\left\{a_r\right\}_{r=0}^{\infty}$ defined by

(*i*) $a_r = (-1)^r$ (*ii*) $a_r = (-1)^r 3^r$ (*iii*) $a_r = 5^r$

(*iv*) $a_r = r + 1$ (*v*) $a_r = 6(r + 1)$ (*vi*) $a_r = C(r + 3, r)$

(*vii*) $a_r = (r + 3)(r + 2)(r + 1)$

(*viii*) $a_r = \dfrac{(-1)^r (r+2)(r+1)}{2!}$ (*ix*) $a_r = 5^r + (-1)^r 3^r + 8C(r + 3, r)$

(*x*) $a_r = (r + 1) 3^r$ (*xi*) $a_r = (r + 3)(r + 1)3^r$.

6. Build a generating function for a_r = the number of integral solution to the equation $e_1 + e_2 + e_3 = r$ if

(*i*) $0 \le e_i \le 3$ for each i

(*ii*) $2 \le e_i \le 5$ for each i

(*iii*) $0 \le e_i$ for each i

(*iv*) $0 \le e_1 \le 6$ and e_1 is even, $2 < e_2 \le 7$ and e_2 is odd, $5 \le e_3 \le 7$.

7. Write a generating function for a_r when a_r is

(*i*) the number of ways of selecting r balls from 3 red balls, 5 blue balls, 7 white balls.

(*ii*) the number of ways of selecting r coins from an unlimited supply of pennies, nickels, dimes and quarters.

(*iii*) the number of r-combinations formed from n letters where the first letter can appear an even number of times up to 12, the second letter can appear an odd number of times upto 7, the remaining letters can occur an unlimited number of times.

(*iv*) the number of ways of obtaining a total of r upon tossing 50 distinguishable dice.

(*v*) the number of integers between 0 and 999 whose sum of digits is r.

8. Find a generating function for a_r = the number of ways of distributing r similar balls into 7 numbered boxes where the second, third, fourth and fifth boxes are nonempty.

9. (*a*) Find a generating function for the number of ways to distribute 30 balls into 5 numbered boxes where each box contains at least 3 balls and at most 7 balls.

(*b*) Factor out x^{15} from the above functions and interpret this revised generating function combinatorially.

10. Build a generating function for a_r = the number of ways to distribute r similar balls into 5 numbered boxes with

(*i*) at most 3 balls in each box.

(*ii*) 3, 6, or 8 balls in each box.

(*iii*) at least 1 ball in each of the first 3 boxes and at least 3 balls in each of the last 2 boxes.

(*iv*) at most 5 balls in box 1, at most 7 balls in the last 4 boxes.

(*v*) a multiple of 5 balls in box 1, a multiple of 10 balls in box 2, a multiple of 25 balls in box 3, a multiple of 50 balls in box 4, and a multiple of 100 balls in box 5.

11. (*a*) Find a generating function for the number of ways to select 6 non consecutive integers from 1, 2, ... , n.

(*b*) Which coefficient do we want to find in case $n = 20$?

(*c*) Which coefficient do we want for general n ?

12. In $(1 + x^5 + x^9)^{10}$ find

(*i*) the coefficient of x^{23} (*ii*) the coefficient of x^{32}.

13. Find the coefficient of x^{12} in $\dfrac{1 - x^4 - x^7 + x^{11}}{(1-x)^5}$.

14. Find the coefficient of x^{10} in

(*a*) $(1 + x + x^2 + ...)^2$ (*b*) $\dfrac{1}{(1-x)^3}$ (*c*) $\dfrac{1}{(1-x)^5}$

(*d*) $\dfrac{1}{(1+x)^5}$ (*e*) $(x^3 + x^4 + ...)^2$

(*f*) $x^4(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + ... + x^{12})$.

15. Find the coeficient of x^{12} in

(*a*) $\dfrac{x^2}{(1-x)^{10}}$ (*b*) $\dfrac{x^5}{(1-x)^{10}}$ (*c*) $(1 - x)^{20}$ (*d*) $(1 + x)^{20}$

(*e*) $(1 + x)^{-20}$ (*f*) $(1 - 4x)^{-5}$ (*g*) $(1 - 4x)^{15}$ (*h*) $(1 + x^3)^{-4}$

(*i*) $\dfrac{x^2 - 3x}{(1-x)^4}$ (*j*) $(1 - 2x)^{19}$.

16. Find a_r be the number of non negative integral solutions to $x_1 + x_2 + x_3 = r$.

(*a*) Find a_{10} if $0 \le x_i \le 4$ for each i

(*b*) Find a_{50} where $2 \le x_1 \le 50$, $0 \le x_2 \le 50$, $5 \le x_3 \le 25$.

17. Let a_r be the number of ways the sum r can be obtained by tossing 50 distinguishable dice. Write a generating function for the sequence $\left\{a_r\right\}_{r=0}^{\infty}$. Then find the number of ways to obtain the sum of 100, that is, find a_{100}.

18. (*a*) Find the coefficient of x^{50} in $(x^{10} + x^{11} + ... + x^{25})\,(x + x^2 + ... + x^{15})\,(x^{20} + x^{21} + ... + x^{45})$

(*b*) Find the coefficient of x^{25} in $(x^2 + x^3 + x^4 + x^5 + x^6)^7$.

19. Find the coefficient of x^{20} in $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + ...)^5$.

20. Find the coefficient of x^{14} in

(*a*) $(1 + x + x^2 + x^3)^{10}$

(*b*) $(1 + x + x^2 + x^3 + x^4 + ... + x^8)^{10}$

(*c*) $(x^2 + x^3 + x^4 + x^5 + x^6 + x^7)^4$.

21. How many ways are these to place an order for 12 chocolate sundaes if there are 5 types of sundaes, and at most 4 sundaes of one type are allowed ?

22. Use generating functions to find the number of ways to select 10 balls from a large pile of red, white and blue balls if

(*a*) the selection has at least 2 balls of each colour

(*b*) the selection has at most 2 red balls, and

(*c*) the selection has an even number of blue balls.

23. (*a*) Find the generating function for the number of ways to select 10 candy bars from large supplies of six different kinds.

(*b*) Find the generating function for the number of ways to select, with repetitions allowed, r objects from a collection of n distinct objects.

24. Find the generating function for the number of integer solutions to the equation $c_1 + c_2 + c_3 + c_4 = 20$ where $-3 \le c_1$, $-3 \le c_2$, $-5 \le c_3 \le 5$, and $0 \le c_4$.

25. Determine the generating function for the number of integer solutions for the following equation.

(*a*) $c_1 + c_2 + c_3 + c_4 = 20$, $0 \le c_i \le 7$ for all $1 \le i \le 4$

(*b*) $c_1 + c_2 + c_3 + c_4 = 20$, $0 \le c_i$ for all $1 \le i \le 4$, with c_2 and c_3 even

(*c*) $c_1 + c_2 + c_3 + c_4 + c_5 = 30$, $2 \le c_1 \le 4$ and $3 \le c_i \le 8$ for all $2 \le i \le 5$

(*d*) $c_1 + c_2 + c_3 + c_4 + c_5 = 30$, $0 \le c_i$ for all $1 \le i \le 5$, with c_2 even and c_3 odd.

Problem Set 1.2

1. Find the closed form of the gererating function for each of the following sequences :

(*a*) 0, 0, 0, 1, 1, 1, ...

(*b*) 0, 0, 1, 2, 3, 4, ...

(*c*) 3, – 3, 3, – 3, 3, – 3, ...

(*d*) 3, 9, 27, 81, ...

(*e*) 1, – 2, 3, – 4, ...

2. Find the generating function for each of the following sequences if G(x) is the generating function for the sequence $\{a_n^-\}$:

(a) 0, 0, 0, a_0, a_1, a_2, ...

(b) 0, 0, a_2, a_3, ...

(c) a_0, 0 , a_1, 0, a_2, 0, ...

(d) a_0, $3a_1$, $9a_2$, $27a_3$, ...

3. Find the closed form of the generating function for each of the following sequence $\{a_n\}$ where :

(a) $a_n = 3$ (b) $a_n = n + 3$ (c) $a_n = 3^n$ (d) $a_n = n(3 + 5n)$

(e) $a_n = n(n - 1)$.

4. Find the coefficients of

(a) x^{10} in $1/(1 - x)^6$ (b) x^{10} in $1/(1 - 3x)$

(c) x^{12} in $x^2/(1 - x)^{10}$ (d) x^{10} in $(x^3 + x^4 + x^5 + ...)^3$.

5. Find the generating function for the sequence A = $\{a_n\}$

$$\text{where} \qquad a_n = \begin{cases} 1 & \text{if} \quad 0 \le n \le 3 \\ 5 & \text{if} \quad 4 \le n \le 7 \\ 0 & \text{if} \quad n \ge 8 \end{cases}.$$

6. Use generating functions to determine eight identical balls can be distributed among three children if each child receives at least two balls and atmost four balls.

7. Find the number of ways in which 9 balls can be distributed among three distinct boxes so that no box will contain more than 4 balls.

8. Find the generating function for each of the following discrete numeric functions :

(i) 1, – 2, 3, – 4, 5, 6, ... (ii) 1, 2/3, 3/9, 4/27, ...

(iii) 1, 1, 2, 2, 3, 3, 4, 4, ... (iv) $0 * 1, 1 * 2, 2 * 3, 3 * 4, ...$

(v) $0 * 5^0, 1 * 5^1, 2 * 5^2, 3 * 5^3, ...$.

9. If x, y and z are digits, then find the number of possible solutions to the following equations

(i) $x + y + z = 10$ (ii) $x - y + z = 17$

(iii) $x + 2y + 3z = 8$ (iv) $x - 2y + 2z = 37$.

10. If x, y and z are positive integers, then find the number of possible solutions to the following equations

(i) $x + y + z = 16$ (ii) $x - y + z = -7$

(iii) $x + 2y + 3z = 30$ (iv) $x - 2y + 2z = 0$.

11. If x, y and z are non-negative integers, then find the number of possible solutions to the following

(i) $x + y + z = 100$ (ii) $x - y + z = 27$

(iii) $x + 2y + 3z = 10$ (iv) $x - 2y + 2z = 71$.

12. In how many ways can we select seven non-consecutive integers from { 1, 2, 3, ... 50} ?

13. Determine the sequence generated by each of the following generating functions.

(*a*) $f(x) = (2x - 3)^3$ (*b*) $f(x) = x^4/(1 - x)$

(*b*) $f(x) = x^3/(1 - x^2)$ (*d*) $f(x) = 1/(1 + 3x)$

(*e*) $f(x) = 1/(3 - x)$ (*f*) $f(x) = 1/(1 - x) + 3x^2 - 11$.

14. (*a*) Find the coefficient of x^7 in $(1 + x + x^2 + x^3 + ...)^{15}$

(*b*) Find the coefficient of x^7 in $(1 + x + x^2 + x^3 + ...)^n$ for $n \in Z^+$.

15. Find the coefficient of x^{20} in $(x^2 + x^3 + x^4 + x^5 + x^6)^5$.

16. Find the coefficient of x^{15} in each of the following

(*a*) $x^3(1 - 2x)^{10}$ (*b*) $(x^3 - 5x)/(1 - x)^3$ (*c*) $(1 + x)^4/(1 - x)^4$.

17. Determine the constant (that is, the coefficient of x^0) in $(3x^2 - (2/x))^{15}$.

18. Find the coefficient of x^{50} in $(x^7 + x^8 + x^9 + ...)^6$.

19. Find the generating functions for the following sequences

(*a*) $\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, ... \binom{8}{8}$

(*b*) $\binom{8}{1}, 2\binom{8}{2}, 3\binom{8}{3}, ... 8\binom{8}{8}$

(*c*) 1, – 1, 1, – 1, 1, – 1, ...

(*d*) 0, 0, 0, 6, – 6, 6, – 6, 6, ...

(*e*) 1, 0, 1, 0, 1, 0, 1, ...

(*f*) $0, 0, 1, a, a^2, a^3, ..., a \neq 0$.

20. Find the partitions of 7.

21. In $f(x) = [1/(1 - x)]\,[1/(1 - x^2)]\,[1/(1 - x^3)]$, the coefficient of x^6 is 7. Interpret this result in terms of partitions of 6.

22. Find the generating function for the number of integer solutions of

(*a*) $2w + 3x + 5y + 7z = n, 0 \leq w, x, y, z$

(*b*) $2w + 3x + 5y + 7z = n, 0 \leq w, 4 \leq x, y ; 5 \leq z$.

23. Find the generating function for the number of partitions of the non negative integer n into summands where

(*a*) each summand must appear an even number of times ;

(*b*) each summand must be even.

24. Find the exponential generating function for the sequence 0! 1! 2! 3! ......

25. Find the exponential generating function for each of the following sequences.

(*a*) 1, – 1, 1, – 1, 1, – 1, ...

(*b*) $1, 2, 2^2, 2^3, 2^4, ...$

(*c*) $1, -a, a^2, -a^3, a^4, ...\ a \in R$.

(*d*) $1, a^2, a^4, a^6, \ldots\ a \in \mathbb{R}$

(*e*) $a, a^3, a^5, a^7, \ldots\ a \in \mathbb{R}$

(*f*) $0, 1, 2(2), 3(2^2), 4(2^3), \ldots$.

26. Find the generating function for the sequence $a_0, a_1, a_2, \ldots\ldots$, where $a_n = \sum_{i=0}^{n}\left(\frac{1}{i!}\right)$, $n \in \mathbb{N}$.

27. Let $f(x)$ be the generating function for the sequence $a_0, a_1, a_2, \ldots\ldots$. For what sequence is $(1 - x) f(x)$ the generating function ?

28. Find the generating function for each of the following sequences

(*a*) 7, 8, 9, 10, ...

(*b*) $1, a, a^2, a^3, a^4, \ldots\ldots\ a \in \mathbb{R}$

(*c*) $1, (1 + a), (1 + a)^2, (1 + a)^3, \ldots\ldots\ a \in \mathbb{R}$

(*d*) $2, 1 + a, 1 + a^2, 1 + a^3, \ldots\ldots\ a \in \mathbb{R}$.

29. Find the coefficient of x^{83} in

$$f(x) = (x^5 + x^8 + x^{11} + x^{14} + x^{17})^{10}.$$

30. (*a*) For what sequence of numbers is

$g(x) = (1 - 2x)^{-5/2}$ the exponential generating function ?

(*b*) Find a and b so that $(1 - ax)^b$ is the exponential genearting function for the sequence 1, 7, 7.11, 7.11.15, ...

Problem Set 1.3

1. Let G denote the set of all 2 × 2 matrices such that the first row is [1 *m*], where *m* is an integer, and the second row is [0 1]. Show that G is an infinite cyclic group under matrix multiplication. Find the generator of this group.

2. Prove that a subgroup of a cyclic group is cyclic.

3. Given a finite set X and a group G of permutation of X, prove that the distinct orbits with respect to G constitute a partition of X.

4. Show that the Burnside-Frobenius theorem holds for X = {*a*, *b*, *c*, *d*} and G = $\{g_1, g_2, g_3, g_4\}$, where g_1 maps each element into itself ; g_2 maps *a* and *b* into each other and *c* and *d* into each other, g_3 maps *a* and *c* into each other and *b* and *d* into each other, g_4 maps *a* and *d* into each other and *b* and *c* into each other.

5. From the Burnside-Frobenius theorem, obtain the number of ways of seating *n* people around a circular table.

6. Use the Burnside-Frobenius theorem to find the number of distinguishable colorings, with respect to the symmetry group of the square, of a 3 × 3 chessboard if 2 cells must be colored black and the others white.

7. Show that if *f* and *g* are permutations, *fg* and *gf* are of the same type.

8. Prove that conjugate permutations have the same number of fixed points.

9. If A = {α, β, γ, δ} and H = $\{h_1, h_2, h_3, h_4\}$ is a group of permutations of A, where

$h_1 = (\alpha)(\beta)(\gamma)(\delta)$ $\qquad h_3 = (\alpha\gamma)(\beta\delta)$

$h_2 = (\alpha\beta)(\gamma\delta)$ $\qquad h_4 = (\alpha\delta)(\beta\gamma)$

find the cycle index of H.

10. Find the cycle index of the group of face permutations induced by the rotational symmetries of the cube.

11. Find the cycle index of the group of vertex permutations induced by the rotational symmetries of the cube.

12. Display the complete group of symmetries of a regular $(2m + 1)$-gon.

13. Display the complete group of symmetries of a regular $2m$-gon.

14. Obtain the cycle index of the dihedral group H_{2n}.

15. A regular tetrahedron has 4 vertices, 4 faces (congruent equilateral triangles), and 6 edges. Find the cycle index of the group of permutations of the 4 vertices (or 4 faces) induced by the rotational symmetries of the regular tetrahedron.

16. Let G be the set of all 3×3 matrices A that have [1 a b] as the first row, [0 1 c] as the second row and [0 0 1] as the third row, the numbers a, b and c are elements of the set [0, 1, 2]. If scalar addition and multiplication are modulo 3, show that G is a group under ordinary matrix multiplication. Determine the cycle index of G.

17. A regular polytope (a solid in which all faces are congruent polygons and each vertex is incident with the same number of faces) with 12 vertices, 20 faces (congruent equilateral triangles) and 30 edges is called a regular icosahedron. Identify the rotational symmetries of this solid, and obtain the cycle indices of the groups of (*a*) vertex permutations and (*b*) face permutations.

18. With respect to the β rotational symmetries of a cube, in how many ways can the faces be painted red, blue, or green, if each color must be used at least once ?

19. Find the number of ways, under the rotational group of coloring the vertices and faces of a regular octahedron so that 4 vertices are red, 2 vertices are blue 4 faces are green, and 4 faces are yellow.

20. Find the number of inequivalent ways of seating 2 men, 2 women, and 1 child at a round dining table.

21. If $G = < x >$ is a cyclic group of order 12, list the orders of x^k for $k = 0, 1, 2, ..., 11$.

22. With respect to the group of rotational symmetries of the cube, in how many ways can 6 edges be colored red and the remaining 6 blue ?

23. Prove that, for every integer r, $r^2(r^2 + 11) \equiv (\text{mod } 12)$.

24. Find the number of (rotationally) distinct ways of painting the faces of a regular dodecahedron in 3 or fewer colors.

25. How many distinct 7-horse merry-to-rounds are there with 2 red horses, 3 white horses, and 2 blue horses ?

26. Find the number of distinguishable ways of coloring the cells of a 3×3 chessboard so that 2 cells are red, 4 cells are white, and 3 cells are blue.

27. Show that the stabilizer G_x is a subgroup of G.

28. Prove that $r^8 + r^4 + 2r^2 + 4r$ is divisible by 8, for all positive integers r.

29. Express the permutation (1 2 3 4)(5 6 7)(1 6 7 2 9)(3 4) as a product of disjoint cycles.

30. Find the cycle index of the group of edge permutations induced by the rotational symmetries of the cube.

31. A complex number θ is a primitive n^{th} root of unity if $\theta^n = 1$, but $\theta^k \neq 1$ for $k = 1, 2, \ldots, n - 1$. Count the primitive n^{th} roots of unity.

32. Use the Burnside-Frobenius theorem to find the number of distinguishable ways to colorings the sides of a square using 2 colors.

33. If $x = (1\ 3\ 5\ 7)(2\ 4\ 6)$ and $y = (1\ 2\ 3\ 4\ 5)$, find the order of xy.

34. Find the number of permutations of type [3 1 0 0 0].

35. Evaluate the cycle indices of the dihedral groups (*a*) H_{12} and (*b*) H_{14}.

36. If *f* is the permutation that maps 1, 2, 3, 4, 5, 6, 7, 8 and 9 into 9, 8, 5, 4, 1, 6, 3, 2 and 7. Write the disjoint-cycle representation of *f*.

37. Find the number of ways, under the rotational group, of coloring a regular tetrahedron so that 2 vertices are red, 2 vertices are blue, 2 faces are green, 2 faces are yellow, 3 edges are black, 3 edges are white.

38. Find the number of (rotationally) distinct ways of painting the faces of a regular icosahedron so that 4 faces are red and the other faces are blue.

39. If the vertices of a square are painted in 3 or fewer colors, in how many patterns will 2 vertices be of 1 color and 2 of another color.

40. Find the number of distinguishable necklaces with 10 stones of at most 2 colors.

41. If $x = (a\ b\ c\ d)$ and $y = (b\ d)$, express x^2, x^3, x^4, xy and x^2y as products of disjoint cycles.

42. Find the number of (rotationally) distinct ways of coloring the vertices of a cube using at most 3 colors.

43. If $x = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$ find the groups generated by x^2 and x^4.

44. Find the number of (rotationally) distinct ways of painting the faces of a cube using 6 colors so that each face is of a different color.

Problem Set 1.4

1. Five salesmen of B, C, D and E of a company are considered for a three member trade delegation to represent the company in an international trade conference ; construct the sample space and find the probability that

(*i*) A is selected (*ii*) A is not selected and (*iii*) Either A or B (not both) is selected.

2. If two dice are thrown, what is the probability that the sum is

(*i*) greater than 8 and (*ii*) neither 7 nor 11 ?

3. An integer is chosen at random from two hundred digits. What is the probability that the integer is divisible by 6 or 8 ?

4. Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers ?

5. A problem in statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively.

6. A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9th one examined is the last defective ?

7. A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelop just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA ?

8. From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred 3 are white and 1 is black ?

9. A speaks truth 4 out of 5 times. A die is tossed. He reports that there is a six. What is the chance that actually there was six ?

10. A bag contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made such that (*i*) coins are replaced before the second trial (*ii*) the coins are not replaced before the second trial. Find the probability that the first drawing will give 4 gold and the second 4 silver coins.

11. Sixty percent of the employees of the XYZ corporation are college graduates of these, ten percent are sales of the employees who did not graduate from college, eighty percent are in sales. What is the probability that

(*i*) an employee selected at random is in sales ?

(*ii*) an employee selected at random is neither in sales nor a college graduate ?

12. A manager has two assistants and the bases his decision on information supplied independently by each one of them. The probability that he makes a mistake in his thinking is 0.005. The probability that an assistant gives wrong information is 0.3. Assuming that the mistakes made by the manager are independent of the information given by the assistants, find the probability that he reaches a wrong decision.

13. A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.

14. The probability that a student passes a physics test is $\frac{2}{3}$ and the probability that he passes both a physics test and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes the English test ?

15. A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced, a second drawing is then made. What is the probability that

(*i*) the first number drawn is even and the second odd ?

(*ii*) the first number is odd and the second even ?

How will your results in (*i*) and (*ii*) be effected if the first counter drawn is not replaced ?

16. A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.

17. A rod of length '*a*' is broken into three parts at random what is the probability that a triangle can be formed from these parts ?

Answers 1.1

1. $1 + x + x^2 + 3x^3 + 3x^4 + 3x^5$ **2.** $(1 + x + x^2 + x^3)(1 + x + ... + x^5)(1 + x + x^2)$.

3. $A(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^9$

$= x^9(1 + x + ... + x^5)^9 = x^9(1 - x^6)^9(1 - x)^{-9}$

the coefficient of x^{16} in $(1 - x^6)^9 (1 - x)^{-9}$,

$a_{25} = C(24, 16) - 9C(18, 10) + C(9, 2) C(12, 4)$.

4. The co-efficient of x^{20} in $(1 + x + x^2 + x^3)^3 \left(\sum_{r=0}^{\infty} x^r\right)^2$ is $C(24, 20) - 3C(20, 16) + 3C(16, 12) - C(12, 8)$.

6. (*i*) $(1 + x + x^2 + x^3)^3$ (*ii*) $(x^2 + x^3 + x^4 + x^5)^3$

(*iii*) $(x + x^2 + ...)^3$ (*iv*) $(1 + x^2 + x^4 + x^6)(x^3 + x^5 + x^7)(x^5 + x^6 + x^7)$.

7. (*i*) $(1 + x + x^2 + x^3)(1 + x + ... + x^5)(1 + x + ... + x^7)$

(*ii*) $(1 + x + ... + x^n + ...)^4$

(*iii*) $(1 + x^2 + ... + x^{12})(x + x^3 + x^5 + x^7)(1 + x + ... + x^n ......)^{n-2}$

(*iv*) $(x + ... + x^6)^{50}$, a_{100}(*v*) $(1 + x + ... + x^9)^3$.

8. $(1 + x + ...)^3 (x + x^2 + ...)^4$.

11. (*a*) $(1 + x + ...)^2 (x + x^2 + ...)^5$, think of the 6 integers chosen as dividers for 7 boxes where the first and last box can be empty and the other 5 boxes are non empty.

(*b*) Coefficient of x^{14}

(*c*) Coefficient of x^{n-6}.

12. (*i*) To solve $e_1 + e_2 + ... + e_{10} = 23$, where $e_i = 0, 5, 9$. This can be done only with one 5, two 9's and seven 0's. Hence the coefficient is $\frac{10!}{1!2!7!}$.

(*ii*) 32 can be obtained only with three 9's, one 5, and 60's. Thus the coefficient of x^{32} is $\frac{10!}{3!1!6!}$.

14. (*a*) $\frac{1}{(1-x)^2} = \sum_{r=0}^{\infty} C(r+1, r)x^r = \sum_{r=0}^{\infty} (r+1)x^r$, coefficient of x^{10} is 11.

(*b*) $\dfrac{1}{(1-x)^3} = \sum_{r=0}^{\infty} C(r+2, r)x^r = \sum_{r=0}^{\infty} \dfrac{(r+2)(r+1)}{2} x^r$, coefficient of x^{10} is (12)(11)/2.

(*c*) C(14, 10).

(*d*) $(-1)^{10}$ C(14, 10) = C(14, 10).

(*e*) $[x^3 (1 + x + x^2 + ...)]^2 = x^6 \left[\dfrac{1}{(1-x)^2}\right]$, coefficient of x^{10} is the coefficient of x^4 in $\dfrac{1}{(1-x)^2}$

$= \sum_{r=0}^{\infty} C(r+1, r)x^r$, coefficient = 5.

(*f*) C(8, 6) – C(4, 2) – C(3, 1).

15. (*a*) C(19, 10) (*b*) C(16, 7) (*c*) C(20, 12) (*d*) C(20, 12)

(*e*) C(31, 12) (*g*) 4^{12} C(15, 12) (*j*) $(-2)^{12}$ C(19, 12).

16. (*a*) C(12, 10) – 3C(7, 5) + 3 (*b*) C(45, 43) – C(24, 22).

17. $(x + x^2 + x^3 + x^4 + x^5 + x^6)^{50} = x^{50} (1 - x^6)^{50} \dfrac{1}{(1-x)^{50}}$, coefficeint of x^{100} is C(99, 50)

– C(44 + 49, 44) C(50, 1) + C(49 + 38, 38) C(50, 2) – C(49 + 32, 32) C(50, 3) ....

18. (*a*) C(21, 19) – C(6, 4) – C(5, 3) (*b*) C(17, 11) – 7C(12, 6) + C(7, 2) C(7, 1).

19. C(14, 9) – C(9, 4).

20. (*a*) $(1 + x + x^2 + x^3)^{10} = \left(\dfrac{1-x^4}{1-x}\right)^{10} = \dfrac{(1-x^4)^{10}}{(1-x)^{10}} = (1 - x^4)^{10} \sum_{r=0}^{\infty} C(r+9, r)x^r$

$= [1 - C(10, 1)x^4 + C(10, 2)x^8 - C(10, 3)x^{12} + ... x^{40}]$

$\sum_{r=0}^{\infty} C(r+9, r)x^r$,

Coefficient of x^{14} is C(23, 9) – C(10, 1) C(19, 10) + C(10, 2) C(15, 6) – C(10, 3) C(11, 2).

(*b*) C(14, 9) – C(9, 4).

21. C(16, 12) – C(5, 1) C(11, 7) + C(5, 2) C(6, 2).

22. (*a*) C(6, 4) (*b*) C(12, 10) – C(9, 7).

23. (*a*) The coefficient of x^{10} in $(1 + x + x^2 + x^3 + ...)^6$.

(*b*) The coefficient of x^r in $(1 + x + x^2 + x^3 + ...)^n$.

24. The answer is the coefficient of x^{31} in the generating function

$$(1 + x + x^2 + x^3 + ...)^3(1 + x + x^2 + ... + x^{10}).$$

25. (*a*) The coefficient of x^{20} in $(1 + x + x^2 + ... + x^7)^4$.

(*b*) The coefficient of x^{20} in $(1 + x + x^2 + ... + x^{20})^2 (1 + x^2 + x^4 + ... + x^{20})^2$
or $(1 + x + x^2 + ...)^2 (1 + x^2 + x^4 + ...)^2$.
(*c*) The coefficient of x^{30} in $(x^2 + x^3 + x^4)(x^3 + x^4 + ... + x^8)^4$.
(*d*) The coefficient of x^{30} in $(1 + x + x^2 + ... + x^{30})^3 (1 + x^2 + x^4 + ... + x^{30})$
$(x + x^3 + x^5 + ... + x^{29})$
or $(1 + x + x^2 + ...)^3 (1 + x^2 + x^4 + ...)(x + x^3 + x^5 + ...)$.

Answers 1.2

1. (*a*) $\frac{x^3}{1-x}$ (*b*) $\frac{x^2}{(1-x)^2}$ (*c*) $\frac{3}{1+x}$ (*d*) $\frac{3}{1-3x}$ (*e*) $\frac{1}{(1-x)^2}$.

2. (*a*) x^3 G(x) (*b*) G(x) – $a_0 - a_1 x$ (*c*) G(x^2) (*d*) G($3x$).

3. (*a*) $\frac{3}{1-x}$ (*b*) $\frac{x}{(1-x)^2} + \frac{3}{1-x}$ (*c*) $\frac{1}{1-3x}$ (*d*) $\frac{3x}{(1-x)^2} + \frac{5x(1+x)}{(1-x)^3}$ (*e*) $\frac{2x^2}{(1-x)^3}$.

4. (*a*) C(15, 10) (*b*) 3^{10} (*c*) C(19, 10) (*d*) 3.

5. $1 + x + x^2 + x^3 + 5x^4 + 5x^5 + 5x^6 + 5x^7$.

6. 6 **7.** 10 **14.** (*a*) $\binom{21}{7}$ (*b*) $\binom{n+6}{7}$

15. $\binom{14}{10} - 5\binom{9}{5} + \binom{5}{2}$

16. (*a*) 0 (*b*) $\binom{14}{12} - 5\binom{16}{14}$ (*c*) $\binom{18}{15} + 4\binom{17}{14} + 6\binom{16}{13} + 4\binom{15}{12} + \binom{14}{11}$.

19. (*a*) $(1 + x)^8$ (*b*) $8(1 + x)^7$ (*c*) $(1 + x)^{-1}$ (*d*) $\frac{6x^3}{(1+x)}$ (*e*) $(1 - x^2)^{-1}$ (*f*) $\frac{x^2}{(1-ax)}$.

20. 7 : 6 + 1 ; 5 + 2 ; 5 + 1 + 1 ; 4 + 3 ; 4 + 2 + 1 ; 4 + 1 + 1 + 1 ; 3 + 3 + 1 ; 3 + 2 + 2 ;
3 + 2 + 1 + 1 ; 3 + 1 + 1 + 1 + 1 ; 2 + 2 + 2 + 1 ; 2 + 2 + 1 + 1 + 1 ; 2 + 1 + 1 + 1 + 1 + 1 ;
1 + 1 + 1 + 1 + 1 + 1 + 1.

21. The number of partitions of 6 into 1's, 2's and 3's is 7.

23. (*a*) and (*b*)

$$(1 + x^2 + x^4 + x^6 + ...)(1 + x^4 + x^8 + ...)(1 + x^6 + x^{12} + ...) ...... = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i}}.$$

24. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = (0\,!)\frac{x^0}{0!} + (1\,!)\frac{x^1}{1!} + (2\,!)\frac{x^2}{2!} + (3\,!)\frac{x^3}{3!} + ...$

25. (*a*) e^{-x} (*b*) e^{2x} (*c*) e^{-ax} (*d*) $e^{a^2 x}$ (*e*) $ae^{a^2 x}$ (*f*) xe^{2x}. **26.** $f(x) = \left[\frac{e^x}{(1-x)}\right]$.

27. $a_0, a_1, -a_0, a_2, -a_1, a_3, -a_2, ...$

28. (*a*) $\dfrac{6}{(1-x)} + \dfrac{1}{(1-x)^2}$ (*b*) $\dfrac{1}{(1-ax)}$

(*c*) $\dfrac{1}{[1-(1+a)x]}$ (*d*) $\dfrac{1}{\left[(1-x)+\dfrac{1}{(1-ax)}\right]}$.

30. (*a*) 1, 5, (5)(7), (5)(7)(9), (5)(7)(9)(11), ... (*b*) $a = 4, b = -\dfrac{7}{4}$.

Answers 1.3

5. $k = \dfrac{n!}{n} = (n-1)!$ **6.** $k = 8$

9. $Z(H\,;x_1, x_2, x_3, x_4) = \dfrac{1}{4}(x_1^4 + 3x_2^2)$

12. $Z(G\,;x_1, x_2, ...\, x_6) = \dfrac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_2^3 + 6x_1^2x_4 + 8x_3^2)$

11. $Z(G\,;x_1, x_2, ..., x_8) = \dfrac{1}{24}(x_1^8 + 9x_2^4 + 8x_1^2x_3^2 + 6x_4^2)$

12. $\{e, f, f^2, ..., f^{2m}, h, hf, hf^2, ..., hf^{2m}\}$ **13.** $\{e, f, f^2, ...\, f^{2m-1}, g, gf, gf^2, gf^{2m-1}\}$

14. $Z(H_{2n}\,;x_1, x_2, ..., x_n\} = \begin{cases} \dfrac{1}{2n}(U+V) & n \text{ even} \\ \dfrac{1}{2n}(U+V') & n \text{ odd} \end{cases}$

15. $Z(G\,;x_1, x_2, x_3, x_4) = \dfrac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2)$

16. $Z(G\,;x_1, x_2, ...\, x_{27}) = \dfrac{1}{27}(x_1^{27} + 26x_3^9)$

17. (*a*) $Z(G\,;x_1, x_2, ...\, x_{12}) = \dfrac{1}{60}(x_1^{12} + 15x_2^6 + 20x_3^4 + 24x_1^2x_5^2)$

(*b*) $Z(G\,;x_1, x_2, ...\, x_{20}) = \dfrac{1}{60}(x_1^{20} + 15x_2^{10} + 20x_1^2x_3^6 + 24x_5^4)$

18. 30

19. 7

20. 21

21. 1, 12, 6, 4, 3, 12, 2, 12, 3, 4, 6, 12

22. 48

24. 9099

25. 30

26. 174

29. (1 7 3)(2 9)(5 6)

30. $Z(G\,;\,x_1, x_2, ... x_{12}) = \frac{1}{24}\,(x_1^{12} + 6x_1^2x_2^5 + 3x_2^6 + 8x_3^4 + 6x_4^3)$

31. $\phi(n)$

32. 21

33. 12

34. 10

35. (*a*) $\frac{1}{12}\,(x_1^6 + 3x_1^2x_2^2 + 4x_2^3 + 2x_3^2 + 2x_6)$ (*b*) $\frac{1}{14}\,(x_1^7 + 7x_1x_2^3 + 6x_7)$

36. (1 9 7 3 5)(2 8)(4)(6)

37. 4

38. 96

39. 6

40. 78

41. (*a c*)(*b d*), (*a d c b*), (*a*)(*b*)(*c*)(*d*), (*ad*)(*bc*), (*ac*)(*b*)(*d*)

42. 333

43. $\{x^2, x^4, x^6, x^8\}$, $\{x^4, x^8\}$

44. 30.

Answers 1.4

1. (*i*) $\frac{3}{5}$ (*ii*) $\frac{2}{5}$ (*iii*) $\frac{3}{5}$

2. (*i*) $\frac{5}{18}$ (*ii*) $\frac{7}{9}$

3. $\frac{1}{4}$

4. 35%

5. $\frac{29}{32}$

6. $\frac{8}{195}$

7. $\frac{4}{11}$

8. 0.14

9. $\frac{4}{9}$

10. (*i*) $\frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$ (*ii*) $\frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{14}C_4}$

11. (*i*) 0.38 (*ii*) 0.08

12. 0.51245

13. 0.5275

14. $\frac{4}{9}$

15. (*i*) $\frac{9}{34}$ (*ii*) $\frac{72}{289}$; $\frac{9}{34}$

16. $\frac{15}{16}$

17. $\frac{1}{4}$

Unit-2

Recurrence Relations

2.1. THE FIRST-ORDER LINEAR RECURRENCE RELATION

Suppose n is a natural number, we define 2^n as

$$2^n = \underbrace{2.2.2.......2}_{n2's}$$

or $\quad 2' = 2$, and for $k \geq 1$, $2^{k+1} = 2.2^k$

We write $\quad 0\,! = 1$ and for $k \geq 0$, $(k + 1)\,! = (k + 1)\,k\,!$

A sequence is a function whose domain is some infinite set of integers (often N) and whose range is a set of real numbers.

The sequence which is the function $f : \mathrm{N} \to \mathrm{R}$ defined by $f(n) = n^2 = 1, 4, 9, 16, ...$ ...(1)

The numbers in the list are called the **terms of the sequence,** the terms are denoted $a_0, a_1, a_2, ...$.

The sequence 2, 4, 8, 16, ... can be defined recursively like : $a_1 = 2$ and for $f \geq 1$, $a_{k+1} = 2a_k$ setting $k = 1, 2, 3, ...$ and $a_1 = 2$ in (1) gives 2, 4, 8, ...

The equation $a_{k+1} = 2a_k$ in (1), which defines one member of the sequence in terms of a previous one, is called a **recurrence relation.** The equation $a_1 = 2$ is called an **initial condition.**

For example, we write

$a_0 = 2$ and for $k \geq 0$, $a_{k+1} = 2a_k$ or we say $a_1 = 2$ and for $k \geq 2$, $a_k = 2a_{k-1}$

In (1), for instance, $a_n = 2^n$, we say that $a_n = 2^n$ **is the solution** to the recurrence relation.

A sequence of numbers like 50, 64, 78, 92, ... where each term is determined by adding the same fixed number to the previous one, is called an **arithmetic sequence.** The fixed number is called **the common difference** of the sequence.

The arithmetic sequence with first term a and common difference d is the sequence defined by

$$a_1 = a \text{ and } k \geq 1,\ a_{k+1} = a_k + d$$

The general arithmetic sequence, takes the form

$$a,\ a + d,\ a + 2d,\ ......$$

and for $n \geq 1$, the nth term of the sequence is $a_n = a + (n - 1)\,d$.

The sum of n terms of the arithmetic sequence with first term a and common difference d is

$$\mathrm{S} = \frac{n}{2}\,[2a + (n - 1)\,d]$$

The geometric sequence with first term a and common ratio r is the sequence defined by

$$a_1 = a \text{ and for } k \geq 1,\ a_{k+1} = ra_k$$

The general geometric sequence, this has the form

$$a,\ ar,\ ar^2,\ ar^3,\ ....$$

the nth term being $a_n = ar^{n-1}$, the sum S of n terms $(r \neq 1)$, $\mathrm{S} = a(1 - r^n)/(1 - r)$.

The Fibonacci sequence,

$$f_1 = 1, f_2 = 1 \text{ and for } k \geq 2, f_{k+1} = f_k + f_{k-1}$$

the nth term of the Fibonacci sequence is the closed integer to the number

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$$

For example, $a_1 = 1$ and for $k > 1$,

$$a_k = \begin{cases} 1 + a_{k/2} & \text{if } k \text{ is even} \\ 1 + a_{3k-1} & \text{if } k \text{ is odd.} \end{cases}$$

A linear homogeneous recurrence relation of degree k **with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots\ldots, c_k$ are real numbes, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** since the right-hand side is a sum of multiples of the previous terms of the sequence. The recurrence relation is **homogeneous** since no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all **constants,** rather than functions that depend on n. The degree is k because a_n is expressed in terms of the previous k terms of the sequene. A consequence of the principle of Mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and k initial conditions

$$a_0 = c_0, a_1 = c_1, \ldots\ldots. a_{k-1} = c_{k-1}.$$

Problem 2.1. *A person invests Rs. 10,000/- @ 12% interest compounded annually. How much will be there at the end of 15 years.*

Solution. Let A_n represents the amount at the end of n years.

So at the end of $n - 1$ years, the amount is A_{n-1}.

Since the amount after n years equals the amount after $n - 1$ years plus interest for the nth year.

Thus the sequence $\{A_n\}$ satisfies the recurrence relation

$$A_n = A_{n-1} + (0.12)\, A_{n-1} = (1.12)\, A_{n-1}, n \geq 1.$$

With initial condition $A_0 = 10{,}000$.

The recurrence relation with the initial condition allow us to compute the value of A_n for any n.

For example,

$$A_1 = (1.12)\, A_0$$
$$A_2 = (1.12)\, A_1 = (1.12)^2\, A_0$$
$$A_3 = (1.12)\, A_2 = (1.12)^3\, A_0$$
$$\vdots$$
$$A_n = (1.12)^n\, A_0$$

which is an explicit formula and the required amount can be derived from the formula by putting $n = 15$.

So, $A_{15} = (1.12)^{15}\, (10000)$.

Problem 2.2. *Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years ?*

Solution. To solve this problem. Let P_n denote the amount in the account after n years.

Since the amount in the account after n years equals the amount in the account after $n - 1$ years plus interest for the nth year, we see that sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11\ P_{n-1} = (1.11)\ P_{n-1}$$

This initial condition is $P_0 = 10{,}000$.

We can use an interative approach to find a formula for P_n.

Note that $P_1 = (1.11)\ P_0$

$P_2 = (1.11)\ P_1 = (1.11)^2\ P_0$

$P_3 = (1.11)\ P_2 = (1.11)^3\ P_0$

....

....

$P_n = (1.11)\ P_{n-1} = (1.11)^n\ P_0$

when we insert the initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^n\ 10{,}000$ is obtained.

We can use mathematical induction to establish its validity. That the formula is valid for $n = 0$ is a consequence of the initial condition.

Now assume that $P_n = (1.11)^n\ 10{,}000$.

Then, from the recurrence relation and the induction hypothesis.

$P_{n+1} = (1.11)\ P_n = (1.11)\ (1.11)^n\ 10{,}000 = (1.11)^{n+1}\ 10{,}000$.

This shows that the explicit formula for P_n is valid.

Inserting $n = 30$ into the formula $P_n = (1.11)^n\ 10{,}000$

Shows that after 30 years the account contains

$P_{30} = (1.11)^{30}\ 10{,}000 = \$228{,}922.97$.

Problem 2.3. *A computer system considers a string of decimal digit a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n the number of valid n-digit codewords. Find a recurrence relation for a_n.*

Solution. Note that $a_1 = 9$ since there are 10 one-digit strings, and only one, namely, the string 0, is not valid.

A recurrence relation can be derived for this sequence by considering how a valid n-digit string can be obtained from strings of $n - 1$ digits. There are two ways to form a valid string with n digits from a string with one fewer digit.

First, a valid string of n digits can obtained by appending a valid string of $n - 1$ digits with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with n digits can be formed in this manner in $9a_{n-1}$ ways.

Second, a valid string of n digits can be obtained by appending a 0 to a string of length $n - 1$ that is not valid.

This produces a string with an even number of 10 digits since the invalid string of length $n - 1$ has an odd number of 0 digits.

The number of ways that this can be done equals the number of invalid ($n-1$)-digit strings.

Since these are 10^{n-1} strings of length $n-1$, and a_{n-1} are valid, these are $10^{n-1} - a_{n-1}$ valid n-digit strings obtained by appending an invalid string of length $n-1$ with a0.

Since all valid strings of length n are produced in one of these two ways, it follows that these are

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$
$$= 8a_{n-1} + 10^{n-1}$$

valid strings of length n.

Problem 2.4. *Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4,$ ...... and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3 ?*

Solution. We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2 \quad \text{and} \quad a_3 = a_2 - a_1 = 2 - 5 = -3$$

We can find a_4, a_5 and each successive term in a similar way.

Problem 2.5. *Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4,$ ...... where $a_n = 3n$ for every non negative integer n. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.*

Solution. Suppose that $a_n = 3n$ for every non negative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2}$

$$= 2[3(n-1)] - 3(n-2) = 3n = a_n.$$

Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every non-negative integer n.

Note that $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$.

Since $2a_1 - a_0 = 2.2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every non-negative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2.5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation.

Problem 2.6. *Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five ?*

Solution. Let a_n denote the number of bit strings of length n that do not have two consecutive 0s.

To obtain a recurrence relation for $\{a_n\}$, note that by the sum rule, the number of bit strings of length n that do not have two consecutive 0s equals the number of such bit strings ending with a 0 plus the number of such bit strings ending what a 1.

We will assume that $n \geq 3$, so that the bit string has at least three bits.

The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n-1$ with no two consecutive 0s with a 1 added at the end.

Consequently, there are a_{n-1} such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their ($n-1$)st bit, otherwise they would end with a pair of 0s.

It follows that the bit strings of length n ending with a 0 that have no two consecutive 0s are precisely the bit strings of length $n-2$ with no two consecutive 0s with 10 added at the end. Consequently, there are a_{n-2} such bit strings.

We conclude that, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

The initial conditions are $a_1 = 2$, since both it strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, since the valid bit strings of length two are 01, 10 and 11.

To obtain a_5, we use the recurrence relation three times to find that

$$a_3 = a_2 + a_1 = 3 + 2 = 5,$$
$$a_4 = a_3 + a_2 = 5 + 3 = 8,$$
$$a_5 = a_4 + a_3 = 8 + 5 = 13.$$

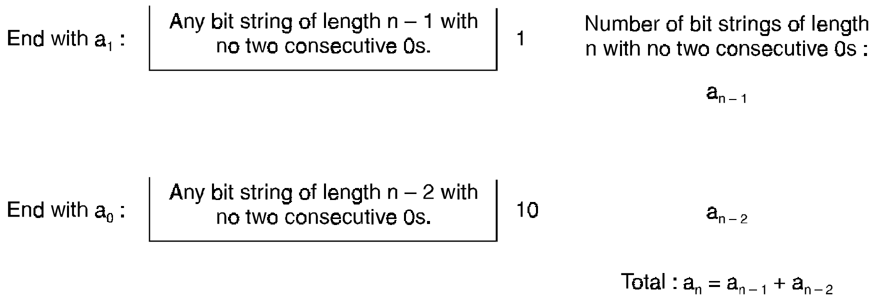


Fig. 4.5. Counting Bit Strings of length n with no two consecutive 0s.

Problem 2.7. *Find a recurrence relation for C_n, the number of ways to parenthesize the product of n + 1 numbers, $x_0 . x_1 . x_2 ...... x_n$ to specify the order of multiplication. For example, $C_3 = 5$ since there are five ways to parenthesize $x_0 . x_1 . x_2 . x_3$ to determine the order of multiplication.*

$((x_0 . x_1) . x_2) . x_3 \quad (x_0 . (x_1 . x_2)) . x_3 \quad (x_0 . x_1) . (x_2 . x_3) \quad x_0 . ((x_1 . x_2) . x_3) \quad x_0 . (x_1 . (x_2 . x_3)).$

Solution. To develop a recurrence relation for C_n, we note that however we insert parentheses in the product $x_0 . x_1 . x_2 .... x_n$, one "." operator remains outside all parenthese, namely, the operator for the final multiplication to be performed.

For example, in $(x_0 . (x_1 . x_2)) . x_3$ it is the final ".", while in $(x_0 . x_1) . (x_2 . x_3)$ it is the second "."

Thie final operator appears between two of the $n + 1$ numbers say, x_k and x_{k+1}.

There are $C_k C_{n-k-1}$ ways to inset parentheses to determine the order of the $n + 1$ numbers to be multiplied when the final operator appears between x_k and x_{k+1}, since there are C_k ways to insert parentheses in the product $x_0 . x_1 ...... x_k$ to determine the order in which these $k + 1$ numbers are to be multiplied and C_{n-k-1} ways to inset parentheses in the product $x_{k+1} x_{k+2} ...... x_n$ to determine the order in which these $n - k$ numbers are to be multiplied.

Since this final operator can appear between any two of the $n + 1$ numbers, it follows that

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + ... C_{n-2} C_1 + C_{n-1} C_0$$
$$= \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$.

2.1.1. Back Tracking Method

In this method, we shall start from a_n and move backward towards a_1 to find a pattern, if any, to solve the problem.

To backtrack, we keep on substituting the definition of a_n, a_{n-1}, a_{n-2} and so on. Until a recognizable pattern appears.

2.1.2. Forward Chaining Method

In this method, we begin from initial (terminating) condition and keep on moving towards the nth term until we get a clear pattern.

2.1.3. Summation Method

To solve a first order linear recurrence relation with constant coefficient.

In this method, we arrange the given equation in the following form : $a_n - ka_{n-1} = f(n)$ and then backtrack till terminating condition.

In the process, we get a number of equations. Add these equations in such a way that all intermediate terms gets cancelled. Finally, we get the required solution.

Problem 2.8. *Solve the recurrence equation* $a_n = a_{n-1} + 3$ *with* $a_1 = 2$.

Solution. (*i*) **Backtracking Method :**

We have $a_n = a_{n-1} + 3$ with $a_1 = 2$

$$\begin{aligned}
a_n &= a_{n-2} + 3 + 3 && (\text{since } a_{n-1} = a_{n-2} + 3)\\
&= a_{n-2} + 2 \times 3\\
&= a_{n-3} + 3 + 2 \times 3 && (\text{since } a_{n-2} = a_{n-3} + 3)\\
&= a_{n-3} + 3 \times 3\\
&= a_{n-4} + 3 + 3 \times 3 && (\text{since } a_{n-3} = a_{n-4} + 3)\\
&= a_{n-4} + 4 \times 3\\
&\text{-- -- -- -- -- -- -- -- -- -- --}\\
&\text{-- -- -- -- -- -- -- -- -- -- --}\\
&= a_{n-(n-1)} + (n-1) \times 3\\
&= a_1 + 3(n-1)\\
&= 2 + 3(n-1) && (\text{since } a_1 = 2 \text{ is the terminating condition})
\end{aligned}$$

$\therefore \quad a_n = 2 + 3(n-1)$

(*ii*) **Forward Chaining Method :**

Given, initial condition : $a_1 = 2$

Now

$$\begin{aligned}
a_1 &= 2\\
a_2 &= a_1 + 3\\
a_3 &= a_2 + 3\\
&= a_1 + 2 \times 3\\
a_4 &= a_3 + 3\\
&= a_1 + 2 \times 3 + 3\\
&= a_1 + 3 \times 3\\
&= a_1 + (4-1) \times 3\\
a_5 &= a_1 + (5-1) \times 3\\
&\text{-- -- -- -- -- -- -- -- -- -- --}\\
&\text{-- -- -- -- -- -- -- -- -- -- --}
\end{aligned}$$

$$a_n = a_1 + (n-1)\,3$$

$\therefore \quad a_n = 2 + 3(n-1)$

(*iii*) **Summation Method :**

The given equation can be rearranged as

$$a_n - a_{n-1} = 3$$
$$a_{n-1} - a_{n-2} = 3$$
$$a_{n-2} - a_{n-3} = 3$$

$$-------------$$
$$-------------$$

$$a_3 - a_2 = 3$$
$$a_2 - a_1 = 3$$

We stop here, since $a_1 = 2$ is given.

Adding all, we get

$$a_n - a_1 = 3 + 3 + 3 + \ldots\ldots + (n-1) \text{ times.}$$
$$= 3(n-1)$$

$\Rightarrow \quad a_n = a_1 + 3(n-1).$

Problem 2.9. *Solve the recurrence equation $t_n = 2t_{n/2} + n$ with $t_1 = 1$.*

Solution. ***Backtracking Method :***

We have $\quad t_n = 2t_{n/2} + n \qquad$ ...(1)

By repeated substitution of the definition of $t_{n/2}$, $t_{n/4}$, $t_{n/8}$ etc. we get the following results :

$$t_n = 2[2t_{n/4} + n/2] + n$$
$$= 2^2\, t_{n/4} + 2 * n/2 + n$$
$$= 2^2\, t_{n/4} + n + n \qquad \text{...(2)}$$
$$t_n = 2^2\,[2\, t_{n/8} + n/4] + n + n$$
$$= 2^3\, t_{n/8} + 2^2 * n/4 + n + n$$
$$= 2^3\, t_{n/8} + n + n + n \qquad \text{...(3)}$$

Equations (1), (2) and (3) can be written as

$$t_n = 2^{\log_2^2}\, t_{n/2} + n \log_2^2, \qquad \text{from (1)}$$

$$t_n = 2^{\log_2^4}\, t_{n/4} + n \log_2^4, \qquad \text{from (2)}$$

$$t_n = 2^{\log_2^8}\, t_{n/8} + n \log_2^8, \qquad \text{from (3)}$$

Similarly, we can write,

$$t_n = 2^{\log_2^n}\, t_{n/n} + n \log_2^n$$

$$t_n = 2^{\log_2^n}\, t_1 + n \log_2^n$$

$$t_n = 2^{\log_2^n} + n \log_2^n \qquad \because \quad t_1 = 1$$

$\therefore \quad t_n = 2^{\log_2^n} + n \log_2^n$

Summation Method :

Arranging the given recurrence equation, we get

$$t_n - 2t_{n/2} = n$$

$$t_{n/2} - 2t_{n/4} = \frac{n}{2}$$

$$t_{n/4} - 2t_{n/8} = \frac{n}{4}$$

– – – – – – – – –

– – – – – – – – –

$$t_4 - 2t_2 = 4$$

$$t_2 - 2t_1 = 2$$

The number equations in this set of equations is $\log_2^n$.

Now, if we multiply the above equations by

$1, 2, 3, 4, \ldots\ldots., 2^{\log_2^n}$ respectively from top to bottom, and then by adding them all together, we get

$$t_n - 2^{\log_2^n} = n + n + n + \ldots\ldots \log_2^n \text{ times.}$$

$$t_n - 2^{\log_2^n} = n \log_2^n$$

$$\Rightarrow \qquad t_n = 2^{\log_2^n} + n \log_2^n.$$

Problem 2.10. *Solve the recurrence equation*

$$t(n) = t(\sqrt{n}) + C \log_2{}^n \quad \text{with} \quad t(1) = 1.$$

Solution. We have $t(n) = t(\sqrt{n}) + \text{C} \log_2{}^n$

$$t(2) = t(2^{1/2}) + \text{C} \log_2{}^2$$

$$= t(2^{1/4}) + \text{C} \log_2^{2^{1/2}} + \text{C} \log_2{}^2$$

$$= t(2^{1/8}) + \text{C} \log_2^{2^{1/4}} + \text{C} \log_2^{2^{1/2}} + \text{C} \log_2{}^2$$

$$= t(2^{1/2n}) + \text{C} \log_2^{2^{1/n}} + \ldots\ldots + \text{C} \log_2^{2^{1/4}} + \text{C} \log_2^{2^{1/2}} + \text{C} \log_2{}^2$$

Since t is defined for $n = 1$ only, we have to let

$t(2^{1/2n}) \longrightarrow t(1)$. This is possible when $n \to \infty$

i.e., when $n \to \infty \quad \Rightarrow \quad \frac{1}{2n} \to 0 \quad \Rightarrow \quad 2^{1/2n} \to 1$

Therefore, $t(2) = t(1) + \left[\text{C} \log_2^2 + \text{C} \log_2^{2^{1/2}} + \text{C} \log_2^{2^{1/4}} + \ldots\ldots\right]$

$$= t(1) + \text{C} \log_2{}^2 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots.\right]$$

$= t(1) + 2\,C \log_2^2$

i.e., $t(2) = t(1) + 2\,C \log_2^2$

Similarly, $t(3) = t(1) = 2\,C \log_2^3$

$t(4) = t(1) + 2\,C \log_2^4$

– – – – – – – – –

– – – – – – – – –

$t(n) = t(1) + 2\,C \log_2^n$

Therefore $t(n) = t(1) + 2\,C \log_2^n$.

Problem 2.11. *Solve the recurrence relation $a_n = a_{n-1} + 3$ with $a_1 = 2$ defines the sequence 2, 5, 8, ...*

Solution. We backtrack the value of a_n by substituting the definition of a_{n-1}, a_{n-2}, and so on until a pattern is clear.

$$a_n = a_{n-1} + 3 \quad \text{or} \quad a_n = a_{n-1} + 3$$
$$= (a_{n-2} + 3) + 3 \qquad = a_{n-2} + 2.3$$
$$= ((a_{n-3} + 3) + 3) + 3 \qquad = a_{n-3} + 3.3$$

Eventually this process will produce

$$a_n = a_{n-(n-1)} + (n-1)\,.\,3$$
$$= a_1 + (n-1)\,.\,3$$
$$= 2 + (n-1)\,.\,3$$

An explicit formula for the sequence is $a_n = 2 + (n-1)\,3$.

Problem 2.12. *Backtrack to find an explicit formula for the sequence defined by the recurrence relation $b_n = 2b_{n-1} + 1$ with initial condition $b_1 = 7$.*

Solution. We begin by substituting the definition of the previous term in the defining formula.

$$b_n = 2b_{n-1} + 1$$
$$= 2\,(2b_{n-2} + 1) + 1$$
$$= 2[2\,(2b_{n-3} + 1) + 1] + 1$$
$$= 2^3\, b_{n-3} + 4 + 2 + 1$$
$$= 2^3\, b_{n-3} + 2^2 + 2^1 + 1.$$

A pattern is emerging with these rewritting of b_n.

(**Note :** There are no set rules for how to rewrite these expressions and a certain amount of experimentation may be necessary.)

The backtracking will end at

$$b_n = 2^{n-1} b_{n-(n-1)} + 2^{n-2} + 2^{n-3} + ... + 2^2 + 2^1 + 1$$
$$= 2^{n-1} b_1 + 2^{n-1} - 1$$
$$= 7\,.\,2^{n-1} + 2^{n-1} - 1 \qquad \text{(using } b_1 = 7\text{)}$$
$$= 8\,.\,2^{n-1} - 1 \quad \text{or} \quad 2^{n+2} - 1.$$

Problem 2.13. *Write down the first six terms of the sequence defined by $a_1 = 1$, $a_{k+1} = 3a_k + 1$ for $k \geq 1$. Guess a formula for a_n and prove that your formula is correct.*

Solution. The first six terms are

$$a_1 = 1$$
$$a_2 = 3a_1 + 1 = 3(1) + 1 = 4$$
$$a_3 = 3a_2 + 1 = 3(4) + 1 = 13$$
$$a_4 = 40,\ a_5 = 121,\ a_6 = 364.$$

Since there is multiplication by 3 at each step, we might suspect that 3^n is involved in the answer.

After trial and error, we guess that $a_n = \frac{1}{2}(3^n - 1)$ and verify this by mathematical induction.

When $n = 1$, the formula gives $\frac{1}{2}(3^1 - 1) = 1$, which is indeed a_1, the first term in the sequence.

Now assume that $k \geq 1$ and that $a_k = \frac{1}{2}(3^k - 1)$.

We wish to prove that $a_{k+1} = \frac{1}{2}(3^{k+1} - 1)$

We have $a_{k+1} = 3a_k + 1 = 3\frac{1}{2}(3^k - 1) + 1$

Using the induction hypothesis.

Hence, $a_{k+1} = \frac{1}{2}3^{k+1} - \frac{3}{2} + 1 = \frac{1}{2}(3^{k+1} - 1)$ as required.

By the principle of mathematical induction, our guess is correct.

Problem 2.14. *A sequence is defined recursively by $a_0 = 1$, $a_1 = 4$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find the first six terms of this sequence. Guess a formula for a_n and establish the validity of your guess.*

Solution. Here there are two initial conditions $a_0 = 1$, $a_1 = 4$.

Also, the recurrence relation, $a_n = 4a_{n-1} - 4a_{n-2}$, defines the general term as a function of two previous terms.

The first six terms of the sequence are

$$a_0 = 1$$
$$a_1 = 4$$
$$a_2 = 4a_1 - 4a_0 = 4(4) - 4(1) = 12$$
$$a_3 = 4a_2 - 4a_1 = 4(12) - 4(4) = 32$$
$$a_4 = 4a_3 - 4a_2 = 4(32) - 4(12) = 80$$
$$a_5 = 4a_4 - 4a_3 = 4(80) - 4(32) = 192.$$

Finding a general formula for a_n requires some ingenuity.

Let us examine some of the first six terms.

We note that $a_3 = 32 = 24 + 8 = 3(8) + 4(8)$

and $a_4 = 80 = 64 + 16 = 4(16) + 16 = 5(16)$

and $a_5 = 192 = 6(32)$.

We guess that $a_n = (n + 1)\, 2^n$.

To prove this, we use the strong form of mathematical induction (with $n_0 = 0$).

When $n = 0$, we have $(0 + 1)\, 2^0 = 1(1) = 1$, in agreement with the given value for a_0.

When $n = 1$, $(1 + 1)\, 2^1 = 4 = a_1$. Now that the formula has been verified for $k = 0$ and $k = 1$.

We may assume that $k > 1$ and that $a_n = (n + 1)\, 2^n$ for all n in the interval $0 \le n < k$.

We wish to prove the formula is valid for $n = k$.

That is, we wish to prove that $a_k = (k + 1)\, 2^k$.

Since $k \ge 2$, we know that $a_k = 4a_{k-1} - 4a_{k-2}$.

Applying the induction hypothesis to $k - 1$ and to $k - 2$ (each of which is in the range $0 \le n < k$).

We have $a_{k-1} = k\, 2^{k-1}$ and $a_{k-2} = (k - 1)\, 2^{k-2}$.

Thus,
$$\begin{aligned} a_k &= 4(k\, 2^{k-1}) - 4(k - 1)\, 2^{k-2} \\ &= 2k\, 2^k - k\, 2^k + 2^k \\ &= k\, 2^k + 2^k = (k + 1)\, 2^k \quad \text{as required.} \end{aligned}$$

By principle of mathematical induction, the formula is valid for all $n \ge 0$.

Problem 2.15. *Solve the recurrence relation $a_n = a_{n-1} + 2$, $n \ge 2$ subject to initial condition $a_1 = 3$.*

Solution. We backtrack the value of a_n by substituting the expression of a_{n-1}, a_{n-2} and so on, until a pattern is clear.

Given $a_n = a_{n-1} + 2$ ...(1)

Replacing n by $n - 1$ in (1), we obtain

$$a_{n-1} = a_{n-2} + 2$$

From (1),
$$\begin{aligned} a_n &= a_{n-1} + 2 = (a_{n-2} + 2) + 2 \\ &= a_{n-2} + 2.2 \qquad \text{...(2)} \end{aligned}$$

Replacing n by $n - 2$ in (1), we obtain

$$a_{n-2} = a_{n-3} + 2$$

So, from (2), $a_n = (a_{n-3} + 2) + 2.2 = a_{n-3} + 3.2$

In general $a_n = a_{n-k} + k\,.\,2$

For $k = n - 1$,
$$\begin{aligned} a_n &= a_{n-(n-1)} + (n - 1)\,.\,2 \\ &= a_1 + (n - 1)\,.\,2 = 3 + (n - 1)\,.\,2 \end{aligned}$$

which is an explicit formula.

2.2. THE SECOND-ORDER LINEAR Homogeneous Recurrence Relation with constant coefficients

Let $k \in Z^+$ and $C_n\ (\ne 0), C_{n-1}, C_{n-2}, \ldots\ldots C_{n-k}\ (\ne 0)$ be real numbers. If a_n, for $n \ge 0$, is a discrete function, then

$$C_n a_n + C_{n-1}\, a_{n-1} + C_{n-2}\, a_{n-2} + \ldots\ldots + C_{n-k}\, a_{n-k} = f(x),\ n \ge k$$

is a linear recurrence relation (with constant coefficients) of order k. When $f(n) = 0$, for all $n \geq 0$, the relation is called homogeneous ; other wise, it is non-homogeneous.

The homogeneous relation of order two :

$$C_n a_n + C_{n-1}a_{n-1} + C_{n-2}a_{n-2} = 0,\ n \geq 2.$$

A solution of the form $a_n = Cr^n$, where $C \neq 0$ and $r \neq 0$ substituting $a_n = Cr^n$ into

$$C_n a_n + C_{n-1}a_{n-1} + C_{n-2}a_{n-2} = 0.$$

We obtain $C_n Cr^n + C_{n-1}Cr^{n-1} + C_{n-2}C_r^{n-2} = 0,$

with C, $r \neq 0$, this becomes

$$C_n r^2 + C_{n-1}r + C_{n-2} = 0,$$

a quadratic equation which is called the characteristic equation.

2.3. THE NON HOMOGENEOUS RECURRENCE RELATIONS

The recurrence relations

$$a_n + C_{n-1}a_{n-1} = f(n),\ n \geq 1 \qquad ...(1)$$

$$a_n + C_{n-1}a_{n-1} + C_{n-2}a_{n-2} = f(n),\ n \geq 2 \qquad ...(2)$$

where C_{n-1} and C_{n-2} are constants, $C_{n-1} \neq 0$, in (1), $C_{n-2} \neq 0$, and $f(n)$ is not identically 0.

Although there is no general method for solving all non homogeneous relations, for certain functions $f(n)$ we shall find a successful technique.

When $C_{n-1} = -1$, (1) gives, for the non homogeneous relation $a_n - a_{n-1} = f(n)$, we have

$$a_1 = a_0 + f(1)$$

$$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$$

$$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$$

$$a_n = a_0 + f(1) + ... + f(n)$$

$$= a_0 + \sum_{i=1}^{n} f(i)$$

We can solve this type of relation in terms of n, if we find a suitable summation formula for $\sum_{i=1}^{n} f(i)$.

(*a*) The non homogeneous first-order relation

$$a_n + C_{n-1}a_{n-1} = kr^n$$

where k is a constant and $n \in Z^+$.

(*b*) If r^n is not a solution of the associated homogeneous relation $a_n + C_{n-1}a_{n-1} = 0$, then $a_n(P) = Ar^n$, where A is a constant. When r^n is a solution of the associated homogeneous relation, then

$$a_n(P) = Bn\ r^n, \text{ for B a constant.}$$

(*c*) The non-homogeneous second order relation

$$a_n + C_{n-1}a_{n-1} + C_{n-2}a_{n-2} = k\ r^n,$$

where k is a constant. Here,

(*i*) $a_n^{(p)} = A\, r^n$, for A a constant, if r^n is not a solution of the associated homogeneous relation,

(*ii*) $a_n^{(p)} = Bn\, r^n$, where B is a constant, if $a_n^{(h)} = C_1 r^n + C_2 r_1^{\,n}$ where $r_1 \neq r$,

(*iii*) $a_n^{(p)} = Cn^2 r^n$, for C a constant, when $a_n^{(h)} = (C_1 + C_2 n)\, r^n$.

2.3.1. Characteristic Equation Method :

This method can be used to solve any constant order linear recurrence equation with constant coefficient. This recurrence relation may be homogeneous or non-homogeneous. Before attempting to solve any such problem, let us first, understand what is characteristic equation for a given recurrence equation and how to find it.

A recurrence equation of the mentioned type can be arrranged in standard form as :

$$\mathbf{A_n + C_1A_{n-1} + C_2A_{n-2} + C_3\, A_{n-3} = R.H.S.} \qquad ...(1)$$

Where C_1, C_2, C_3 are constant coefficients and R.H.S. has one of the following forms :

Form	**Examples**
Homogeneous	0
A constant to the *n*th power	$2^n, \pi^{-s}, 2^{-n}, \sqrt{2^n}$
A polynomial in *n*	$3, n^2, n^2 - n, n^3 + 2n - 1$
A product of a constant to the *n*th power and a polynomial in *n*	$2^n (n^2 + 2n - 1), (n-1)\, n^6, n\, 6^n$
A linear combination of any of the above	$(2^n + 3^{n/2})\, (n^2 + 2n - 1) + 5$

In the recurrence equation (1), assigning R.H.S. = 0, we get

$$A_n + C_1A_{n-1} + C_2A_{n-2} + C_3A_{n-3} = 0 \qquad ...(2)$$

This equation (2) gives the homogeneous part of the given recurrence equation. Every recurrence equation has a homogeneous part. If the recurrence relation is homogeneous then it has only homogeneous part and solving such equation is one step process. On the other hand, if the given recurrence equation is non-homogeneous then its homogeneous part is obtained by assigning R.H.S. equal to zero.

A characteristic equation corresponds to homogeneous part of the given recurrence relation.

The characteristic equation of (2) is given as :

$$x^3 + C_1x^2 + C_2x + C_3 = 0 \qquad ...(3)$$

This has been obtained by the following **procedure :**

(*i*) Find the order of the recurrence equation here it is 3.

(*ii*) Take any variable (say *x*) and substitute

A_n, A_{n-1}, A_{n-2}, by x^3, x^2, x respectively in the homogeneous part of the recurrence equation.

Equation so obtained is called **characteristic equation** of the given recurrence equation.

Example (1) : The characteristic equation of the recurrence equation $C_n = 3C_{n-1} - 2C_{n-2}$ is given by

$$x^2 - 3x + 2 = 0.$$

Example (2) : The characteristic equation of the recurrence equation $f_n = f_{n-1} + f_{n-2}$ is given by

$$x^2 - x - 1 = 0.$$

Example (3) : The characteristic equation of the recurrence equation

$$A_n - 5A_{n-1} + 6A_{n-2} = 2^n + n$$

is given by $x^2 - 5x + 6 = 0.$

Theorem 2.1. If the characteristic equation $x^2 - r_1x - r_2 = 0$ of the recurrence equation $a_n = r_1a_{n-1} + r_2a_{n-2}$ has two distinct roots S_1 and S_2 then $a_n = uS_1^n + vS_2^n$ is the closed form formula for the sequence where u and v depend on the initial condition.

Proof. Since S_1 and S_2 are roots of

$$x^2 - r_1x - r_2 = 0 \quad ...(1)$$

We have $S_1^2 - r_1S_1 - r_2 = 0$ ...(2)

$$S_2^2 - r_1S_2 - r_2 = 0 \quad ...(3)$$

Since u and v are dependent on the initial conditions

We have $a_1 = uS_1 + vS_2$

and $a_2 = uS_1^2 + vS_2^2$

Now,

$$\begin{aligned} a_n &= uS_1^n + vS_2^n \\ &= uS_1^{n-2}S_1^2 + vS_2^{n-2}S_2^2 \\ &= uS_1^{n-2}[r_1S_1 + r_2] + vS_2^{n-2}[r_1S_2 + r_2] \quad \text{(from (2) and (3))} \\ &= r_1uS_1^{n-1} + r_2uS_1^{n-2} + r_1vS_2^{n-1} + r_2vS_2^{n-2} \\ &= r_1[uS_1^{n-1} + vS_2^{n-1}] + r_2[uS_1^{n-2} + vS_2^{n-2}] \\ &= r_1a_{n-1} + r_2a_{n-2} \end{aligned}$$

i.e., $a_n = uS_1^n + vS_2^n$ in an explicit formula for the given recurrence relation.

Theorem 2.2. If the characteristic equation $x^2 - r_1x - r_2 = 0$ of the recurrence equation $a_n = r_1a_{n-1} + r_2a_{n-2}$ has single roots then $a_n = uS^n + v_nS^n$ is the closed form formula for the sequence where u and v depend on the initial condition.

Proof. Since S is roots of $x^2 - r_1x - r_2 = 0$ ...(1)

We have $S^2 - r_1S - r_2 = 0$ ...(2)

Now,

$$\begin{aligned} a_n &= uS^n + vnS^n \\ &= uS^n + v(n-1)S^n + vS^n \\ &= uS^{n-2}S^2 + v(n-1)S^{n-2}S^2 + vS^n \\ &= [uS^{n-2} + v(n-1)S^{n-2}]S^2 + vS^n \\ &= [uS^{n-2} + v(n-1)S^{n-2}](r_1S + r_2) + vS^n \\ &= r_1[uS^{n-1} + v(n-1)S^{n-1}] + r_2[uS^{n-2} + v(n-1)S^{n-2}] + vS^n \\ &= r_1[uS^{n-1} + v(n-1)S^{n-1}] + r_2[uS^{n-2} + v(n-2)S^{n-2}] + r_2vS^{n-2} + vS^n \\ &= r_1a_{n-1} + r_2a_{n-2} + vS^{n-2}[r_2 + S^2] \quad ...(3) \end{aligned}$$

From equation (2) we have

$$S = \frac{r_1 \pm \sqrt{r_1^2 + 4r_2}}{2} \qquad \because \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

∵ S is an equal root, we have

$$\sqrt{r_1^2 + 4r_2} = 0 \qquad \text{or} \qquad S = \frac{r_1}{2} \qquad \text{or} \qquad r_1 = 2S.$$

Now substituting $r_1 = 2S$ in equation (2), we get

$$S^2 - 2S^2 - r_2 = 0$$

$$-S^2 - r_2 = 0$$

$$S^2 + r_2 = 0 \qquad \text{...(4)}$$

Using the result from equation (4) in equation (3), we have

$$a_n = r_1 a_{n-1} + r_2 a_{n-2}$$

Therefore, the explicit formula for the recurrence equation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ is

$$a_n = uS^n + v_n S^n$$

with initial conditions

$$a_1 = uS + vS \quad \text{and} \quad a_2 = uS^2 + 2vS^2.$$

Note. (*i*) In general, if $S_1, S_2, S_3, ... S_r$ are r distinct roots of the characteristic equation of a rth order recurrence equation then its explicit formula is given by

$$a_n = u_1 S_1^{\,n} + u_2 S_2^{\,n} + u_3 S_3^{\,n} + ... + u_r S_r^{\,n}$$

where $u_1, u_2, u_3, ... u_r$ depend on initial conditions.

(*ii*) If S is r times equal roots of the characteristic equation of a rth order recurrence equation then its explicit formula is given by

$$a_n = u_1 S^n + u_2 n S^n + u_3 n^2 S^n + ... + u_r n^{r-1} S^n$$

where $u_1, u_2, u_3, ... u_r$ depend on initial conditions.

(*iii*) A combination of 1 and 2 is also possible,

i.e., some roots of a characteristic equation are distinct and some are equal.

In that case, for distinct roots we use method outlined in 1 and for repeated roots we use method mentioned in 2.

There are two parts to the total solution. The homogeneous part of the solution depends only on what is on the left of the total solution depends on what is on the R.H.S. and has the same form as the R.H.S. We will calculate the two parts separately and add them to form the total solution.

There are four steps in the process as listed below :

Step 1 : Find the homogeneous solution to the homogeneous equation. This results when you set the R.H.S. to zero. If it is already zero, skip the next two steps and go directly to the step 4. Your answer will contains one or more undetermined coefficients whose values cannot be determined until step 4.

Step 2 : Find the particular solution by guessing a form similar to the R.H.S. This step does not produce any additional undetermined coefficients, nor does it eliminate those from step 1.

Step 3 : Combine the homogeneous and particular solution.

Step 4 : Use boundary or initial conditions to eliminate the undetermined constants from the step 1.

We shall now discuss the way to find a particular solution : The form of a particular solution has nothing to do with the order of the recurrence relation. It only depends on the form of the R.H.S. of recurrence equation expressed in standard form. Guess a solution of the same form but with undetermined coefficients, which have to be calculated. We find their values by substituting the guessed particular solution into the recurrence equation. You may find the following table useful for guessing a particular solution :

R.H.S.	Guessed Particular Solution
17 (constant)	C (constant)
π^n (constant to the *n*th power)	$C\pi^n$ (constant to the *n*th power)
$2^n + 5^n + 3$ (linear combination)	$D2^n + E5^n + F$ (same linear combination)
$5n^3$ (polynomial in n)	$An^3 + Bn^2 + Cn + D$ (Decreasing polynomial)
$5n^3 - 1$ (polynomial in n)	$An^3 + Bn^2 + Cn + D$ (Decreasing polynomial)
$3n^2 5^n$ (linear combination)	$5^n (Bn^2 + Cn + D)$ (linear combination)

Notice how a polynomial in n produces a decreasing polynomial with all orders, down to, and including the constant term. Without those extra terms, usually the coefficients cannot be determined successfully.

Theorem 2.3. If $\{a_n^{(P)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots\ldots + C_k a_{n-k} + F(n), \text{ then every solution is of the form}$$

$\{a_n^{(P)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots\ldots + C_k a_{n-k}.$$

Proof. Since $\{a_n^{(P)}\}$ is a particular solution of the non-homogeneous recurrence relation.

We know that $a_n^{(P)} = C_1 a_{n-1}^{(P)} + C_2 a_{n-2}^{(P)} + \ldots\ldots + C_k a_{n-k}^{(P)} + F(n)$

Now suppose that $\{b_n\}$ is a second solution of the non-homogeneous recurrence relation, so that

$$b_n = C_1 b_{n-1} + C_2 b_{n-2} + \ldots\ldots + C_k b_{n-k} + F(n)$$

Substracting the first of these two equations from the second shows that

$$b_n - a_n^{(P)} = C_1(b_{n-1} - a_{n-1}^{(P)}) + C_2(b_{n-2} - a_{n-2}^{(P)}) + \ldots\ldots + C_k(b_{n-k} - a_{n-k}^{(P)})$$

It follows that $\{b_n - a_n^{(P)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.

Consequently, $b_n = a_n^{(P)} + a_n^{(h)}$ for all n.

Theorem 2.4. Suppose that $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots\ldots + C_k a_{n-k} + F(n),$$

where $C_1, C_2, \ldots\ldots, C_k$ are real number and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots\ldots + b_1 n + b_0) S^n$.

where $b_0, b_1, ..., b_t$ and S are real numbers. When S is a not root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(P_t n^t + P_{t-1} n^{t-1} + ... + P_1 n + P_0)\, S^n$$

When S is a root of this characteristic equation and its multiplicity is m, theorem is a particular solution of the form

$$n^m\, (P_t n^t + P_{t-1} n^{t-1} + ... + P_1 n + P_0)\, S^n.$$

Problem 2.16. *What is the solutions of recurrence relation*

$$a_n = a_{n-1} + 2a_{n-2} \quad \text{with} \quad a_0 = 2 \text{ and } a_1 = 7\ ?$$

Solution. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n \text{ for some constants } \alpha_1 \text{ and } \alpha_2.$$

From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$

$$a_1 = 7 = \alpha_1 \,.\, 2 + \alpha_2 \,.\, (-1)$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3.2^n - (-1)^n.$$

Problem 2.17. *What is the solution of the recurrence relation*

$$a_n = 6a_{n-1} - 9a_{n-2} \text{ with initial conditions } a_0 = 1 \text{ and } a_1 = 6\ ?$$

Solution. The only root of $r^2 - 6r + 9 = 0$ is $r = 3$.

Hence, the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2.

Using the initial conditions, it follows that

$$\alpha_0 = 1 = \alpha_1$$

$$a_1 = 6 = \alpha_1 \,.\, 3 + \alpha_2 \,.\, 3$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n\, 3^n.$$

Problem 2.18. *Find the solution to the recurrence relation*

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial conditions $a_0 = 2$, $a_1 = 5$ *and* $a_2 = 15$.

Solution. The characteristic polynomial of this recurrence relation is $r^3 - 6r^2 + 11r - 6 = 0$

The characteristic roots are $r = 1$, $r = 2$ and $r = 3$

Since $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$

Hence, the solutions to this recurrence relation are of the form $a_n = \alpha_1 \,.\, 1^n + \alpha_2 \,.\, 2^n + \alpha_3 \,.\, 3^n$.

To find the constants α_1, α_2 and α_3, use the initial conditions.

This gives $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$

$a_1 = 5 = \alpha_1 + \alpha_2 \,.\, 2 + \alpha_3 \,.\, 3$

$a_2 = 15 = \alpha_1 + \alpha_2 \,.\, 4 + \alpha_3 \,.\, 9$

When these three simultaneous equations are solved for α_1, α_2 and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$ and $\alpha_3 = 2$.

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with $a_n = 1 - 2^n + 2 \,.\, 3^n$.

Problem 2.19. *Find the solution to the recurrence relation*

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution. The characteristic equation of this recurrence relation is $r^3 + 3r^2 + 3r + 1 = 0$

Since $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation.

The solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0} (-1)^n + \alpha_1, 1^n (-1)^n + \alpha_{1,0}\, n^2 (-1)^n$$

To find the constant $\alpha_{1,0}$, $\alpha_{1,1}$ and $\alpha_{1,2}$, use the initial conditions.

This gives $a_0 = 1 = \alpha_{1,0}$

$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$

$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$

The simultaneous solution of these three equations is

$$\alpha_{1,0} = 1,\ \alpha_{1,1} = 3, \text{ and } \alpha_{1,2} = -2.$$

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with $a_n = (1 + 3n - 2n^2)(-1)^n$.

Problem 2.20. *Find the explicit formula for the Fibonacci numbers.*

Solution. Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = \dfrac{(1+\sqrt{5})}{2}$ and $r_2 = \dfrac{(1-\sqrt{5})}{2}$.

Therefore, it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constant α_1 and α_2.

The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants.

We have $f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = \frac{1}{\sqrt{5}} \text{ and } \alpha_2 = -\frac{1}{\sqrt{5}}.$$

Conseqently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Problem 2.21. *Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?*

Solution. To solve this linear non-homogeneous recurrence relation with constant coefficiently we need to solve its associated linear homogeneous equation and to find a particular solution for the given non-homogeneous equation.

The associated linear homogeneous equation is $a_n = 3a_{n-1}$.

Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

We now find a particular solution. Since $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n, say, $P_n = cn + d$ where c and d are constants.

To determine whether there are any solutions of this form, suppose that $P_n = cn + d$ is such a solution.

Then the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.

Simplifying and combining like terms gives $(2 + 2c)n + (2d - 3c) = 0$

It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$

This shows that $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.

Consequently, $a_n^{(P)} = -n - \frac{3}{2}$ is a particular solution.

All solutions are of the form

$$a_n = a_n^{(P)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \,.\, 3^n, \text{ where } \alpha_n \text{ is a constant.}$$

To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general solution. We find that $3 = -1 - \frac{3}{2} + 3\alpha$, which implies that $\alpha = \frac{11}{6}$.

The solution we seek is $a_n = -n - \frac{3}{2} + \left(\frac{11}{6}\right) 3^n$.

Problem 2.22. *Find all solutions of the recurrence relation*

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution. This is a linear homogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ are

$$a_n^{(h)} = \alpha_1 . 3^n + \alpha_2 . 2^n,$$

where α_1 and α_2 are constants.

Since $F(n) = 7^n$, a reasonable trial solution is $a_n^{(P)} = C . 7^n$, where C is a constant.

Substituting the terms of this sequence into the recurrence relation implies that

$$C . 7^n = 5C . 7^{n-1} - 6C . 7^{n-2} + 7^n$$

Factoring out 7^{n-2}, this equation becomes

$$49C = 35C - 6C + 49, \text{ which implies that } 20C = 49 \text{ or that } C = \frac{49}{20}$$

Hence, $a_n^{(P)} = \left(\frac{49}{20}\right) 7^n$ is a particular solution.

All solutions are of the form

$$a_n = \alpha_1 . 3^n + \alpha_2 . 2^n + \left(\frac{49}{20}\right) 7^n.$$

Problem 2.23. *What form does a particular solution of the linear non-homogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n\,3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)\,3^n$?*

Solution. The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

Its characteristic equation $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root 3 of multiplicity two with F(n) of the P(n) S^n, where P(n) is a polynomial and S is a constant, we need to ask whether S is a root of this characteristic equation.

Since S = 3 is a root with multiplicity $m = 2$ but S = 2 is not a root.

The particular solution has the form $P_0 n^2 3^n$ if $F(n) = 3^n$ the form $n^2 (P_1 n + P_0)\, 3^n$ if $F(n) = n\, 3^n$, the form $(P_2 n^2 + P_1 n + P_0)\, 2^n$ if $F(n) = n^2 2^n$ and the form $n^2 = (P_2 n^2 + P_1 n + P_0)\, 3^n$ if $F(n) = (n^1 + 1)\, 3^n$.

Problem 2.24. *Solve the recurrence equation*

$$a_r - 7a_{r-1} + 10a_{r-2} = 2^r \text{ with initial condition } a_0 = 0 \text{ and } a_1 = 6.$$

Solution. The general solution, also called **homogeneous** solution, to the problem is given by homogeneous part of the given recurrence equation.

The homogeneous part of the equation

$$a_r - 7a_{r-1} + 10a_{r-2} = 2^r \quad ...(1)$$

is

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad ...(2)$$

The characteristic equation of (2) is given by

$$x^2 - 7x + 10 = 0$$

$$\Rightarrow \quad (x - 2)(x - 5) = 0$$

$$\Rightarrow \quad x = 2 \text{ and } x = 5$$

Since the two roots 2 and 5 of characteristic equation are distinct, the homogeneous solution is given by

$$a_r = A2^r + B5^r \quad ...(3)$$

Now, **particular solution** is given by

$$a_r = rC2^r$$

Substituting the value of a_r in equation (1), we get

$$C[r\,2^r - 7(r-1)\,2^{r-1} + 10\,(r-2)\,2^{r-2}] = 2^r$$

or, $$C[4r - 7(r-1)\,2 + 10(r-2)]\,2^{r-2} = 2^r$$

or, $$C[4r - 14(r-1) + 10r - 20] = 4$$

or, $$C[4r - 14r + 14 + 10r - 20] = 4$$

or, $$-6C = 4 \quad \Rightarrow \quad C = -\frac{2}{3}$$

Particular solution is $a_r = -\frac{2}{3}\,r\,2^r$.

The complete, also called total, solution is obtained by combining the homogeneous and particular solutions. This is given as

$$a_r = A2^r + B5^r - \frac{2}{3}r2^r \qquad ...(4)$$

The equation (4) contains two undetermined coefficients A and B, which are to be determined. To find this, we use the given initial conditions for $r = 0$ and $r = 1$.

Since values of a_0 and a_1 are given.

Putting $r = 0$ and $r = 1$ in equation (4), we get the following equations (5) and (6) respectively.

$$a_0 = A2^0 + B5^0 - \frac{2}{3} * 0 * 2^0$$

or $$0 = A + B \qquad ...(5) \quad (\because \; a_0 = 0 \text{ is given})$$

and $$a_1 = A2^1 + B5^1 - \frac{2}{3} * 1 * 2^1$$

or $$6 = 2A + 5B - \frac{4}{3} \qquad (\because \; a_1 = 6 \text{ is given})$$

or $$2A + 5B = \frac{22}{3} \qquad ...(6)$$

Solving equation (5) and (6), we get

$$A = -\frac{22}{9} \text{ and } B = \frac{22}{9}$$

Replacing A and B in equation (4) by its respective values, we get the closed form formula for the given recurrence equation. Thus,

$$a_r = -\frac{22}{9}2^r + \frac{22}{9}5^r - \frac{2}{3}r2^r$$

or $$a_r = \frac{22}{9}\,[5^r - 2^r] - \frac{2}{3}r2^r.$$

Problem 2.25. *Solve the recurrence equation*

$$A_n - A_{n-1} - A_{n-2} = 2n \text{ with } A_0 = 0 \text{ and } A_1 = 1.$$

Solution. The homogeneous part of the equation

$$A_n - A_{n-1} - A_{n-2} = 2n \qquad ...(1)$$

is
$$A_n - A_{n-1} - A_{n-2} = 0 \qquad ...(2)$$

The characteristic equaion of (2) is given as

$$x^2 - x - 1 = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+4}}{2}$$

or
$$x = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x = \frac{1-\sqrt{5}}{2}.$$

Thus, two roots of the characteristic equation are distinct. So, the homogeneous solution is given by

$$A_n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n \qquad ...(3)$$

The particular solution is given by the guess

$$A_n = Cn + D$$

Substituting this value in equation (1), we get

$$Cn + D - C(n-1) - D - C(n-2) - D = 2n$$

or
$$(Cn - Cn - Cn) + D + C - D + 2C - D = 2n$$

$\Rightarrow \quad -Cn + 3C - D = 2n$

$\Rightarrow \quad -C = 2,\ 3C - D = 0$

$\Rightarrow \quad C = -2,\ D = -6$

$\therefore \quad A_n = -2n - 6$ is the particular solution.

Combining homogeneous and particular solutions, we get total solution as

$$A_n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n - 2n - 6 \qquad ...(4)$$

Equation (4) contains two undetermined coefficients A and B. Which are to be determined.

To find this we use the given initial conditions for $n = 0$ and $n = 1$, since values of A_0 and A_1 are given.

Putting $n = 0$ and $n = 1$ in equation (4), we get the following equations (5) and (6) respectively.

$$A_0 = A\left[\frac{1+\sqrt{5}}{2}\right]^0 + B\left[\frac{1-\sqrt{5}}{2}\right]^0 - 2 * 0 - 6$$

or
$$0 = A + B = 0 \qquad ...(5) \quad (\because \quad A_0 = 0 \text{ is given})$$

or $$A_1 = A\left[\frac{1+\sqrt{5}}{2}\right]^1 + B\left[\frac{1-\sqrt{5}}{2}\right]^1 - 2 * 1 - 6$$

or $$1 = A\left[\frac{1+\sqrt{5}}{2}\right] + B\left[\frac{1-\sqrt{5}}{2}\right] - 8 \qquad ...(6) \quad (\because \quad A_1 = 1 \text{ is given})$$

Solving (5) and (6), we get

$$B - \frac{1-\sqrt{5}}{1+\sqrt{5}}\ B = 6 - \frac{18}{1+\sqrt{5}}$$

or $$B\ \frac{1+\sqrt{5}-1+\sqrt{5}}{1+\sqrt{5}} = \frac{6+6\sqrt{5}-18}{1+\sqrt{5}}$$

or $$B = \frac{3(\sqrt{5}-2)}{\sqrt{5}} \text{ and } A = \frac{3(\sqrt{5}+2)}{\sqrt{5}}.$$

Replacing A and B in equation (4) by its respective values, we get the closed form formula for the given recurrence equation. Therefore, the final solution is

$$A_n = \frac{3(\sqrt{5}+2)}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n + \frac{3(\sqrt{5}-2)}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n - 2n - 6.$$

Problem 2.26. *Solve the recurrence equation*

$$a_n - 8a_{n-1} + 16a_{n-2} = 0 \text{ with } a_2 = 16 \text{ and } a_3 = 80.$$

Solution. The given equation is homogeneous, so only the homogeneous part of the solution is required.

In this case, we do not have to try for the particular solution and hence no combining of the particular part with the homogeneous part is needed.

Here, we will go to the step 4 from the step 1, skipping the steps 2 and 3.

The characteristic equation of the given recurrence equation is $x^2 - 8x + 16 = 0$

or $(x-4)^2 = 0 \quad \Rightarrow \quad x = 4, 4.$

Since the two roots of characteristic equation are equal, the homogeneous solution is given by

$$a_n = (u + vn)\ 4_n \qquad ...(1)$$

The equation (1) contains two undetermined coefficients u and v, which are to be determined.

To find this, we use the given initial conditions for $n = 2$ and $n = 3$.

Since values of a_2 and a_3 are given.

Putting $n = 2$ and $n = 3$ in equation (1), we get the following equations (2) and (3) respectively. When $n = 2$,

$$a_2 = (u + 2v)\ 4^2 \quad \text{or} \quad 16 = 16\ (u + 2v)$$

or $$u + 2v = 1 \qquad ...(2)$$

and when $n = 3$.

$$a_3 = (u + 3v)\, 4^3 \qquad \text{or} \qquad 80 = 64\,(u + 3v)$$

or $$u + 3v = \frac{5}{4} \qquad ...(3)$$

Solving equation (2) and (3), we have

$$u = \frac{1}{2} \quad \text{and} \quad v = \frac{1}{4}$$

Replacing u and v their respective values in (1), we have the solution in the form of closed formula.

Therefore, $a_n = \left[\frac{1}{2} + \frac{n}{4}\right] 4^n$.

Problem 2.27. *Solve the recurrence equation*

$$a_n = a_{n-1} + 2(n-1) \text{ with } a_0 = 1.$$

Solution. The given equation is non-homogeneous of order 1.

This can also be solved either by backtracking or by summation method.

Here, we shall solve it by characteristic equation method.

The characteristic equation of the given recurrence equation is $x - 1 = 0$ or $x = 1$.

The characteristic equation has only one root, so the homogeneous solution is given by

$$a_n = A * 1^n \quad ...(1)$$

Now to find particular solution, we guess

$$a_n = Bn + D$$

Now substituting this guess for a_n and a_{n-1} in the given recurrence equation, we get

$$Bn + D - B(n-1) - D = 2n - 2$$

or $$B = 2n - 2 \quad \Rightarrow \quad 0 = 2, \text{ this is impossible.}$$

Notice coefficient of n in L.H.S. is zero and in R.H.S., it is 2.

This has happened because of our assumption (guess).

Now we will modify our guess and take a polynomial in n of a degree one higher than what we took earlier as a guess to the particular solution.

Let $$a_n = Bn^2 + Cn + D$$

Now substituting this guess for a_n and a_{n-1} in the given recurrence equation, we get

$$Bn^2 + Cn + D - B\,(n-1)^2 - C(n-1) - D = 2n - 2$$

or $$2Bn + (C - B) = 2n - 2$$

$\Rightarrow$ $2B = 2$ and $C - B = -2$

$\Rightarrow$ $B = 1$ and $C = -1$ and $D = 0$

$\therefore$ $a_n = n^2 - n$, is the particular solution. Here, we have assigned $D = 0$.

The complete solution is then given by combining homogeneous and particular solution as

$$a_n = A + n^2 - n \qquad ...(2)$$

To determine the value of A, we use the initial condition $a_0 = 1$.

We have from equation (2), a_0 = A or A = 1 when $n = 0$.

Therefore $a_n = n^2 - n + 1$.

Problem 2.28. *Solve the recurrence equation*

$$a_n = 5a_{n-1} + 6a_{n-2} = 2^n + n \quad \textit{with}$$

initial condition $a_1 = 0$ and $a_2 = 10$.

Solution. The general to the problem is given by the homogeneous part of the given recurrence equation.

The homogeneous part of the equation

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n \qquad ...(1)$$

is

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \qquad ...(2)$$

The characteristic equation of (2) is given as

$$x^2 - 5x + 6 = 0$$

$$\Rightarrow \quad (x - 2)(x - 3) = 0$$

$$\Rightarrow \quad x = 2 \text{ and } x = 3$$

The two roots 2 and 3 of characteristic equation are distinct. So the homogeneous solution is given by

$$a_n = \text{A}2^n + \text{B}3^n \qquad ...(3)$$

Let particular solution be

$$a_n = \text{C}n\,2^n + \text{D}n + \text{E}.$$

Substituting this particular solution in equation (1) for a_n, a_{n-1} and a_{n-2}, we get

$$\text{C}n\,2^n + \text{D}n + \text{E} - 5\,[\text{C}(n-1)\,2^{n-1} + \text{D}\,(n-1) + \text{E}]$$
$$+ 6[\text{C}(n-2)\,2^{n-2} + \text{D}(n-2) + \text{E}] = 2^n + n$$

or

$$\text{C}[n\,2^n - 5(n-1)\,2^{n-1} + 6(n-2)\,2^{n-2}] + \text{D}[n - 5\,(n-1) + 6(n-2)]$$
$$+ \text{E}[1 - 5 + 6] = 2^n + n$$

or

$$\text{C}[4n - 10(n-1) + 6(n-2)]\,2^{n-2} + \text{D}\,[n - 5n + 5 + 6n - 12]$$
$$+ 2\text{E} = 2^n + n$$

or

$$\text{C}[4n - 10n + 10 + 6n - 12]\,2^{n-2} + \text{D}[2n - 7] + 2\text{E} = 2^n + n$$

or

$$\text{C}[-2]\,2^{n-2} + \text{D}[2n - 7] + 2\text{E} = 2^n + n$$

or

$$-2\text{C} = 4 \quad 2\text{D} = 1 \quad \text{and} \quad -7\text{D} + 2\text{E} = 0$$

or

$$\text{C} = -2 \quad \text{D} = \frac{1}{2} \quad \text{and} \quad \text{E} = \frac{7}{4}.$$

Therefore, particular solution is

$$a_n = -2n\,2^n + \frac{n}{2} + \frac{7}{4}$$

Total solution is given by

$$a_n = \text{A}2^n + \text{B}3^n - 2n2^2 + \frac{n}{2} + \frac{7}{4} \qquad ...(4)$$

Using initial conditions for $n = 1$ and $n = 2$, we get the following two equations (5) and (6) respectively.

$$a_1 = A2^1 + B3^1 - 2 * 1 * 2^1 + \frac{1}{2} + \frac{7}{4}$$

or $$0 = 2A + 3B - 4 + \frac{1}{2} + \frac{7}{4} \qquad (\because \quad a_1 = 0)$$

or $$2A + 3B = \frac{7}{4} \qquad ...(5)$$

and $$a_2 = A2^2 + B3^2 - 2 * 2 * 2^2 + \frac{2}{2} + \frac{7}{4}$$

or $$10 = 4A + 9B - 16 + 1 + \frac{7}{4} \qquad (\because \quad a_2 = 10)$$

or $$4A + 9B = \frac{93}{4} \qquad ...(6)$$

$$A = -9 \qquad \text{and} \quad B = \frac{79}{12}$$

Solving equation (5) and (6), we get

Replacing the values of A and B in equation (4), we get the closed form formula for the given recurrence equation

$$a_n = -9 * 2^n + \frac{79}{12} 3^n - 2n\, 2^n + \frac{n}{2} + \frac{7}{4}.$$

Problem 2.29. *Solve the recurrence equation*

$$C_n = 3C_{n-1} - 2C_{n-2} \text{ for } n \geq 3 \text{ with } C_1 = 5 \text{ and } C_2 = 3.$$

Solution. The given equation is **homogeneous** so only homogeneous part of the solution is required.

The characteristic equation of the given recurrence equation is

$$x^2 - 3x + 2 = 0$$

or $$(x - 1)(x - 2) = 0$$

$\Rightarrow$ $$x = 1 \text{ and } x = 2.$$

The two roots of the characteristic equation are distinct. So the homogeneous solution is given by

$$C_n = u * 1^n + v * 2^n \qquad ...(1)$$

The equation (1) contains two undetermined coefficients, u and v, which are to be determined.

To find this, we use the given initial conditions for $n = 1$ and $n = 2$.

Since the values of C_1 and C_2 are given.

Putting $n = 1$ and $n = 2$ in equation (1), we get the following equation (2) and (3) respectively.

When $n = 1$

$$C_1 = u * 1 + v * 2 \qquad \text{or} \qquad 5 = u + 2v$$

or $u + 2v = 5$ ...(2)

and when $n = 2$,

$C_2 = u * 1^2 + v * 2^2$

or $3 = u + 4v$

or $u + 4v = 3$ ...(3)

By solving equation (2) and (3), we have

$u = 7$ and $v = -1$

Replacing u and v by their respective values in (1), we have the solution in the form of closed form formula.

Therefore $C_n = 7 - 2^n$.

Problem 2.30. *Solve the recurrence equation*

$$T_n = T_{\sqrt{n}} + C \log_2 n \quad \text{with } T = 1$$

by characteristic equation method.

Solution. The given equation can be written in standard form as

$$T_n - T_{\sqrt{n}} = C \log_2 n \qquad ...(1)$$

The homogeneous solution of (1) is given by the equation.

$$T_n - T_{\sqrt{n}} = 0 \qquad ...(2)$$

The characteristic equation of (2) is

$$x - \sqrt{x} = 0$$

$\Rightarrow$ $x^2 - x = 0$ or $x(x - 1) = 0$

$\Rightarrow$ $x = 0$ and $x = 1$

The homogeneous solution is then given by

$$T_n = A * 0^n + B * 1^n \qquad ...(3)$$

Guess the particular solution as

$$T_n = ZC \log_2 n$$

Using this particular solution in equation (1), we get

$$ZC \log_2 n - ZC \log_2 n^{1/2} = C \log_2 n$$

or

$$\left(Z - \frac{Z}{2}\right) C \log_2 n = C \log_2 n$$

$$Z - \frac{Z}{2} = 1 \quad \text{or} \quad Z = 2$$

$\therefore$ Particular solution is $T_n = 2C \log_2 n$

Combining homogeneous and particular solution, we get the complete solution, which is :

$$T_n = A * 0^n + b * 1^n + 2C \log_2 n$$

$\Rightarrow \quad T_n = B + 2C \log_2 n$

For $n = 1$, the initial condition is given.

Putting $n = 1$ in the above equation, we can find the value of undetermined coefficient B as below :

When $n = 1$, $T_1 = B + 2C \log_2 1$,

$\Rightarrow \quad 1 = B + 2C * 0$ $\qquad (\because \quad T_1 = 1 \text{ and } \log_2 1 = 0)$

$\Rightarrow \quad B = 1$

Replacing B by 1, in the complete solution,

We have $\quad T_n = 1 + 2C \log_2 n$.

Problem 2.31. *Solve the recurrence equation*

$$a_r = \sqrt{a_{r-1} + \sqrt{a_{r-2} + \sqrt{a_{r-3}} + \sqrt{........}}} \text{ with } a_0 = 4.$$

Solution. By squaring the given equation, we have

$$a_r^2 = a_{r-1} + \sqrt{a_{r-2} + \sqrt{a_{r-3} + \sqrt{a_{r-4}} + \sqrt{........}}}$$

$$\because \quad a_r^2 = a_{r-1} + a_{r-1} \qquad \left(\because \quad a_{r-1} = \sqrt{a_{r-2} + \sqrt{a_{r-3} + \sqrt{a_{r-4} + \sqrt{.......}}}}\right)$$

$$a_r^2 = 2a_{r-1}$$

$$\Rightarrow \quad a_r^2 - 2a_{r-1} = 0 \qquad ...(1)$$

Let us replace a_r with a new sequence term.

Let $\quad b_r = \log_2 a_r \qquad \Rightarrow \quad a_r = 2^{br}$

Substituting the value of a_r in terms of the new sequence variable b_r in equation (1), we get

$$(2^{br})^2 - 2 * 2^{br-1} = 0$$

$$2^{2br} - 2^{b_{r-1}+1} = 0$$

This is possible only if $2b_r - b_{r-1} + 1 = 0$ ...(2)

With initial condition $b_0 = \log_2 a_0 = \log_2 4 = 2$

$\Rightarrow \quad b_0 = 2$

Now, to solve the equation (2), which is of first order, linear and non-homogeneous.

The characteristic equation of (2) is

$$2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

$\therefore$ Homogeneous solution is

$$b_r = A\left(\frac{1}{2}\right)^r \qquad ...(3)$$

For particular solution, we guess (Assume) b_r = B. Substituting this value for sequence term b_r and b_{r-1} in equation (2), we have

$$2B - B = 1 \quad \Rightarrow \quad B = 1$$

∴ Particular solution is $b_r = 1$

Then, the total solution is

$$b_r = A\left(\frac{1}{2}\right)^r + 1$$

Using the initial condition $b_0 = 2$, we get A = 1

∴ The solution to the changed equation is

$$b_r = \left(\frac{1}{2}\right)^r + 1$$

Since $b_r = \log_2 a_r$, $a_r = 2^{br}$

∴ Final solution is given by

$$a_r = 2^{\left(\frac{1}{2}\right)^r + 1}.$$

Problem 2.32. *Solve the recurrence equation*

$a_n + 9a_{n-2} = 0$ with initial condition $a_0 = 0$ and $a_1 = 1$.

Solution. The characteristic equation for the given recurrence equation is given by

$$x^2 + 9 = 0 \quad \Rightarrow \quad x^2 = -9$$

$$\Rightarrow \quad x = \pm 3i$$

∴ The homogeneous solution is

$$a_n = A(-3i)^n + B(3i)^n$$

When the values of the coefficients are determined at the very end, the resulting sequence values a_n will be real.

Using the initial condition for $n = 0$ and $n = 1$, we get the following two equations.

$$0 = A(-3i)^n + B(3i)^0$$

$$\Rightarrow \quad A + B = 0 \qquad ...(1)$$

and

$$1 = A(-3i)^1 + B(3i)^1$$

$$\Rightarrow \quad -3iA + 3iB = 1 \qquad ...(2)$$

Now solving equation (1) and (2), we get values of A and B which are

$$A = \frac{-1}{6i} = \frac{i}{6} \quad \text{and} \quad B = \frac{-i}{6}$$

Substituting the value of A and B in homogeneous solution we get

$$a_n = \frac{i}{6}(-3i)^n + \frac{-i}{6}(3i)^n.$$

Problem 2.33. *Find an explicit formula for the sequence defined by $C_n = 3C_{n-1} - 2C_{n-2}$ with initial conditions*

$$C_1 = 5 \text{ and } C_2 = 3.$$

Solution. The recurrence relation $C_n = 3C_{n-1} - 2C_{n-2}$ is a linear homogeneous relation of degree 2.

Its associated equation is $x^2 = 3x - 2$.

Rewriting this as $x^2 - 3x - 2 = 0$. We see there are two roots, 1 and 2.

We can find u and v so that $C_1 = u(1) + v(2)$ and $C_2 = u(1)^2 + v(2)^2$.

Solving this 2×2 system yields $u = 7$ and $v = -1$.

We have $C_n = 7.1^n + (-1) . 2^n$ or $C_n = 7 - 2^n$.

Note that using $C_n = 3C_{n-1} - 2C_{n-2}$ with initial conditions $C_1 = 5$ and $C_2 = 3$ gives 5, 3, – 1, – 9 as the first four terms of the sequence.

The formula $C_n = 7 - 2^n$ also produces 5, 3, – 1, – 9 as the first four terms.

Problem 2.34. *Solve the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$ with initial conditions $d_1 = 1.5$ and $d_2 = 3$.*

Solution. The associated equation for this linear homogeneous relation is

$$x^2 - 2x + 1 = 0$$

This equation has one (multiple) root, 1.

Thus, $d_n = u(1)^n + vn(1)^n$.

Using this formula and the initial conditions,

$$d_1 = 1.5 = u + v(1) \text{ and}$$
$$d_2 = 3 = u + v(2)$$

We find that $u = 0$ and $v = 1.5$, then

$$d_n = 1.5\, n.$$

Problem 2.35. *The recurrence relation $f_n = f_{n-1} + f_{n-2}$, $f_1 = f_2 = 1$, defines the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ...*

The initial conditions are $f_1 = 1$ and $f_2 = 1$,

$f_n \le \left(\frac{5}{3}\right)^n$. *This gives a bound on how fast the Fibonacci numbers grow.*

Solution. (By strong induction)

Basic step : Here n_0 is 1, P(1) is $1 \le \frac{5}{3}$ and this is clearly true.

Induction step : We use P(j), $j \le k$ to show

$$P(k+1) : f_{k+1} \le \left(\frac{5}{3}\right)^{k+1}.$$

Consider the left-hand side of P(k + 1)

$$f_{k+1} = f_k + f_{k-1} \leq \left(\frac{5}{3}\right)^k + \left(\frac{5}{3}\right)^{k-1}$$

$$= \left(\frac{5}{3}\right)^{k-1} \left(\frac{5}{3} + 1\right)$$

$$= \left(\frac{5}{3}\right)^{k-1} \left(\frac{8}{3}\right)$$

$$< \left(\frac{5}{3}\right)^{k-1} \left(\frac{5}{3}\right)^2$$

$$= \left(\frac{5}{3}\right)^{k+1}$$, the right-hand side of P(k + 1).

Problem 2.36. *Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$ given $a_0 = 1$, $a_1 = 4$.*

Solution. The characteristic polynomial, $x^2 - 5x + 6$, has distinct roots $x_1 = 2$, $x_2 = 3$.

The solution is given by $a_n = C_1(2^n) + C_2(3^n)$.

Since $a_0 = 1$, we must have $C_1(2^0) + C_2(3^0) = 1$, and

since $a_1 = 4$. We have $C_1(2^1) + C_2(3^1) = 4$.

Therefore $C_1 + C_2 = 1$

$2C_1 + 3C_2 = 4$.

Solving, we have $C_1 = -1$, $C_2 = 2$.

The solution is $a_n = -2^n + 2(3^n)$.

Problem 2.37. *Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, $n \geq 2$ with initial conditions $a_0 = 1$, $a_1 = 4$.*

Solution. The characteristic polynomial, $x^2 - 4x + 4$, has the repeated root $x = 2$.

Hence, the solution is $a_n = C_1(2^n) + C_2n(2^n)$.

The initial conditions yield $C_1 = 1$, $2C_1 + 2C_2 = 4$. So $C_2 = 1$.

Thus, $a_n = 2^n + n(2^n) = (n + 1)2^n$.

Problem 2.38. *Find a formula for the nth term of the Fibonacci sequence.*

Solution. To simplify the algebra which follows, we take for initial conditions

$a_0 = a_1 = 1$ rather than $a_1 = a_2 = 1$.

Hence, the recurrence relation to be solved is

$a_n = a_{n-1} + a_{n-2}$, $n \geq 2$.

We must remember that the nth term will be a_{n-1}.

The characteristic polynomial, $x^2 - x - 1$, has distinct root $\frac{1 \pm \sqrt{5}}{2}$.

Hence, the solution to our recurrence relation is

$$a_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The initial conditions give $C_1 + C_2 = 1$

$$C_1 \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Yielding $C_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)$ and $C_2 = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)$.

Thus, the solution is

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

The nth term of the Fibonacci sequence is

$$a_{n-1} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Problem 2.39. *Solve* $a_n = 2a_{n-1} + 3a_{n-2} + 5^n$, $n \geq 2$, *given* $a_0 = -2$, $a_1 = 1$.

Solution. In looking for a particular solution, it seems reasonable to try $P_n = a(5^n)$. Substituting into the recurrence relation, we get

$$a(5^n) = 2a(5^{n-1}) + 3a(5^{n-2}) + 5^n$$

$$25a = 10a + 3a + 25$$

$$12a = 25, \quad \text{so } a = \frac{25}{12}$$

and $P_n = \frac{25}{12}(5^n)$ is a particular solution.

Next we solve the homogeneous recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}$$

The characteristic polynomial, $x^2 - 2x - 3$, has disinct roots -1 and 3.

The solution $q_n = C_1(-1)^n + C_2(3^n)$.

The given recurrence relation has the solution

$$P_n + q_n = \frac{25}{12}(5^n) + C_1(-1)^n + C_2(3^n)$$

The initial conditions give

$$a_0 = -2 = \frac{25}{12} + C_1 + C_2$$

$$a_1 = 1 = \frac{25}{12}(5) - C_1 + 3C_2.$$

Hence $C_1 = -\frac{17}{24}$, $C_2 = -\frac{27}{8}$.

The solution is $a_n = \frac{25}{12}(5^n) - \frac{17}{24}(-1)^n - \frac{27}{8}(3^n)$.

Problem 2.40. *Solve the recurrence relation $a_n = -3a_{n-1} + n$, $n \geq 1$ where $a_0 = 1$.*

Solution. Since $f(n) = n$ is linear, we try a linear function for P_n.

That is, we set $P_n = a + b_n$ and attempt to determine a and b.

Putting this expression for P_n in the given recurrence relation, we obtain

$$a + b_n = -3[a + b(n-1)] + n$$
$$= -3a + 3b + (1 - 3b)\,n.$$

This equation will hold if $a = -3a + 3b$ and $b = 1 - 3b$, which is the same as $a = \frac{3}{16}$, $b = \frac{1}{4}$

We conclude that $P_n = \frac{3}{16} + \frac{1}{4}n$ is a particular.

Solution to the recurrence, ignoring initial conditions.

The corresponding homogeneous recurrence relation in this case is $a_n = -3a_{n-1}$, whose characteristic polynomial is $x^2 + 3x$.

The characteristic roots are -3 and 0.

The solution to the homogeneous recurrence relation is $q_n = C_1(-3)^n + C_2(0^n) = C_1(-3)^n$.

Thus, $P_n + q_n = \frac{3}{16} + \frac{1}{4}n + C_1(-3)^n$.

Since $a_0 = 1$, $\frac{3}{16} + \frac{1}{4}(0) + C_1(-3)^0 = 1$

That is, $\frac{3}{16} + C_1 = 1$.

Thus, $C_1 = \frac{13}{16}$ and the solution is $a_n = \frac{3}{16} + \frac{1}{4}n + \frac{13}{16}(-3)^n$.

Problem 2.41. *Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial condition $a_0 = 1$ and $a_1 = -1$.*

Solution. The associated homogeneous recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \qquad ...(1)$$

Let $a_n = r^n$ be a solution of (1)

The characteristic equation is $r^2 - 5r + 6 = 0$

$\Rightarrow \quad r = 3, 2$

So the solution of (1) is $a_n^{(h)} = C_1\, 3^n + C_2\, 2^n$.

To find the particular solution of the given equation, let $a_n^{(P)} = A$. Substituting in the given equation

$$A - 5A + 6 = 2 \quad \Rightarrow \quad A = 1.$$

$a_n^{(P)} = 1$ which is a particular solution.

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(P)} = C_1\, 3^n + C_2\, 2^n + 1$ ...(2)

To find C_1 and C_2, put $n = 0$ and $n = 1$ in (2)

or $\quad a_0 = C_1 + C_2 + 1$

$1 = C_1 + C_2 + 1$

$\Rightarrow \quad C_1 + C_2 = 0$ ...(3)

Again $\quad a_1 = 3C_1 + 2C_2 + 1$

$-1 = 3C_1 + 2C_2 + 1$

or $\quad 3C_1 + 2C_2 = -2$ ...(4)

Solving (3) and (4), we get $C_1 = -2$ and $C_2 = 2$

Putting the values of C_1 and C_2 in (3), the required solution is

$$a_n = -2.3^n + 2.2^n + 1.$$

Problem 2.42. *What form does a particular solution of the linear homogeneous recurrence relation* $a_{n+2} - 6a_{n+1} + 9a_n = f(n)$ *have*

When $f(n) = 3^n, f(n) = n \,.\, 3^n$ *and* $f(n) = (n^2 + 1)\, 3^n$?

Solution. Let $a_n = r^n$ be a solution of the associated homogeneous recurrence relation.

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

The characteristic equation $r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0$,

has a single root 3 of multiplicity two.

Thus if $f(n) = 3^n$, the particular solution has the form $A_0 n^2 3^n$.

If $f(n) = n\, 3^n$, the particular solution has the form $n^2\,(A_0 + A_1 n)\, 3^n$.

If $f(n) = a(n^2 + 1)\, 3^n$, the particular solution has the form $n^2\,(A_0 + A_1 n + A_2 n^2)\, 3^n$.

Problem 2.43. *Solve* $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$.

Solution. Let $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_{n+2} - 4a_{n+1} + 4a_n = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r - 2)^2 = 0$$

has a single root 2 of multiplicity two. So the solution of associated homogeneous relation is

$a_n^h = (C_1 + C_2 n)\, 2^n$.

To find the particular solution of the given relation, we note $b = 2$ is a characteristic equation with multiplicity $s = 2$.

So, the particular solution has the form $a_n^{(P)} = A_0 n^2 \,.\, 2^n$.

Substituting in the given relation, we get

$$A_0(n+2)^2\, 2^{n+2} - 4\, A_0(n+1)^2\, 2^{n+1} + 4\, A_0 n^2 \,.\, 2^n = 2^n$$

$\Rightarrow \quad 4A_0(n+2)^2 - 8A_0(n+1)^2 + 4A_0 n^2 = 1 \qquad \Rightarrow \quad A_0 = \dfrac{1}{8}$

Hence the general solution is

$$a_n = a_n^{(h)} + a_n^{(P)} = (C_1 + C_2 n)\, 2^n + \left(\frac{1}{8}\right) n^2 \,.\, 2^n.$$

Problem 2.44. *Solve the following* $y_{n+2} - y_{n+1} - 2y_n = n^2$.

Solution. Substituting $y_n = r^n$ in the associated homogeneous recurrence relation.

The characteristic equation is $r^2 - r - 2 = 0$

$\Rightarrow \quad (r+1)(r-2) = 0$

This gives $r = -1,\ r = 2$

The solution of the associated homogeneous recurrence relation is $y_b = C_1(-1)^n + C_2 \,.\, 2^n$.

Let the particular solution of the given equation is

$$A_n = A_0 + A_1 n + A_2 n^2 \qquad \text{(since } f(n) \text{ is a polynomial of degree 2)}$$

Substituting in the given equation, we have

$$A_0 + A_1(n+2) + A_2(n+2)^2 - [A_0 + A_1(n+1) + A_2(n+1)^2] - 2(A_0 + A_1 n + A_2 n^2) = n^2$$

$$(-2A_0 + A_1 + A_2) + (-2A_1 + 2A_2)\, n - 2A_2 n^2 = n^2$$

On comparing the coefficients of like powers of n,

We have
$$-2A_0 + A_1 + 3A_2 = 0 \qquad \text{...(1)}$$
$$-2A_1 + 2A_2 = 0 \qquad \text{...(2)}$$
$$-2A_2 = 1 \qquad \text{...(3)}$$

From (3) $\quad A_2 = -\dfrac{1}{2}$

We have from (2) $\quad A_1 = A_2 = -\dfrac{1}{2}$

From (1) $\quad -2A_0 - \dfrac{1}{2} - \dfrac{3}{2} = 0 \qquad \Rightarrow \quad A_0 = -1$

Therefore, particular solution of given recurrence relation is

$$a_n^{(P)} = -1 - \left(\frac{1}{2}\right)n - \left(\frac{1}{2}\right)n^2$$

Hence the general solution of the given recurrence relation is

$$a_n = C_1(-1)^n + C_2 \,.\, 2^n - 1 - \frac{n}{2} - \left(\frac{1}{2}\right)n^2.$$

Problem 2.45. *Solve the recurrence relation*

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$$

Solution. Let $a_n = r^n$ be a solution of the given equation.

The characteristic equation is

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$\Rightarrow \quad r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 = 0$$

$$r^2(r-2) - 6r(r-2) + 9(r-2) = 0$$

$$(r-2)(r^2 - 6r + 9) = 0$$

$$(r-2)(r-3)^2 = 0$$

which gives $r = 2, 3, 3$.

So the general solution is

$$a_n = (b_1 + b_2 n)\, 3^n + b_2 2^n.$$

Problem 2.46. *Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.*

Solution. The given recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

i.e., $$a_n - a_{n-1} - 2a_{n-2} = 0 \qquad ...(1)$$

is a 2nd order linear homogeneous recurrence relation with constant coefficients.

Let, $a_n = r^n$ is a solution of (1).

The characteristic equation is

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

or $r = 2, -1$ distinct real roots.

So the general solution is $a_n = b_1(2)^n + b_2(-1)^n$

Again $a_0 = 0$ implies $b_1 + b_2 = 0$

Add $a_1 = 1$ implies $2b_1 - b_2 = 1$

The solution of these two equations are $b_1 = \frac{1}{3}$ and $b_2 = -\frac{1}{3}$

Hence the explicit solution is given by $a_n = \left(\frac{1}{3}\right) 2^n - \left(\frac{1}{3}\right)(-1)^n.$

Problem 2.47. *Solve the recurrence relation of the Fibonacci sequence of numbers* $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ *with the initial condition* $f_0 = 1, f_1 = 1$.

Solution. Let $f_n = r^n$ is a solution of the given equation, the characteristic equation is $r^2 - r - 1 = 0$

$$r = \frac{(1 \pm \sqrt{5})}{2}$$

i.e., $$r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$$ are two distinct roots.

So the general solution is

$$a_n = b_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + b_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

Again $a_0 = 1$ implies $b_1 + b_2 = 1$

$$a_1 = 1 \quad \text{implies} \quad b_1 \left(\frac{1+\sqrt{5}}{2}\right) + b_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solving these two equations we get

$$b_1 = \frac{1}{\sqrt{5}},\ b_2 = -\frac{1}{\sqrt{5}}$$

Thus the nth Fibonacci number is given explicitly by

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Problem 2.48. *Solve the recurrence relation* $a_n = 4(a_{n-1} - a_{n-2})$.

Solution. The given relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$ ...(1)

Let $a_n = r^n$ is a solution of (1)

Then the characteristic equation is $r^2 - 4r + 4 = 0$ which gives $r = 2, 2$

Thus the general solution is $a_n = (b_1 + nb_2)\, 2^n$

So, $a_0 = b_1$ and $a_1 = (b_1 + b_2)^2$

Now $a_1 = 1$ gives $b_1 = 1$ and $a_1 = 1$ gives $2(b_1 + b_2) = 1$ $\Rightarrow$ $b_2 = -\frac{1}{2}$

So, the general solution is $a_n = \left(1 - \frac{1}{2}n\right) 2^n$.

2.4. THE METHOD OF GENERATING FUNCTIONS

One of the uses of generating function method is to find the closed form formula for a recurrence (equation) relation. Before using this method, ensure that the given recurrence equation is in linear form.

A non-linear recurrence equation cannot be solved by the Generating Function Method. Use substitution of **variable technique** to convert a **non-linear recurrence (equation) relation into linear.**

Solving a recurrence (equation) relation using generating function method involves two **steps process.**

Step 1 : Find generating function for the sequence for which the general term is given by recurrence (equation) relation.

Step 2 : Find coefficient of x^2 or $\frac{x^2}{n!}$ depending upon whether the generating function is binomial or exponential to get a_n, the general term of the sequence.

The value so obtained will be an algebraic formula for a_n, expressed in terms of n which is the position of a_n is sequence.

A generating function is a polynomial expression of the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$$

in which the coefficients a_i are all zero after a certain point, a generating function usually has infinitely many non-zero terms. There is an obvious correspondence between generating functions and sequences.

$$a_0, a_1, a_2, ...$$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ... \quad \leftrightarrow \quad a_0, a_1, a_2, a_3, ...$$

*If $f(x) = a_0 + a_1x + a_2x^2 + ...$

and $g(x) = b_0 + b_1x + b_2x^2 + ...$ then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ...$$

$$f(x)g(x) = (a_0b_0) + (a_1b_0 + a_0b_1)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + ...$$

The coefficient of x^n in the product $f(x)g(x)$ is the infinite sum

$$a_0b_n + a_1b_{n-1} + a_2b_{n-2} + ... + a_nb_0.$$

Problem 2.49. *If $f(x) = 1 + x + x^2 + ... + x^n + ...$ and*

$$g(x) = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + ...,$$

find $f(x) + g(x)$ and $f(x)g(x)$.

Solution. $f(x) + g(x)$

$$= (1 + x + x^2 + ... + x^n + ...) + (1 - x + x^2 - x^3 + ... + (-1)^n x^n + ...)$$

$$= (1 + 1) + (1 - 1)x + (1 + 1)x^2 + ... + (1 + (-1)^n) x^n + ...$$

$$= 2 + 2x^2 + 2x^4 + ...$$

$$f(x)g(x) = (1 + x + x^2 + ... + x^n + ...) . (1 - x + x^2 - x^3 + ... + (-1)^n x^n + ...)$$

$$= 1 + [1(-1) + 1(1)]x + [1(1) + 1(-1) + 1(1)] x^2 + ...$$

$$= 1 + x^2 + x^4 + x^6 + ...$$

Problem 2.50. *Solve the recurrence relation $a_n = 3a_{n-1}$, $n \geq 1$, given $a_0 = 1$.*

Solution. Consider the generating function

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$

of the sequence $a_0, a_1, a_2, ...$ multiplying by $3x$ and writing the product $3xf(x)$ below $f(x)$ so that terms involving x^n match, we obtain

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$
$$3xf(x) = 3a_0x + 3a_1x^2 + ... + 3a_{n-1}x^n + ...$$

Subtracting gives

$$f(x) - 3xf(x) = a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)\, x^2 + ... + (a_n - 3a_{n-1})\, x^n + ...$$

Since $a_0 = 1$, $a_1 = 3a_0$.

In general, $a_n = 3a_{n-1}$, this says that.

$$(1 - 3x)\, f(x) = 1.$$

Thus, $f(x) = \dfrac{1}{1-3x}$

We have $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ ...(∗)

Using (∗), $f(x) = 1 + 3x + (3x)^2 + ... + (3x)^n + ...$

$$= 1 + 3x + 9x^2 + ... + 3^nx^n + ...$$

We conclude that a_n, which is the coefficient of x^n in $f(x)$, must equal 3^n.

We have $a_n = 3^n$ as the solution to our recurrence relation.

Problem 2.51. *Solve the recurrence relation* $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$, *given* $a_0 = 3$, $a_1 = -2$.

Solution. Letting $f(x)$ be the generating function of the sequence in question.

We have
$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$
$$2xf(x) = 2a_0x + 2a_1x^2 + ... + 2a_{n-1}x^n + ...$$
$$x^2 f(x) = a_0x^2 + ... + a_{n-2}x^n + ...$$

Therefore, $f(x) - 2xf(x) + x^2f(x)$

$$= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 + a_0)\, x^2 + ... + (a_n - 2a_{n-1} + a_{n-2})\, x^n + ...$$
$$= 3 - 8x.$$

Since $a_0 = 3$, $a_1 = -2$ and $a_n - 2a_{n-1} + a_{n-2} = 0$ for $n \geq 2$.

So, $(1 - 2x + x^2)\, f(x) = 3 - 8x$

$$(1 - x)^2 f(x) = 3 - 8x$$

$$\Rightarrow \qquad f(x) = \frac{1}{(1-x)^2}\,(3 - 8x)$$
$$= (1 + 2x + 3x^2 + ... + (n + 1)x^n + ...)\,(3 - 8x)$$
$$= 3 - 2x - 7x^2 - 12x^3 + ... + [3(n + 1) - 8n]x^n + ...$$
$$= 3 - 2x - 7x^2 - 12x^3 + ... + (-5n + 3)\, x^n + ...$$

Therefore $a_n = 3 - 5x$ is the desired solution.

Problem 2.52. *Solve the recurrence relation*

$$a_n = -3a_{n-1} + 10a_{n-2},\ n \geq 2,\ \text{given } a_0 = 1,\ a_1 = 4.$$

Solution. Letting $f(x)$ be the generating function of the sequence in question.

We have $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$

$3xf(x) = 3a_0x + 3a_1x^2 + ... + 3a_{n-1}x^2 + ...$

$10x^2f(x) = 10a_0x^2 + ... + 10a_{n-2}x^n + ...$

Therefore, $f(x) = 3xf(x) - 10x^2 f(x)$

$= a_0 + (a_1 + 3a_0)\, x + (a_2 + 3a_1 - 10a_0)\, x^2 + ...$

$+ (a_n + 3a_{n-1} - 10a_{n-2})\, x^n + ...$

$= 1 + 7x$

Since $a_0 = 1$, $a_1 = 4$ and $a_n + 3a_{n-1} - 10a_{n-2} = 0$ for $n \geq 2$.

So $(1 + 3x - 10x^2) f(x) = 1 + 7x$

$$\Rightarrow \qquad f(x) = \frac{1+7x}{1+3x-10x^2} = \frac{1+7x}{(1+5x)(1-2x)}$$

At this point, it is useful to recall the method of partial fractions.

We get $\dfrac{1}{(1+5x)(1-2x)} = \dfrac{A}{1+5x} + \dfrac{B}{1-2x} = \dfrac{A(1-2x)+B(1+5x)}{(1+5x)(1-2x)}$

Equating numerators,

$$1 = (A + B) + (-2A + 5B)\, x$$

$$\Rightarrow \qquad A + B = 1$$

$$-2A + 5B = 0$$

Solving for A and B, we get $A = \dfrac{5}{7}$, $B = \dfrac{2}{7}$

Therefore, $\dfrac{1}{(1+5x)(1-2x)} = \dfrac{5}{7}\left(\dfrac{1}{1+5x}\right) + \dfrac{2}{7}\left(\dfrac{1}{1-2x}\right)$

So $f(x) = \dfrac{1+7x}{(1+5x)(1-2x)}$

$$= \frac{5}{7}\left(\frac{1}{1+5x}\right)(1 + 7x) + \frac{2}{7}\left(\frac{1}{1-2x}\right)(1 + 7x)$$

$$= \frac{5}{7}\,[1 + (-5x) + (-5x)^2 + ...]\,(1 + 7x) + \frac{2}{7}\,[1 + (2x) + (2x)^2 + ...]\,(1 + 7x)$$

$$= \frac{5}{7}\,[1 - 5x + 25x^2 + ... + (-5)^n]\, x^n)\,(1 + 7x)] + \frac{2}{7}\,[1 + 2x + 4x^2 + ...\, 2^nx^n + ...]]\,(1 + 7x).$$

$$= \frac{5}{7}(1 + 2x - 10x^2 + ... + [(-5)^n + 7(-5)^{n-1}]\, x^n + ...)$$

$$+ \frac{2}{7}\,(1 + 9x + 18x^2 + ... + [2^n + 7(2^{n-1})\, x^n\,]+ ......)$$

$$= \frac{5}{7}(1 + 2x - 10x^2 + ... + 2(-5)^{n-1}x^n + ...)$$

$$+ \frac{2}{7}(1 + 9x + 18x^2 + ... + 9(2^{n-1})\, x^n + ...)$$

$$= 1 + 4x - 2x^2 + ... + \left[-\frac{2}{7}(-5)^n + \frac{9}{7}(2^n)\right] x^n + ...$$

Hence $a_n = -\frac{2}{7}(-5)^n + \frac{9}{7}(2^n)$ is the desired solution.

Problem 2.53. *Solve the recurrence relation $a_n = -a_{n-1} + 2n - 3$, $n \geq 1$, given $a_0 = 1$.*

Solution. Let $f(x)$ be the generating function of the sequence $a_0, a_1, a_2, ...$

Then $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$

$xf(x) = a_0x + a_1x^2 + ... + a_{n-1}x^n + ...$

Therefore, $f(x) + xf(x)$

$$= a_0 + (a_1 + a_0)x + (a_2 + a_1)x^2 + ... + (a_n + a_{n-1})x^n + ...$$

given that, $a_0 = 1$ and $a_n + a_{n-1} = 2n - 3$.

Thus $a_1 + a_0 = 2(1) - 3 = -1$

$a_2 + a_1 = 2(2) - 3 = 1$, and so on.

We obtain $(1 + x)\, f(x) = 1 - x + x^2 + ... + (2n - 3)x^n + ...$

So $f(x) = \dfrac{1}{1+x}\,(1 - x + x^2 + ... + (2x - 3)\, x^n + ...)$

$$= (1 + (-x) + (-x)^2 + ... + (-x)^n + ...) \,.\, (1 - x + x^2 + ... + (2n - 3)x^n + ...)$$

$$= (1 - x + x^2 - x^3 + ... + (-x)^n + ...) \,.\, (1 - x + x^2 + ... + (2n - 3)\, x^n + ...)$$

$$= 1 - 2x + 3x^2 + [(2n - 3) - (2n - 5) + ... + (-1)^{n-1}(-1) + (-1)^n]\, x^n + ...$$

Now a_n is the coefficient of x^n, the term in the square brackets.

$$a_n = \left\{\sum_{k=0}^{n-1} (-1)^k\, [2n - (2k + 3)]\right\} + (-1)^n$$

$$= (2n - 3)\sum_{k=0}^{n-1} (-1)^k - 2\sum_{k=0}^{n-1} (-1)^k\, k + (-1)^n.$$

We have it to you to verify that

$$\sum_{k=0}^{n-1} (-1)^k\, k = \begin{cases} -\dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

So, if n is even, $a_n = 0 - 2\left(-\frac{n}{2}\right) + 1 = n + 1,$

if n is odd, $a_n = (2n - 3) - 2\left(\frac{n-1}{2}\right) - 1$

$$= 2n - 3 - n + 1 - 1 = n - 3$$

The solution is $a_n = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$

Note that this solution could also be written as

$$a_n = 2(-1)^n + n - 1.$$

Problem 2.54. *Suppose a is a real number. Show that* $\frac{1}{1-ax}$ *is the generating function for a certain geometric sequence.*

Solution. We have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$ (1)

Replacing x by ax in (1), we see that

$$\frac{1}{1-ax} = 1 + (ax) + (ax)^2 + (ax)^3 + \ldots\ldots$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \ldots\ldots$$

From this, we see that $\frac{1}{1-ax}$ is the generating function for the sequence $1, a, a^2, a^3, \ldots\ldots$ which is the geometric sequence with first term 1 and common ratio a.

Problem 2.55. *Prove that* $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n+1)x^n + \ldots\ldots$.

Solution. $(1 - x)^2 (1 + 2x + 3x^2 + \ldots\ldots + (n + 1)x^n + \ldots\ldots)$

$$= (1 - 2x + x^2)(1 + 2x + 3x^2 + \ldots\ldots + (n + 1)x^n + \ldots\ldots)$$

$$= 1 + [1(2) - 2(1)]x + [1(3) - 2(2) + 1(1)]x^2 + \ldots\ldots$$

$$\ldots\ldots + [1(n + 1) - 2(n) + 1(n - 1)]x^n + \ldots\ldots.$$

$$= 1, \quad \text{since } n + 1 - 2n + n = 0.$$

Problem 2.56. *Solve the following recurrence equation using generating function*

$$a_n = a_{n-1} + 2(n - 1),\ a_0 = 1.$$

Solution. We will solve this using binomial generating function. Let $f(x)$ be the binomial generating function for the sequence a of which general term is given by the given recurrence equation.

Therefore, we have $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n.$

or $$f(x) = a_0 + \sum_{n=1}^{\infty} [a_{n-1} + 2(n-1)]x^n .$$

$$= a_0 + \sum_{n=1}^{\infty} a_{n-1}x^n + 2 \sum_{n=1}^{\infty} (n-1)x^n$$

$$= a_0 + x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 2x \sum_{n=1}^{\infty} (n-1)x^{n-1}$$

$$= a_0 + x \sum_{m=0}^{\infty} a_m x^m + 2x\,[0 * x^0 + 1 * x^1 + 2 * x^2 + ...] \qquad [n - 1 = m]$$

$$= a_0 + x\,f(x) + 2x^2\,[1 + 2x + 3x^2 + ...]$$

$\because \quad f(x) = \sum_{m=0}^{\infty} a_m x^m$

or $$(1 - x)\,f(x) = a_0 + \frac{2x^2}{(1-x)^2} = 1 + \frac{2x^2}{(1-x)^2} \qquad (\because \quad a_0 = 1)$$

$$\Rightarrow \quad f(x) = \frac{1 - 2x + 3x^2}{(1-x)^3}$$

$$f(x) = \frac{1}{(1-x)^3} - \frac{2x}{(1-x)^3} + \frac{3x^2}{(1-x)^3}$$

is the generating function of the equation.

The value of a_n is given by the coefficient of x^n is $f(x)$.

Therefore

Coeff. of x^n = coeff. of x^n in $\frac{1}{(1-x)^3}$ – $2x$ * coeff. of x^{n-1} in $\frac{1}{(1-x)^3}$ + $3x^2$ * coeff of x^{n-2} in $\frac{1}{(1-x)^3}$

$$= {}^{3+n-1}C_n - 2 * {}^{3+(n-1)-1}C_{n-1} + 3 * {}^{3+(n-2)-1}C_{n-2}$$

or $$a_n = {}^{n+2}C_n - 2 * {}^{n+1}C_{n-1} + 3 * {}^{n}C_{n-2}$$

$$= \frac{n^2 + 3n + 2}{2} - (n^2 + n) + \frac{3(n^2 - n)}{2}$$

$$= \frac{n^2 + 3n + 2 - 2n^2 - 2n + 3n^2 - 3n}{2}$$

$$= \frac{2n^2 - 2n + 2}{2}$$
$$= n^2 - n + 1.$$

Therefore, $a_n = n^2 - n + 1$.

Problem 2.57. *Solve the following recurrence equation using generating function*

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 2 \text{ and } f_0 = f_1 = 1.$$

Solution. We will solve this using binomial generating function.

Let $f(x)$ be the binomial generating function for the sequence f of which general term is given by the above recurrence equation.

Then, we have $f(x) = \sum_{n=0}^{\infty} f_n x^n$

$$= f_0 + f_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= f_0 + f_1 x + \sum_{n=2}^{\infty} [f_{n-1} + f_{n-2}]\, x^n$$

$$= f_0 + f_1 x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

$$= f_0 + f_0 x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \qquad (\because \ f_0 = f_1)$$

$$= f_0 + x \sum_{n=1}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} f_{n-2} x^{n-2}$$

$$= f_0 + x \sum_{n-1=0}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n-2=0}^{\infty} f_{n-2} x^{n-2}$$

$$= f_0 + x\, f(x) + x^2 f(x)$$
$$= 1 + x\, f(x) + x^2 f(x)$$

$$= \frac{1}{(1 - x - x^2)}$$ is the Generating Function of the equation.

Using partial fraction, we can write $f(x)$ as

$$f(x) = \frac{1}{\sqrt{5}} \left[\frac{A}{1 - x_1 x} - \frac{B}{1 - x_2 x} \right]$$

where $x_1 = \dfrac{1+\sqrt{5}}{2}$ and $x_2 = \dfrac{1-\sqrt{5}}{2}$

where $A = \dfrac{1+\sqrt{5}}{2}$ and $B = \dfrac{1-\sqrt{5}}{2}$

Therefore, coefficient of $x^n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$

i.e.,
$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Problem 2.58. *Solve the following recurrence equation using generating function*

$$a_n - 8a_{n-1} + 16a_{n-2} = 0 \quad \text{for } n \geq 4$$

and
$$a_2 = 16,\ a_3 = 80.$$

Solution. We will use binomial generating function to solve this problem.

Let $f(x)$ be the binomial generating function for the sequence a of which general term is given by the above recurrence equation. One thing to be noticed here is that the sequence term starts from $n = 2$, so summation will be used accordingly, $f(x)$ is given as,

$$f(x) = \sum_{n=n}^{\infty} a_n x^n$$

$$= a_2x^2 + a_3x^3 + \sum_{n=4}^{\infty} a_n x^n$$

$$= a_2x^2 + a_3x^3 + \sum_{n=4}^{\infty} [8a_{n-1} - 16a_{n-2}]\, x^n$$

$$= a_2x^2 + a_3x^3 + 8\sum_{n=4}^{\infty} a_{n-1}x^n - 16\sum_{n=4}^{\infty} a_{n-2}x^n$$

$$= a_2x^2 + a_3x^3 + 8\,[a_3x^4 + a_4x^5 + ...] - 16\,[a_2x^4 + a_3x^5 + ...]$$

$$= a_2x^2 + a_3x^3 + 8\,[a_2x^2 + a_3x^3 + ...]\,x - 8a_2x^3 - 16[a_2x^2 + a_3x^3 + ...]\,x^2$$

$$= a_2x^2 + a_3x^2 + 8x\,f(x) - 8a_2x^3 - 16x^2 f(x)$$

or $(1 - 8x + 16x^2)\,f(x) = a_2x^2 + a_3x^3 - 8a_2x^3.$

Now, substituting the value of a_2 and a_3 in the above equation, we get,

$$(1 - 8x + 16x^2)\,f(x) = 16x^2 + 80x^3 - 8 * 16x^3$$

$$\Rightarrow \quad f(x) = \frac{16x^2 - 48x^3}{1 - 8x + 16x^2} = 16x^2\,\frac{1-3x}{(1-4x)^2}$$

Therefore, $f(x) = 16x^2 \dfrac{1-3x}{(1-4x)^2}$

To find the general term a_n, we will have to find the coefficient of x^n from $f(x)$.

Using partial fraction method, we can write

$$\frac{1-3x}{(1-4x)^2} = \frac{A}{1-4x} + \frac{B}{(1-4x)^2}$$

Solving the above equation for A and B, we get A = 1, B = 1.

Therefore, the Generating Function can be written as

$$f(x) = 16x^2 \left[\frac{1}{1-4x} + \frac{x}{(1-4x)^2}\right]$$

Hence, coefficient of x^n in $f(x)$ is given as

16 ∗ (coefficient of x^{n-2} in first term + coefficient of x^{n-3} in second term).

This is equal to $16 * (4^{n-2} + (n-2)\, 4^{n-3})$

or $$a_n = 16 * 4^{n-2} \left[1 + \frac{n-2}{4}\right]$$

$$= 4^n \left[\frac{1}{2} + \frac{n}{4}\right].$$

Problem 2.59. *Solve the following recurrence equation using generating function*

$C_n = 3C_{n-1} - 2C_{n-2} = 0,$ *for* $n \geq 3,$

and $C_1 = 5,\ C_2 = 3.$

Solution. We will again use binomial generating function to solve this problem.

Let $f(x)$ be the binomial generating function for the sequence C of which general term is given by the above recurrence equation.

Since the sequence term starts from $n = 1$, so the summation will be used accordingly, $f(x)$ is given by,

$$f(x) = \sum_{n=1}^{\infty} C_n x^n = C_1x + C_2x^2 + \sum_{n=3}^{\infty} C_n x^n$$

$$= C_1x + C_2x^2 + \sum_{n=3}^{\infty} (3C_{n-1} - 2C_{n-2})x^n, \qquad \text{(by definition of } C_n)$$

$$= C_1x + C_2x^2 + 3\sum_{n=3}^{\infty} C_{n-1}x^n - 2\sum_{n=3}^{\infty} C_{n-2}x^n.$$

$$= C_1x + C_2x^2 + 3[C_2x^3 + C_3x^4 + C_4x^5 + ...] - 2[C_1x^3 + C_2x^4 + C_3x^5 + ...]$$

$$= C_1x + C_2x^2 + 3[C_1x + C_2x^2 + C_3x^3 + ...]x$$
$$- 3C_1x^2 - 2[C_1x + C_2x^2 + C_3x^3 + ...]x^2$$
$$= C_1x + C_2x^2 + 3x\,f(x) - 3C_1x^2 - 2x^2 f(x).$$

or $\quad (1 - 3x + 2x^2)\, f(x) = C_1x + C_2x^2 - 3C_1x^2$
$$= 5x + 3x^2 - 2 * 5x^2$$

Therefore, $\quad f(x) = \dfrac{5x - 12x^2}{(1-x)(1-2x)} = x\,\dfrac{5-12x}{(1-x)(1-2x)}$

To find the general term a_n, we shall have to find the coefficient of x^n from $f(x)$.

Using partial fraction method, we can write

$$\frac{5-12x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

Solving the above equation for A and B, we get A = 7, B = – 2

Therefore, the Generating Function can be written as

$$f(x) = x\left[\frac{7}{1-x} - \frac{2}{1-2x}\right]$$

Hence, coefficient of x^n in $f(x)$ is given as

(coefficient of x^{n-1} in first term + coefficient of x^{n-2} in second term)

This is equal to

$$7 - 2 * 2^{n-1}$$

or $\quad a_n = 7 - 2^n.$

Problem 2.60. *Solve the following recurrence equation using generating function*

$$a_n - 7a_{n-1} + 10a_{n-2} = 2^n, \text{ for } n \geq 2, \quad \text{and} \quad a_0 = 0,\ a_1 = 6.$$

Solution. Let $f(x)$ be the binomial generating function for the sequence a of which general term is given by the above recurrence equation.

Then, $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 x + a_1 x + \sum_{n=2}^{\infty} a_n x^n .$$

$$= a_0 x + a_1 x + \sum_{n=2}^{\infty} (7a_{n-1} = 10a_{n-2} + 2^n)\ x^n \qquad \text{(by def. } n \text{ of } a_n\text{)}$$

$$= a_0 x + a_1 x + 7\sum_{n=2}^{\infty} a_{n-1}x^n - 10\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 2^n x^n .$$

$$= a_0 x + a_1 x + 7x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} - 10x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \sum_{n=2}^{\infty} 2^n x^n .$$

$$= a_0 x + a_1 x + 7x\, f(x) - 10x^2 f(x) + \frac{4x^2}{1-2x} \qquad \left(\because \sum_{n=2}^{\infty} 2^n x^n = \frac{4x^2}{1-2x} \right)$$

or $$(1 - 7x + 10x^2)\, f(x) = a_0 x + a_1 x + \frac{4x^2}{1-2x}$$

$$= 6x + \frac{4x^2}{1-2x}$$

Therefore, $f(x) = \dfrac{6x - 8x^2}{(1-2x)^2\,(1-5x)} = x\, \dfrac{6-8x}{(1-2x)^2\,(1-5x)}$

We will have to find the coefficient of x^n from $f(x)$.
Using partial fraction method, we can write

$$\frac{6x - 8x^2}{(1-2x)^2\,(1-5x)} = \frac{A}{1-5x} + \frac{B}{1-2x} + \frac{Cx}{(1-2x)^2}$$

Solving the above equation for A, B and C, we get

$$A = \frac{22}{9}, \quad B = -\frac{22}{9} \quad \text{and } C = -\frac{4}{3}$$

Therefore, the Generating Function can be written as,

$$f(x) = \frac{22}{9}\left[\frac{1}{1-5x}\right] - \frac{22}{9}\left[\frac{1}{1-2x}\right] - \frac{4}{3}\left[\frac{x}{(1-2x)^2}\right]$$

Hence, coefficient of x^n in $f(x)$ is given as
(coefficient of x^n in first term + coefficient of x^n in second term + coefficient of x^n in third term).

This is equal to $\dfrac{22}{9} 5^n - \dfrac{22}{9} 2^n - \dfrac{4}{3} n 2^{n-1}$

Therefore, $a_n = \dfrac{22}{9}\,[5^n - 2^n] - \dfrac{2}{3} n 2^n.$

Problem 2.61. *Solve the following recurrence equation using exponential generating function*
$d_n = (n-1)(d_{n-1} + d_{n-2})$ *for* $n \geq 3$ *and* $d_1 = 0,\ d_2 = 1.$

Solution. Let $f(x)$ be the exponential generating function of the above recurrence relation, then $f(x)$ is given as

$$f(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} \qquad ...(1)$$

Differentiating (1) with respect to x, we get

$$f'(x) = \sum_{n=0}^{\infty} d_n \frac{x^{n-1}}{(n-1)!} \qquad ...(2)$$

Replacing d_n by its definition, in the equation (2), we get

$$f'(x) = \sum_{n=0}^{\infty} (n-1)\ (d_{n-1} + d_{n-2})\ \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} (n-1)\ d_{n-1}\ \frac{x^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty} (n-1)\ d_{n-2}\ \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} d_{n-1}\ \frac{x^{n-1}}{(n-2)!} + \sum_{n=0}^{\infty} d_{n-2}\ \frac{x^{n-1}}{(n-2)!}$$

$$= x \sum_{n=0}^{\infty} d_{n-1}\ \frac{x^{n-2}}{(n-2)!} + x \sum_{n=0}^{\infty} d_{n-2}\ \frac{x^{n-2}}{(n-2)!}$$

$$= x \left[d_0 \cdot \frac{x^{-1}}{(-1)!} + d_1 \cdot \frac{x^0}{(0)!} + d_2 \frac{x^1}{1!} + d_3 \cdot \frac{x^2}{2!} + ... \right]$$

$$+ x \left[d_0 + d_1 x + d_2 \cdot \frac{x^2}{2!} + d_3 \cdot \frac{x^3}{3!} + ... \right]$$

$$= xf'(x) + xf(x)$$

or $\quad (1 - x) f'(x) = xf(x)$

$$\Rightarrow \qquad \frac{f'(x)}{f(x)} = \frac{x}{1-x}$$

$$\Rightarrow \qquad \frac{f'(x)}{f(x)} = \frac{1}{1-x} - 1 \qquad ...(3)$$

Integrating both sides of equation (3), we get

$$\log f(x) = -\log(1 - x) - x$$

$$\log f(x)\ (1 - x) = -x$$

$$f(x)\ (1 - x) = e^{-x}$$

$$\Rightarrow \quad f(x) = \frac{1}{1-x} \cdot e^{-x}$$

This is the generating function for the given recurrence equation.

To find the general term from this generating function we will have to find the coefficient of $\dfrac{x^n}{n\,!}$,

which is given as $d_n = n\,!\left(1 - \dfrac{1}{1\,!} + \dfrac{2}{2\,!} - \ldots\ldots. (-1)^n \dfrac{1}{n\,!}\right)$.

Problem 2.62. *Use generating functions to solve the recurrence relation*

(i) $a_n = 3a_{n-1} + 2,\ a_0 = 1$

(ii) $a_n - 9a_{n-1} + 20a_{n-2} = 0,\ a_0 = -3,\ a_1 = -10$

(iii) $a_{n+2} - 2a_{n+1} + a_n = 2^{2n},\ a_0 = 2,\ a_1 = 1.$

Solution. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where G($x$) is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x\,G(x) + 2\left[\frac{1}{1-x} - 1\right]$$

Since $x\,G(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$

$$\therefore \quad G(x) - 3xG(x) = 1 + \frac{2x}{1-x} \qquad (a_0 = 1)$$

or $$G(x) = \frac{1+x}{(1-x)(1-3x)}$$

$$= \frac{2}{1-3x} - \frac{1}{1-x} \qquad \text{(by partial fraction)}$$

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = 2 \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n .$$

Hence $a_n = 2 \,.\, 3^n - 1$, which is the required solution.

(*ii*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where G($x$) is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

or $$[G(x) - a_0 - a_1 x] - 9x\,[G(x) - a_0] + 20x^2\,G(x) = 0$$

$G(x)\ [1 - 9x + 20x^2] = a_0 + a_1x + 9a_0x$

$$G(x) = \frac{a_0 + a_1x - 9a_0x}{1 - 9x + 20x^2}$$

$$= \frac{-3 - 10x + 27x}{1 - 9x + 20x^2}$$

$$= \frac{-3 + 17x}{(1 - 5x)(1 - 4x)} \qquad (\because \quad a_1 = -3 \text{ and } a_1 = -10)$$

or $$G(x) = \frac{2}{1 - 5x} - \frac{5}{1 - 4x} \qquad \text{(by partial fraction)}$$

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = 2\ \Sigma 5^n x^n - 5\ \Sigma 4^n x^n$$

Hence $a_n = 2.5^n - 5.4^n$, which is the required solution.

(*iii*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 1 to ∞, we get

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2 \sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2^n x_n$$

or $$\frac{G(x) - a_0 - a_1x}{x^2} - 2\left(\frac{G(x) - a_0}{x}\right) + G(x) = \frac{1}{1 - 2x}$$

$$\frac{G(x) - 2 - x}{x^2} - 2\left(\frac{G(x) - 2}{x}\right) + G(x) = \frac{1}{1 - 2x}$$

$$(x^2 - 2x + 1)\ G(x) = 2 + 3x + \frac{x^2}{1 - 2x}$$

$$G(x) = \frac{2}{(1 - x)^2} + \frac{3x}{(1 - x)^2} + \frac{x^2}{(1 - 2x)(1 - x)^2}$$

By partial fraction, $\frac{x^2}{(1 - 2x)(1 - x^2)} = \frac{1}{1 - 2x} - \frac{1}{(1 - x)^2}$

$$G(x) = \frac{1}{(1 - x)^2} + \frac{3x}{(1 - x)^2} + \frac{1}{1 - 2x}$$

$$\Sigma a_n x^n = \Sigma(n + 1)x^n + 3\Sigma n x^n + \Sigma 2^n x^n$$

Hence $a_n = (n + 1) + 3n + 2^n$
$= 1 + 4n + 2^n$.

Problem 2.63. *Find the sequence $\{y_x\}$ having the generating function G given by*

$$G(x) = \frac{3}{1-x} + \frac{1}{1-2x}.$$

Solution. If the generating function

$$G(x) = \frac{1}{1-x}, \text{ the sequence is } \{y_x\} = \{1\}$$

Therefore, if the generating function is

$$G(x) = \frac{3}{1-x}, \text{ the sequence is } \{y_n\} = \{3\}$$

Also, if the generating function is

$$G(x) = \frac{1}{1-ax}, \text{ the sequence is } \{y_n\} = \{a^n\}$$

Therefore, if the generating function is

$$G(x) = \frac{1}{1-2x}, \text{ the sequence is } \{y_n\} = \{2^n\}$$

Hence the required sequence is $\{y_n\} = \{3 + 2^n\}$

Second Method :

We have $$G(x) = \frac{3}{1-x} + \frac{1}{1-2x}$$
$$= 3(1-x)^{-1} + (1-2x)^{-1}$$
$$= 3(1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots) + (1 + 2x + 2^2x^2 + \ldots\ldots + 2^nx^n + \ldots\ldots)$$
$$= (3 + 1) + (3 + 2)x + (3 + 2^2) + \ldots\ldots + (3 + 2^n)\, x^n + \ldots\ldots$$
$$= \sum_{n=0}^{\infty} (3 + 2^n)\, x^n$$
$$= \sum_{n=0}^{\infty} y_n x^n, \text{ where } y_n = 3 + 2^n$$

Therefore, the required sequence is $\{y_n\} = \{3 + 2^n\}$.

Problem 2.64. *Find the generating function which will give the number of solutions in integers of $x + y + z = 5$ if*

(i) each variable is non negative.

(ii) each variable is non negative and atmost 3.

(iii) *each variable is at least 2 and at most 4.*

(iv) $0 \le x \le 5,\ 2 \le y \le 6,\ 5 \le z \le 8$, *x is even and y is odd.*

Solution. (*i*) The number of solutions is the coefficient of x^5 in the generating function

$$(1 + x + x^2 + x^3 + x^4 + x^5)^3$$

(*ii*) The number of solution is the coefficient of x^5 in the generating function $(1 + x + x^2 + x^3)$

(*iii*) The number of solution is the coefficient of x^5 in the generating function $(x^2 + x^3 + x^4)^3$

(*iv*) The number of solution is the coefficient of x^5 in the generating function

$$(1 + x^2 + x^4)(x^3 + x^5)(x^5 + x^6 + x^7 + x^8).$$

Problem 2.65. *Use generating functions to find the number of ways to select r objects of n different kinds if at least one object of each kind is selected.*

Solution. Let us consider the sequence $\{a_n\}$ where a_n denotes the number of ways to select r objects of n different kinds which includes at least one object of each kind.

Let $G(x) = \Sigma a_r x^r$ be the generating function. Each of the n kind objects constitutes the factor

$$(x + x^2 + ...)$$

Now, $(x + x^2 + x^3 + ...)^n = x^n (1 + x + x^2 + ...)^n$

$$= x^n (1 - x)^{-n} = x^n\, \Sigma C(n + r - 1, r)x^r$$

$$= \Sigma C(n + r - 1, r)x^{n + r}$$

$$= \Sigma C(z - 1, z - n)x^z$$

$$= \Sigma C(r - 1, r - n)x^r$$

Hence the number of ways to select r objects of n different kinds which includes atleast one object of each kind is $C(r - 1, r - n)$.

Problem 2.66. *Solve the recurrence relation $a_k = 3a_{k-1}$ for k = 1, 2, 3, ... and initial condition $a_n = 2$.*

Solution. Let G(x) be the Generating Function for the sequence $\{a_k\}$, that is,

$$G(x) = \sum_{k=0}^{\infty} a_k x^k .$$

First note that $$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$$

Using the recurrence relation, we see that

$$G(x) - 3x\, G(x) = \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})x^k$$

$$= 2,$$

Since $a_0 = 2$ and $a_k = 3a_{k-1}$.

Thus $G(x) - 3xG(x) = (1 - 3x)\,G(x) = 2$.

Solving for G(x) shows that $G(x) = \dfrac{2}{(1-3x)}$.

Using the identity $\dfrac{1}{(1-ax)} = \sum_{k=0}^{\infty} a^k x^k$

We have $$G(x) = 2\sum_{k=0}^{\infty} 3^k a^k = \sum_{k=0}^{\infty} 2\,.\,3^k x^k\,.$$

Consequently, $a_k = 2.3^k$.

Problem 2.67. *Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords to length n. The sequence $\{a_n\}$ satisfies the recurrence relation.*

$a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_1 = 9$. Use the generating functions do find an explicit formula for a_n.

Solution. To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$; when we assign this value to a_0 and use the recurrence relation.

We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$, which is consistent with our original initial condition.

(It also makes sense because there is one codeword of length 0-the emply string.)

We multiply both sides of the recurrence relation by x^n to obtain $a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots\ldots$

We sum both sides of the last equation starting with $n = 1$, to find that

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n)$$

$$= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$

$$= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n$$

$$= 8x\,G(x) + \frac{x}{(1-10x)}$$

Therefore, we have

$$G(x) - 1 = 8x\,G(x) + \frac{x}{(1-10x)}$$

Solving for G(x) shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

Using a, with $a = 8$ and $a = 10$, once, which gives

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n \;.$$

Consequently, we have shown that

$$a_n = \frac{1}{2}\;(8^n + 10^n).$$

Problem 2.68. *Use generating functions to show that*

$$\sum_{k=0}^{n} C(n, k)^2 = C(2n, n)$$

Whenever n is a positive integer.

Solution. First note that by the Binomial Theorem C($2n$, n) is the coefficient of x^n in $(1 + x)^{2n}$. However, we also have

$$(1 + x)^{2n} = [(1 + x)^n]^2$$
$$= [C(n, 0) + C(n, 1)x + C(n, 2)x^2 + \ldots\ldots + C(n, n)x^n]^2$$

The coefficient of x^n in this expression is

$$C(n, 0)C(n, n) + C(n, 1)\,C(n, n-1) + C(n, 2)\,C(n, n-2) + \ldots\ldots + C(n, n)\,C(n, 0).$$

This equals $\sum_{k=0}^{n} C(n, k)^2$, since $C(n, n-k) = C(n, k)$

Since both C(2n, n) and $\sum_{k=0}^{n} C(n,k)^2$ represent the coefficient of x^n in $(1 + x)^{2n}$, they must be equal.

(Table of equivalent)

Expression for Generating Functions

If $\quad A(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - .... - a_{k-1} x^{k-1}$$

$$\sum_{n=k}^{\infty} a_{n-1} x^n = x(A(x) - a_0 - a_1 x - .... - a_{k-2} x^{k-2})$$

$$\sum_{n=k}^{\infty} a_{n-2} x^n = x^2(A(x) - a_0 - a_1 x - .... - a_{k-3} x^{k-3})$$

$$\sum_{n=k}^{\infty} a_{n-3} x^n = x^3(A(x) - a_0 - a_1 x - .... - a_{k-4} x^{k-4})$$

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$$\sum_{n=k}^{\infty} a_{n-k} x^n = x^k(A(x)).$$

Theorem 2.5. *If* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of numbers which satisfy the linear recurrence relation with constant coefficients.*

$a_n + C_1 a_{n-1} + ... + C_k a_{n-k} = 0$ *where* $C_k \neq 0$*, and* $n \geq k$*, then the generating function* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *equals* $\frac{P(x)}{Q(x)}$*, where* $P(x) = a_0 + (a_1 + C_1 a_0)x + ... + (a_{k-1} + C_1 a_{k-2} + ... + C_{k-1} a_0)x^{k-1}$ *and* $Q(x) = 1 + C_1 x + ... + C_k x^k$*.*

Conversely, given such polynomials P(x) and Q(x), where P(x) has degree less than k, and Q(x) has degree k, there is a sequence $\{a_n\}_{n=0}^{\infty}$ whose generating function is $A(x) = \frac{P(x)}{Q(x)}$.

Table of Generating Functions

Sequence a_n	*Generating Function A(x)*
1. $C(k, n)$	$(1+x)^k$
2. 1	$\frac{1}{1-x}$
3. a^n	$\frac{1}{1-ax}$
4. $(-1)^n$	$\frac{1}{1+x}$
5. $(-1)^n a^n = (-a)^n$	$\frac{1}{1+ax}$
6. $C(k-1+n, n)$	$\frac{1}{(1-x)^k}$
7. $C(k-1+n, n)a^n$	$\frac{1}{(1-ax)^k}$
8. $C(k-1+n, n)(-a)^n$	$\frac{1}{(1+ax)^k}$
9. $n+1$	$\frac{1}{(1-x)^2}$
10. n	$\frac{1}{(1-x)^2}$
11. $(n+2)(n+1)$	$\frac{2}{(1-x)^3}$
12. $(n+1)(n)$	$\frac{2x}{(1-x)^3}$
13. n^2	$\frac{x(1+x)}{(1-x)^3}$
14. $(n+3)(n+2)(n+1)$	$\frac{6}{(1-x)^4}$
15. $(n+2)(n+1)(n)$	$\frac{6x}{(1-x)^4}$
16. n^3	$\frac{x(1+4x+x^2)}{(1-x)^2}$

Contd.

17. $(n+1)a^n$	$\dfrac{1}{(1-ax)^2}$
18. na^n	$\dfrac{ax}{(1-ax)^2}$
19. n^2a^n	$\dfrac{(ax)(1+ax)}{(1-ax)^3}$
20. n^3a^n	$\dfrac{(ax)(1+4ax+a^2x^2)}{(1-ax)^4}$

Problem 2.69. *Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$.*

Solution. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Next multiply each term in the recurrence relation by x^n and sum from 2 to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Replace each infinite sum by an expression from the table of equivalent expressions :

$$[A(x) - a_0 - a_1x] - 7x\,[A(x) - a_0] + 10x^2[A(x)] = 0$$

Then simplify

$$A(x)(1 - 7x + 10x^2) = a_0 + a_1x - 7a_0x$$

or
$$A(x) = \frac{a_0 + (a_1 - 7a_0)x}{1 - 7x + 10x^2} = \frac{a_0 + (a_1 - 7a_0)x}{(1-2x)(1-5x)}$$

Decompose $A(x)$ as a sum of partial fractions :

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-5x}$$

where C_1 and C_2 are constants, as yet undetermined Express $A(x)$ as a sum of familiar series :

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-5x} = C_1\sum_{n=0}^{\infty} 2^n x^n + C_2\sum_{n=0}^{\infty} 5^n x^n .$$

Express a_n as coefficient of x^n in $A(x)$ and in the sum of the other series :

$$a_n = C_1 2^n + C_2 5^n$$

Now the constants C_1 and C_2 are uniquely determined once values for a_0 and a_1 are given.

For example, if $a_0 = 10$, and $a_1 = 41$, we may use the form of the general solution $a_n = C_1 2^n + C_2 5^n$, and left $n = 0$ and $n = 1$ to obtain the two equations.

$C_1 + C_2 = 10$ and $2C_1 + 5C_2 = 41$ which determine the values $C_1 = 3$ and $C_2 = 7$. Thus, in this case the unique solution of the recurrence relation is

$$a_n = (3)2^n + (7)5^n.$$

Problem 2.70. *Solve the recurrence relation*

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0 \text{ for } n \geq 3.$$

Solution. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then multiply by x^n and sum from 3 to ∞ since $n \geq 3$.

Thus $$\sum_{n=3}^{\infty} a_n x^n - 9 \sum_{n=3}^{\infty} a_{n-1} x^n + 26 \sum_{n=3}^{\infty} a_{n-2} x^n - 24 \sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

Replace the infinite sums by equivalent expressions

$$(A(x) - a_0 - a_1x - a_2x^2) - 9x(A(x) - a_0 - a_1x) + 26x^2(A(x) - a_0) - 24x^3A(x) = 0$$

Simplify, $$A(x)(1 - 9x + 26x^2 - 24x^3) = a_0 + a_1x + a_2x^2 - 9a_0x - 9a_1x^2 + 26a_0x^2$$

or $$A(x) = \frac{a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 26a_0)x^2}{1 - 9x + 26x^2 - 24x^3}$$

Now $1 - 9x + 26x^2 - 24x^3 = (1 - 2x)(1 - 3x)(1 - 4x)$ so that there are constants C_1, C_2, C_3 such that

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-3x} + \frac{C_3}{1-4x}$$

$$= C_1 \sum_{n=0}^{\infty} 2^n x^n + C_2 \sum_{n=0}^{\infty} 3^n x^n + C_3 \sum_{n=0}^{\infty} 4^n x^n$$

$$= \sum_{n=0}^{\infty} (C_1 2^n + C_2 3^n + C_3 4^n) x^n$$

Thus, $a_n = C_12^n + C_23^n + C_34^n$ and C_1, C_2 and C_3 can be determined once the initial conditions for a_0, a_1 and a_2 are specified.

Assume that the initial conditions are $a_0 = 0$, $a_1 = 1$ and $a_2 = 10$

Then $$A(x) = \frac{x + x^2}{(1-2x)(1-3x)(1-4x)} = \frac{C_1}{(1-2x)} + \frac{C_2}{(1-3x)} + \frac{C_3}{(1-4x)}$$

and $$C_1(1 - 3x)(1 - 4x) + C_2(1 - 2x)(1 - 4x) + C_3(1 - 2x)(1 - 3x) = x + x^2$$

Let $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and find that $C_1 = \frac{3}{2}$, $C_2 = -4$ and $C_3 = \frac{5}{2}$

Thus, in this case, $a_n = \frac{3}{2}(2^n) - 4(3^n) + \frac{5}{2}(4^n)$.

Problem 2.71. *Solve* $a_n - 6a_{n-1} + 12a_{n-2} - 8_{n-3} = 0$ *by generating functions.*

Solution. $A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{P(x)}{Q(x)}$

$$= \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 12a_0)x^2}{1 - 6x + 12x^2 - 8x^3}$$

But since $1 - 6x + 12x^2 - 8x^3 = (1 - 2x)^3$

We use partial fractions to calculate that there are constants C_1, C_2, C_3 such that

$$A(x) = \frac{C_1}{(1-2x)} + \frac{C_2}{(1-2x)^2} + \frac{C_3}{(1-2x)^3}$$

$$= C_1 \sum_{n=0}^{\infty} 2^n x^n + C_2 \sum_{n=0}^{\infty} \binom{n+1}{n} 2^n x^n + C_3 \sum_{n=0}^{\infty} \binom{n+2}{n} 2^n x^n$$

$$= \sum_{n=0}^{\infty} \left[C_1 2^n + C_2 (n+1) 2^n + C_3 \frac{(n+2)(n+1)}{2} 2^n \right] x^n$$

so that $$a_n = C_1 2^n + C_2(n+1)2^n + C_3 \frac{(n+2)(n+1)}{2} 2^n.$$

Problem 2.72. *Solve* $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ *for* $n \geq 3$.

Solution. Here, if $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=3}^{\infty} a_n x^n - 8 \sum_{n=3}^{\infty} a_{n-1} x^n + 21 \sum_{n=3}^{\infty} a_{n-2} x^n - 18 \sum_{n=3}^{\infty} a_{n-3} x^n = 0,$$

$$(A(x) - a_0 - a_1 x - a_2 x^2) - 8x(A(x) - a_0 - a_1 x) + 21x^2(A(x) - a_0) - 18x^2 A(x) = 0$$

or $$A(x) = \frac{a_0 + (a_1 - 8a_0)x + (a_2 - 8a_1 + 21a_0)x^n}{1 - 8x + 21x^2 - 18x^3}$$

But since $1 - 8x + 21x^2 - 18x^3 = (1 - 2x)(1 - 3x)^2$

We see that there are constants C_1, C_2, C_3 such that

$$A(x) = \frac{C_1}{(1-2x)} + \frac{C_2}{(1-3x)} + \frac{C_3}{(1-3x)^2}$$

Then, $$A(x) = \sum_{n=0}^{\infty} [C_1 2^n + C_2 3^n + C_3 n^3 \, C(n+1, n)] \, x^n$$

or $$a_n = C_1 2^n + C_2 3^n + C_3(n+1)3^n.$$

Problem Set 2.1

1. Find the first four terms of the sequence defined by each of the recurrence relations and initial conditions

(*a*) $a_n = 2a_{n-1} + n, \quad n \geq 2, a_1 = 1$

(*b*) $a_n = a_{n-1}^2, \quad n \geq 2, a_1 = 12$

(*c*) $a_n = n(a_{n-1})^2, \quad n \geq 1, a_0 = 1$

(*d*) $b_k = b_{k-1} + 2b_{k-2}, \quad k \geq 2, b_0 = 1, b_1 = 1$

(*e*) $b_k = kb_{k-1} + k^2 b_{k-2}, \quad k \geq 2, b_0 = 1, b_1 = 1.$

2. Show that the sequence 2, 3, 4, 5, ..., $2n + 1$, ... for $n \geq 0$, satisfies the recurrence relation $a_k = 2a_{k-1} - a_{k-2}, k \geq 2.$

3. Show that the sequence 0, 1, 3, 7, ..., $2^m - 1$, ... for $n \geq 0$, satisfies the recurrence relation $a_k = 3a_{k-1} - 2a_{k-2}, k \geq 2.$

4. Show that $a_n = C_1 + C_2 2^n - n$ is a solution of the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 1.$

5. Show that $a_n = 0$, $a_n = 4^n$, $a_n = n4^n$ and $a_n = 2.4^n + 3n4^n$ are all solutions of the same recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}.$$

6. Show that $a_n = C_1 2^n + C_2 4^n$ is a solution of the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 0.$

7. Show that $a_n = (C_1 + C_2 n) \,.\, 2^n$ is a solution of the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0.$

8. Prove that if $S_n = aS_{n-1}$ for $n \geq 1$ and $a \neq 0$ then $S_n = a^n S_0$.

9. Use the iterative method to find the solution to each of the following recurrence relations and initial condtions.

(*a*) $a_n = a_{n-1} + 2, a_0 = 1$

(*b*) $a_n = 2a_{n-1} + 1, a_1 = 7$

(*c*) $a_n = a_{n-1} + n, a_1 = 4$

(*d*) $a_n = na_{n-1}, a_0 = 5.$

10. Solve the recurrence relations together with initial conditions given

(*a*) $a_n = a_{n-1} + 6a_{n-2} : n \geq 2, \quad a_0 = 1, a_1 = 1$

(*b*) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}, \quad a_0 = 0, a_3 = 3, a_5 = 10$

(*c*) $a_n = 7a_{n-1} - 10a_{n-2}, \quad a_0 = 4, a_1 = 17$

(*d*) $a_n - 8a_{n-1} + 16a_{n-2} = 0, \quad a_2 = 16, a_3 = 80$

(*e*) $a_n = a_{n-1} + 2a_{n-2}, \quad a_0 = 2, a_1 = 7.$

11. Solve each of the following recurrence relations that satisfies the initial conditions.

(*a*) $a_n - a_{n-1} = n, \quad a_0 = 2$

(*b*) $a_n - 2a_{n-1} = 6n, \quad a_1 = 2$

(*c*) $a_n - a_{n-1} = 6_{n+1}, \quad a_0 = 1$

(*d*) $a_n - a_{n-1} = 3^n, \quad a_0 = 1$

(*e*) $a_{n+1} - 2a_n = 3 + 4^n, \quad a_0 = 2$

(*f*) $a_{n+2} + 2a_{n+1} - 15a_n = 6n + 10, \quad a_r = 1, a_1 = -\frac{1}{2}$

(*g*) $a_n - 4a_{n-1} + 3a_{n-2} = 2^n, \quad a_1 = 1, a_2 = 11.$

12. Given that $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and $a_3 = 12$ satisfy the recurrence relation $a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$, determine a_n.

13. Solve the following recurrence equations (relations) :

(*a*) $a_r - 7a_{r-1} + 10a_{r-2} = 0$ with $a_0 = 0$ and $a_1 = 3$

(*b*) $a_r - 4a_{r-1} + 4a_{r-2} = 0$ with $a_0 = 1$ and $a_1 = 6$

(*c*) $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$ with $a_0 = 0$ and $a_1 = 1$

(*d*) $a_r + 6a_{r-1} + 9a_{r-2} = 3$ with $a_0 = 0$ and $a_1 = 1$

(*e*) $a_r = a_{r-1} + a_{r-2} = 0$ with $a_0 = 0, a_1 = 2$

(*f*) $a_r - a_{r-1} - a_{r-2} = 0$ with $a_0 = 1, a_1 = 1$

(*g*) $a_r - 2a_{r-1} + 2a_{r-2} - a_{r-3} = 0$ with $a_0 = 2$, $a_1 = 1$ and $a_2 = 1$.

14. Use the backtracking method to find an explicit formula for the sequence defined by the recurrence equation and initial conditions.

(*a*) $a_n = 2.5 * a_{n-1}$ with $a_1 = 4$

(*b*) $b_n = b_{n-1} - 2$ with $b_1 = 0$

(*c*) $C_n = C_{n-1} + n$ with $C_1 = 4$

(*d*) $d_n = -1.1 * d_{n-1}$ with $d_1 = 5$

(*e*) $e_n = 5e_{n-1} + 3$ with $e_1 = 2$

(*f*) $g_n = ng_{n-1}$ with $g_1 = 6$.

15. Solve each of the recurrence equations :

(*a*) $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 2$ and $a_2 = 6$

(*b*) $b_n = -3b_{n-1} - 2b_{n-2}$ with $b_1 = -2$ and $b_2 = 4$

(*c*) $C_n = -6C_{n-1} - 9C_{n-2}$ with $C_1 = 2.5$ and $C_2 = 4.7$

(*d*) $d_n = 4d_{n-1} - 4d_{n-2}$ with $d_1 = 1$ and $d_2 = 7$

(*e*) $e_n = 2e_{n-2}$ with $e_1 = \sqrt{2}$ and $e_2 = 6$

(*f*) $g_n = 2g_{n-1} - 2g_{n-2}$ with $g_1 = 1$ and $g_2 = 4$.

16. Solve the difference (recurrence) equation

$$a_r - ra_{r-1} = r! \quad \text{for } r \geq 1 \text{ and given that } a_0 = 2.$$

(**Hint :** Use change of variable method. Let $b_r = a_r/r!$)

17. Solve the difference equation

$$a_r^2 - 2a_{r-1}^2 = 1 \quad \text{for } r \geq 1 \text{ and given that } a_0 = 2.$$

(**Hint :** Use change of variable method. Let $b_r = a_r^2$)

18. Solve the difference equation

$$ra_r + ra_{r-1} - a_{r-1} = 2^r \quad \text{for } r \geq 1 \text{ and given that } a_0 = 273$$

(**Hint :** Use change of variable method. Let $b_r = ra_r$)

19. Find the closed formula for the Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2}$$

20. Given that $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and $a_3 = 12$ satisfy the recurrene equation $a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$ and determine a_r.

21. Determine the particular solution for the difference equations

(*a*) $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$ (*b*) $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$

(*c*) $a_n - 2a_{n-1} = 7n^2$ (*d*) $a_n - 2a_{n-1} = 7^n$.

22. Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$, $n \geq 2$, given $a_0 = 1$, $a_1 = 3$.

23. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $n \geq 2$ given $a_0 = -5$, $a_1 = 3$.

24. Solve the recurrence relation $a_n = -8a_{n-1} - a_{n-2}$, $n \geq 2$ given $a_0 = 0$, $a_1 = 1$.

25. Solve the recurrence relation $a_{n+1} = 2a_n + 3a_{n-1}$, $n \geq 1$ given $a_0 = 0$, $a_1 = 8$.

26. (*a*) Solve the recurrence relation $a_n = -2a_{n-1} + 15a_{n-2}$, $n \geq 2$ given $a_0 = 1$, $a_1 = -1$.

(*b*) Solve the recurrence relation $a_n = -2a_{n-1} + 15a_{n-2} + 24$, $n \geq 2$ given $a_0 = 1$, $a_1 = -1$.

27. (*a*) Solve the recurrence relation $a_n = 4a_{n-1}$, $n \geq 1$ given $a_0 = 1$.

(*b*) Solve the recurrence relation $a_n = 4a_{n-1} + 8^n$, $n \geq 1$, given $a_0 = 1$.

28. (*a*) Solve the recurrence relation $a_n = 4a_{n-1} - 9$, $n \geq 1$ given $a_0 = 4$.

(*b*) Solve the recurrence relation $a_n = 4a_{n-1} + 3n2^n$, $n \geq 1$, given $a_0 = 4$.

29. Solve the recurrence relation $a_n = 5a_{n-1} - 2a_{n-2} + 3n^2$, $n \geq 2$ given $a_0 = 0$, $a_1 = 3$.

30. Solve the recurrence relation $a_n = -6a_{n-1} - 9a_{n-2}$, $n \geq 2$ given $a_0 = 1$, $a_1 = -4$.

31. (*a*) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$ given $a_0 = 2$, $a_1 = 11$.

(*b*) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 3n$, $n \geq 2$, given $a_0 = 2$, $a_1 = 14$.

32. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + n$, $n \geq 2$, given $a_0 = 5$, $a_1 = 9$.

33. Solve the following recurrence relation

(*a*) $a_{n+1} = -8a_n - 16a_{n-1}$, $n \geq 1$, given $a_0 = 5$, $a_1 = 17$

(*b*) $a_{n+1} = -8a_n - 16a_{n-1} + 5$, $n \geq 1$, given $a_0 = 2$, $a_1 = -1$

(*c*) $9a_n = 6a_{n-1} - a_{n-2}$, $n \geq 2$, given $a_0 = 3$, $a_1 = -1$

(*d*) $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$, given $a_0 = 40$, $a_1 = 37$

(*e*) $a_n = -5a_{n-1} + 6a_{n-2}$, $n \geq 2$, given $a_0 = 5$, $a_1 = 19$

(*f*) $a_{n+1} = 7a_n - 10a_{n-1}$, $n \geq 2$, given $a_1 = 10$, $a_2 = 29$

(*g*) $a_n = -6a_{n-1} + 7a_{n-2}$, $n \geq 2$, given $a_0 = 32$, $a_1 = -17$.

34. Find a recurrence relation for the balance B(k) owed at the end of k months on a loan of \$5000 at a rate of 7% if a payment of \$100 is made each month.

[**Hint :** Express B(k) in terms of B(k – 1), the monthly interest is (0.07/12 B(k – 1)]

35. (*a*) Find a recurrence relation for the balance B(*k*) owed at the end of *k* months on a loan at a rate of *r* if a payment P is made on the loan each month.

[**Hint.** Express B(*k*) in terms of B(*k* – 1) and note that the monthly interest rate is *r*/12.]

(*b*) Determine what the monthly payment P should be so that the loan is paid off after T months.

36. (*a*) Find a recurrence relation for the number of bit strings of length *n* that contain three of consecutive 0'*s*.

(*b*) What are the initial conditions ?

(*c*) How many bit strings of length seven contain three consecutive 0's ?

37. (*a*) Find a recurrence relation for the number of bit strings of length *n* that contain a pair of consecutive 0's.

(*b*) What are the initial conditions ?

(*c*) How many bit strings of length seven conatin two consecutive 0's ?

38. (*a*) Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0's.

(*b*) What are the initial conditions ?

(*c*) How many bit strings of length seven do not contain three consecutive 0's ?

39. (*a*) Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.

(*b*) What are the initial conditions ?

(*c*) How many ways can this person climb a flight of eight stairs ?

Problem Set 2.2

1. Apply the generating function technique to solve the following recurrence relation with given initial conditions

(*a*) $a_{n+1} - (n+1)a_n = 0,\ a_0 = 1$ (*b*) $a_n - 4a_{n-2} = 0,\ a_0 = 0,\ a_1 = 1$

(*c*) $a_{n+2} - 3a_{n+1} + 2a_n = 0,\ a_0 = 2,\ a_1 = 3$ (*d*) $a_{n+2} - 5a_{n+1} + 6a_n = 2,\ a_0 = 1,\ a_1 = 2$

(*e*) $a_{n+2} + 4a_{n+1} + 4a_n = 0,\ a_0 = 1,\ a_1 = 0$ (*f*) $a_n - 7a_{n-1} + 10a_{n-2} = 0,\ a_0 = 3,\ a_1 = 0.$

2. Apply the generating function technique to solve the following recurrence relations :

(*a*) $a_{n+2} + 2a_{n+1} + a_n = 1 + n$

(*b*) $a_{n+2} - 2a_{n+1} + a_n = 2^n$

(*c*) $a_n = 4(a_{n-1} - a_{n-2}) + 2^n$

(*d*) $a_n = 3a_{n-1} - 4n + 3(2^n)$

(*e*) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-2} + 3^n.$

3. Solve the following recurrence equations using generating function method :

(*a*) $A_n = A_{n-1} + A_{n-2} + 2n$ given that $A_0 = 0$ and $A_1 = 1$

(*b*) $a_n + a_{n-1} = 3n2^n$ given that $a_0 = 0$

(*c*) $a_n = 5a_{n-1} - 3$ given that $a_0 = 1$

(*d*) $h_n = 4h_{n-2}$ given that $h_0 = 0$ and $h_1 = 1$

(*e*) $a_n = \begin{cases} 0 & \text{for } n = 0 \\ 2a_{n-1} + 3 & \text{for } n > 0 \end{cases}$

(*f*) $a_n = \begin{cases} 16 & \text{for } n = 2 \\ 60 & \text{for } n = 0 \\ 8a_{n-1} - 16a_{n-2} & \text{for } n > 0 \end{cases}$

(*g*) $a_n - 2a_{n-1} - 4a_{n-2} = 4^n$ given that $a_0 = 1$ and $a_1 = 7$

(*h*) $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$ given that $a_1 = 0$ and $a_2 = 10$

(*i*) $3a_{n+2} - 8a_{n+1} - 3a_n = 3^n - 2n + 1$

(*j*) $a_{n+2} - 6a_{n+1} + 8a_n = 3n^2 + 2 - 5 * 3^n$.

4. Using the method of generating functions, solve the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$, given $a_0 = 1$.

5. Use the method of generating function to solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, given $a_0 = 1$, $a_1 = 4$.

6. Use the generating function to find a formula for a_n given $a_0 = 5$ and $a_n = a_{n-1} + 2^n$ for $n \geq 1$.

7. Use the method of generating function to solve the recurrence relation $a_n = 3a_{n-1} + 1$, $n \geq 1$, given $a_0 = 1$.

8. Use the method of generating functions to solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, $n \geq 2$, given $a_0 = 1$, $a_1 = 4$.

9. Solve the recurrence relation, using generating function method, $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$, with the initial conditions $a_0 = 2$, $a_1 = 8$.

10. Solve each of the following using generating functions

(*a*) $a_n = -5a_{n-1}$, $n \geq 1$, given $a_0 = 2$

(*b*) $a_n = -5a_{n-1} + 3$, $n \geq 1$, given $a_0 = 2$.

(*c*) $a_n = 4a_{n-1} - 3a_{n-2}$, $n \geq 2$, given $a_0 = 2$, $a_1 = 5$.

(*d*) $a_n = -10a_{n-1} - 25a_{n-2}$, $n \geq 2$, given $a_0 = 1$, $a_1 = 25$

(*e*) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, $n \geq 3$, given $a_0 = 1$, a_1, $= 3$, $a_2 = 6$.

(*f*) $a_n = a_{n-1} + a_{n-2} - a_{n-3}$, $n \geq 3$, given $a_0 = 2$, $a_1 = -1$, $a_2 = 3$.

11. let a_n denote the number of n-digit numbers, each of whose digits is 1, 2, 3, or 4 and in which the number of 1's even

(*a*) Find a recurrence relation for a_n (*b*) Find a_n explicit formula for a_n.

12. Let $a_n = r\, a_{n-1} + S\, a_{n-2} + f(n)$, $n \geq 2$, be a second order recurrence relation with constant coefficients show that if P_n satisfies this recurrence relation for $n \geq 2$ and q_n satisfies the associated homogeneous recurrence relation $a_n = ra_{n-1} + Sa_{n-2}$ for $n \geq 2$, then $P_n + \boldsymbol{q_n}$ satisfies the given relation for $n \geq 2$.

13. Find the general solutions to each of the following non-linear recurrence relation :

(a) $a_n - 5a_{n-1} + 6a_{n-2} = 1$ (b) $a_n - 6a_{n-1} + 8a_{n-2} = 3$

(c) $a_{n+2} - 2a_{n+1} + a_n = 3n + 5$ (d) $a_{n+2} - 5a_{n+1} + 6a_n = n^2$

(e) $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$ (f) $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$

(g) $a_n - 4a_{n-1} + 4a_{n-2} = (n + 1)\, 2^n$.

14. Show that $a_n = \left(\frac{1}{6}\right) 5^n$ is a solution of the homogeneous linear recurrence relation

$a_{n+2} - 5a_{n+1} + 6a_n = 2^n$.

15. Show that $a_n = -2^{n+1}$ is a solution of the non homogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

16. Solve the recurrence relation $a_n = 7a_{n-1}$, $n \geq 1$ and $a_2 = 98$.

17. Find a_{12} is $a_{n+1}^2 = 5a_n^2$, $a_n > 0$ for $n \geq 0$ and $a_0 = 2$.

18. A bank pays 6% (annual) interest on savings, compounding the interest monthly. In Bonnie deposits \$1000 on the first day to May, how much will this deposit be worth a year later ?

19. Solve the relation $a_n = n \, . \, a_{n-1}$, where $n \geq 1$ and $a_0 = 1$.

20. Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ where $n \geq 2$ and $a_0 = -1$, $a_1 = 8$.

21. Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$ where $n \geq 0$ and $F_0 = 0$, $F_1 = 1$.

22. For $n \geq 0$, let S = {1, 2, 3, ...} (when $n = 0$, $S \neq 0$) and let a_n denote the number of subsets of S that contain no consecutive integers. Find and solve a recurrence relation for a_n.

23. Find a recurrence relation for the number of binary sequences of length n that have no consecutive 0's.

24. Solve the recurrence relation $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

25. Determine $(1 + \sqrt{3i})^{10}$.

26. Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$, $n \geq 2$ and $a_0 = 1$, $a_1 = 2$.

27. Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

28. Solve the recurrence relation $a_n - a_{n-1} = 3n^2$, where $n \geq 1$ and $a_0 = 7$.

29. Solve the recurrence relation $a_n - 3a_{n-1} = 5(7^n)$ where $n \geq 1$ and $a_0 = 2$.

30. Solve the recurrence relation $a_n - 3a_{n-1} = 5(3^n)$ where $n \geq 1$, and $a_0 = 2$.

31. Pauline takes out a loan of S dollars that is to be paid back in T periods of time. If r is the interest rate per period for the loan, what (constant) payment P must she make at the end of each period ?

32. Solve the recurrence relation $a_{n+2} - 4a_{n+1} + 3a_n = -200$, $n \geq 0$, $a_0 = 3000$, $a_1 = 3300$.

33. Solve the recurrence relation $a_n - 3a_{n-1} = n$, $n \geq 1$, $a_0 = 1$.

34. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ for substitution.

Answers 2.1

1. (*a*) 1, 4, 11, 26 (*b*) 2, 4, 16, 256 (*c*) 1, 1, 2, 12
(*d*) 1, 1, 3, 5 (*e*) 1, 1, 6, 27

9. (*a*) $1 + 2n$ (*b*) $2^{n+2} - 1$ (*c*) $3 + n(n + 1)/2$
(*d*) $5n\,!$

10. (*a*) $a_n = 3^n$ (*b*) $n(n - 1)/2$ (*c*) $a_n = 2^n + 3.5^n$
(*d*) $a_n = (2 + n)4^{n-1}$ (*d*) $a_n = 3.2^n - (-1)^n$

11. (*a*) $a_n = n(n + 1)/2 + 2$ (*b*) $a_n = 10.2^n - 6n - 12$ (*c*) $a_n = (n + 1)^2$
(*d*) $a_n = 3^{n+1} - 1/2$ (*e*) $a_n = (5/6)(-2)^n + 1 + 1/6 \,.\, 4^n$
(*f*) $a_n = 5/8\,(-5)^n + (11/8)3^n - 1/2\, n - 1$ (*g*) $a_n = 3^{n+1} - 2^{n+2}$.

12. $a_n = n \,.\, 2^{n-1}$ **22.** $a_n = 3^n$ **23.** $a_n = -5(3^n) + 6n(3^n)$

24. $a_n = \dfrac{1}{2\sqrt{15}}(-4 + \sqrt{5})^n - \dfrac{1}{2\sqrt{15}}(4 - \sqrt{15})^n$

25. $a_n = -2(-1)^n + 2(3^n) = 2[(-1)^{n+1} + 3^n]$

26. (*a*) $a_n = \dfrac{1}{2}(3^n) + \dfrac{1}{2}(-5)^n$ (*b*) $a_n = 2(3^n) + (-5)^n - 2$

27. (*a*) $a_n = 4^n$, (*b*) $a_n = 2(8^n) - 4^n$

28. (*a*) $a_n = 4^n + 3$ (*b*) $a_n = 10(4^n) - (3n + 6)2^n$

34. $B(k) = (1 + (0.07/12))\,B(k - 1) - 100$, with $B(0) = 5000$

37. (*a*) $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for $n \geq 2$. (*b*) $a_0 = 0, a_1 = 0$ (*c*) 94.

38. (*a*) $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$ (*b*) $a_0 = 1, a_1 = 2, a_2 = 4$ (*c*) 81

39. (*a*) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ (*b*) $a_0 = 1, a_1 = 1$ (*c*) 34.

Answers 2.2

1. (*a*) $a_n = n\,!$ (*b*) $a_n = 0$ if n is even $= 2^{n-1}$ if n is odd
(*c*) $a_n = 1 + 2^n$ (*d*) $a_n = 1 - 2^n + 3^n$
(*e*) $a_n = 2^n(1 - n)$ (*f*) $a_n = 13(5)^n - 10(2)^n$.

2. (*a*) $a_n = (C_0 + C_7 n)(-1)^n + \dfrac{1}{6}(2n - 1)$ (*b*) $a_n = C_0 + C_7 n + 2^n$

(*c*) $a_n = \left(C_0 + C_1 n + \dfrac{1}{2}n^2\right) 2^n$ (*d*) $a_n = C3^n + 2n + 3(1 - 2^{n+1})$

(*e*) $a_n = (C_0 + C_1 n + C_2 n^2)\, 2^n + 27.3^n$

4. $a_n = 2^n$ **5.** $a_n = -2^n + 2.3^n$

6. $a_n = 2^{n+1} + 3$

10. (*a*) $a_n = 2(-5)^n$

(*e*) $a_n = -\frac{1}{2} + \left(-\frac{1}{6}\right)(-1)^n + \frac{5}{3}(2^n)$

(*f*) $a_n = \frac{1}{4}(2n+1) + \frac{7}{4}(-1)^n$.

13. (*a*) $a_n = C_1 3^n + C_2 n^2 + \frac{1}{2}$

(*b*) $a_n = C_1 2^n + C_2 4^n + 1$

(*c*) $a_n = C_1 + C_2 n + \frac{1}{2}n^3 + n^2$

(*d*) $a_n = C_1 2^n + C_2 3^n + \frac{1}{2}n^2 + \frac{3n}{2} + \frac{5}{2}$

(*e*) $a_n = C_1 + C_2 n + \frac{1}{2}n^3 + n^2$

(*f*) $a_n = \frac{C_1}{2^n} + C_2 3^n - \left(\frac{4}{3}\right)2^n$

(*g*) $a_n = (C_1 + C_2 n)\, 2^n + n^2 \,.\, 2\left(\frac{n}{6}+1\right)$.

Unit-3 Graphs and Trees

3.1 WHAT IS A GRAPH ? DEFINITION

A graph G consists of a set of objects V = {v_1, v_2, v_3, ...} called **vertices** (also called **points** or **nodes**) and other set E = {e_1, e_2, e_3, ...} whose elements are called **edges** (also called **lines** or **arcs**).

The set V(G) is called the **vertex set** of G and E(G) is the **edge set.**

Usually the graph is denoted as **G = (V, E)**

Let G be a graph and {*u*, *v*} an edge of G. Since {*u*, *v*} is 2-element set, we may write {*v*, *u*} instead of {*u*, *v*}. It is often more convenient to represent this edge by *uv* or *vu.*

If *e* = *uv* is an edge of a graph G, then we say that *u* and *v* are **adjacent** in G and that *e* joins *u* and *v*. (We may also say that each that of *u* and *v* is adjacent to or with the other).

For example :

A graph G is defined by the sets

V(G) = {*u, v, w, x, y, z*} and E(G) = {*uv, uw, wx, xy, xz*}.

Now we have the following graph by considering these sets.

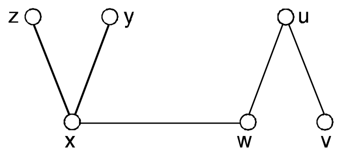


Fig. 3.1

Every graph has a diagram associated with it. The vertex *u* and an edge *e* are **incident** with each other as are *v* and *e*. If two distinct edges say *e* and *f* are **incident** with a common vertex, then they are adjacent edges.

A graph with *p*-vertices and *q*-edges is called a **(*p, q*) graph.**

The (1, 0) graph is called **trivial graph.**

In the following figure the vertices *a* and *b* are adjacent but *a* and *c* are not. The edges *x* and *y* are adjacent but *x* and *z* are not.

Although the edges *x* and *z* intersect in the diagram, their intersection is not a vertex of the graph.

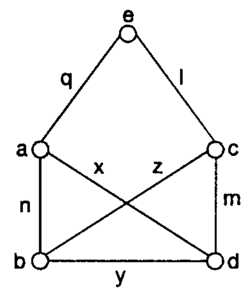


Fig. 3.2

Examples :

(1) Let V = {1, 2, 3, 4} and E = {{1, 2}, {1, 3}, {3, 2}. {4, 4}}.
Then G(V, E) is a graph.

(2) Let V = {1, 2, 3, 4} and E = {{1, 5}, {2, 3}}.
Then G(V, E) is not a graph, as 5 is not in V.

(3)

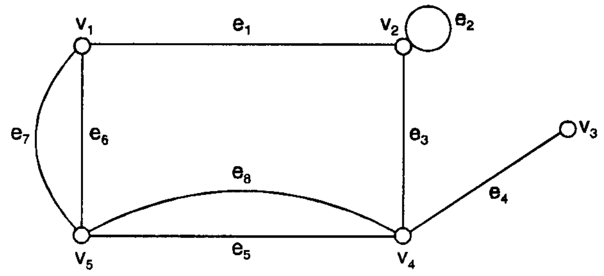


Fig. 3.3

A graph with 5-vertices and 8-edges is called a **(5, 8) graph.**

3.2 DIRECTED AND UNDIRECTED GRAPHS

3.2.1. Directed graph

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices.

In other words, if each edge of the graph G has a direction then the graph is called **directed graph.**

In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.

Figure 3.4(a) is an example of a directed graph.

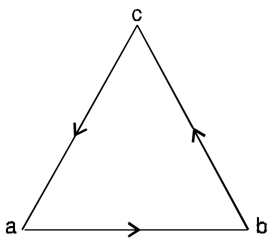


Fig. 3.4(*a*). Directed graph.

Suppose $e = (u, v)$ is a directed edge in a digraph, then (*i*) u is called the **initial vertex** of e and v is the terminal vertex of e

(*ii*) e is said to be **incident** from u and to be incident to v.

(*iii*) u is adjacent to v, and v is adjacent from u.

3.2.2. Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in$ E is associated with an unordered pair of vertices.

In other words, if each edge of the graph G has no direction then the graph is called **un-directed graph.**

Figure 3.4(*b*) is an example of an undirected graph.

We can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

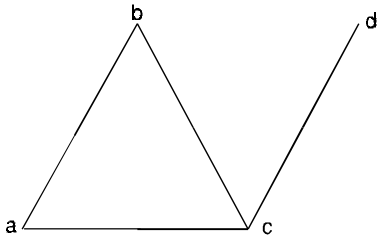


Figure 3.4(*b*). Un-directed graph.

3.3 BASIC TERMINOLOGIES

3.3.1 Loop : An edge of a graph that joins a node to itself is called **loop** or **self loop.**

i.e., a loop is an edge (v_i, v_j) where $v_i = v_f$.

3.3.2. Multigraph

In a multigraph no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_r$ and v_j, v_f.

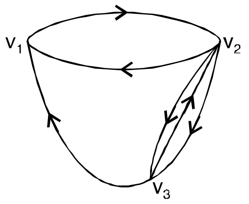


Fig. 3.5(*a*)

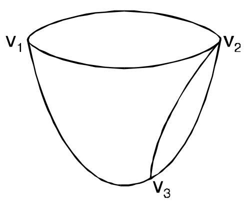


Fig. 3.5(*b*)

In Figure 3.5(*a*), there are two parallel edges associated with v_2 and v_3.

In Figure 3.5(*b*), there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3.

3.3.3. Pseudo graph

A graph in which loops and multiple edges are allowed, is called a pseudo graph.

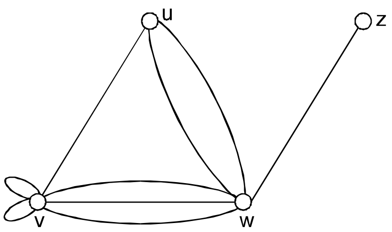


Fig. 3.6(*a*)

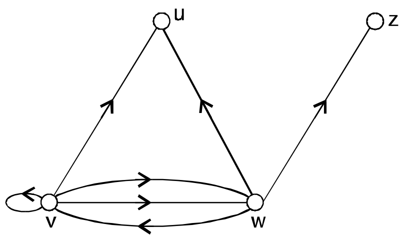


Fig. 3.6(*b*)

3.3.4. Simple graph

A graph which has neither loops nor multiple edges. *i.e.,* where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph.**

Figure 3.4(*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

3.3.5. Finite and Infinite graphs

A graph with finite number of vertices as well as a finite number of edges is called a **finite graph.** Otherwise, it is an **infinite graph.**

3.4 DEGREE OF A VERTEX

The number of edges incident on a vertex v_i with **self-loops counted twice** (is called the **degree of a vertex** v_i and is denoted by $\mathbf{deg_G(v_i)}$ or $\mathbf{deg}\ \boldsymbol{v_i}$ or $\boldsymbol{d(v_i)}$.

The degrees of vertices in the graph G and H are shown in Figure 3.7(*a*) and 3.7(*b*).

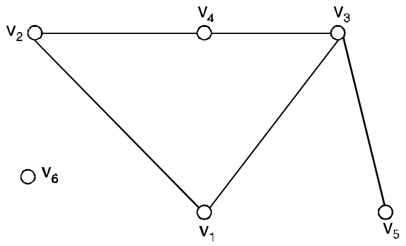


Fig. 3.7(*a*)

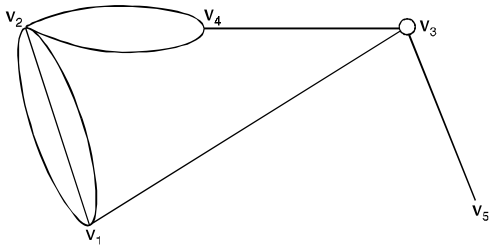


Fig. 3.7(*b*)

In G as shown in Figure 3.7(*a*),

$\deg_G(v_2) = 2 = \deg_G(v_4) = \deg_G(v_1)$, $\deg_G(v_3) = 3$ and $\deg_G(v_5) = 1$ and

In H as shown in Figure 3.7(*b*),

$\deg_H(v_2) = 5$, $\deg_H(v_4) = 3$, $\deg_H(v_3) = 5$, $\deg_H(v_1) = 4$ and $\deg_H(v_5) = 1$.

The degree of a vertex is some times also referred to as its **valency.**

3.5 ISOLATED AND PENDENT VERTICES

3.5.1. Isolated vertex

A vertex having **no incident edge** is called an **isolated vertex.**

In other words, isolated vertices are those with zero degree.

3.5.2. Pendent or end vertex

A vertex of **degree one,** is called a **pendent vertex** or an **end vertex.**

In the above Figure, v_5 is a pendent vertex.

3.5.3. In degree and out degree

In a graph G, the out degree of a vertex v_i of G, denoted by out $\deg_G(v_i)$ or $\deg_G^+(v_i)$, is the number of edges beginning at v_i and the in degree of v_i, denoted by in $\deg_G(v_i)$ or $\deg_G^{-1}(v_i)$, is the number of edges ending at v_i.

The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree **is called a sink.** Since each edge has an initial vertex and terminal vertex.

3.5.4. The Handshaking Theorem 3.1

If G = (*v*, E) be an undirected graph with *e* edges.

Then $\sum_{v \in V} \deg_G(v) = 2e$

i.e., the sum of degrees of the vertices is an undirected graph is even.

Proof : Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equal twice the number of edges.

Note : This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary : In a non directed graph, the total number of odd degree vertices is even.

Proof : Let G = (V, E) a non directed graph.

Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G(v_i) = \sum_{v_i \in U} \deg_G(v_i) + \sum_{v_i \in W} \deg_G(v_i)$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G(v_1) = \sum_{v_i \in W} \deg_G(v_1) \qquad ...(1)$$

Now $\sum_{v_i \in W} \deg_G(v_i)$ is also even

Therefore, from (1) $\sum_{v_i \in W} \deg_G(v_i)$ is even

$\therefore$ The no. of odd vertices in G is even.

Theorem 3.2. If V = {v_1, v_2, ... v_n} is the vertex set of a non directed graph G,

$$\text{then } \sum_{i=1}^{n} \deg(v_i) = 2|E|$$

If G is a directed graph, then $\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |E|$

Proof : Since when the degrees are summed.

Each edge contributes a count of one to the degree of each of the two vertices on which the edge is incident.

Corollary (1) : In any non directed graph there is an even number of vertices of odd degree.

Proof : Let W be the set of vertices of odd degree and let U be the set of vertices of even degree.

Then $\sum_{v \in V(G)} \deg(v) = \sum_{v \in W} \deg(v) + \sum_{v \in U} \deg(v) = 2|E|$

Certainly, $\sum_{v \in U} \deg(v)$ is even,

Hence $\sum_{v \in W} \deg(v)$ is even,

Implying that | W | is even.

Corollary (2) : If $k = \delta(G)$ is the minimum degree of all the vertices of a non directed graph G, then

$$k\,|\,V\,| \le \sum_{v \in V(G)} \deg(v) = 2\,|\,E\,|$$

In particular, if G is a k-regular graph, then

$$k\,|\,V\,| = \sum_{v \in V(G)} \deg(v) = 2\,|\,E\,|.$$

Problem 3.1. *Show that, in any gathering of six people, there are either three people who all know each other or three people none of whom knows either of the other two (six people at a party).*

Solution. To solve this problem, we draw a graph in which we represent each person by a vertex and join two vertices by a solid edge if the corresponding people know each other, and by a dotted edge if not. We must show that there is always a solid triangle or a dotted triangle.

Let v be any vertex. Then there must be exactly five edges incident with v, either solid or dashed, and so at least three of these edges must be of the same type.

Let us assume that there are three solid edges (see figure 3.8) ; the case of atleast three dashed edges is similar.

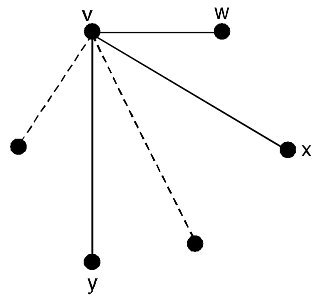


Fig. 3.8.

If the people corresponding to the vertices w and x know each other, then v, w and x form a solid triangle, as required.

Similarly, if the people corresponding to the vertices w and y, or to the vertices x and y, know each other, then we again obtain a solid triangle.

These three cases are shown in Figure 3.9.

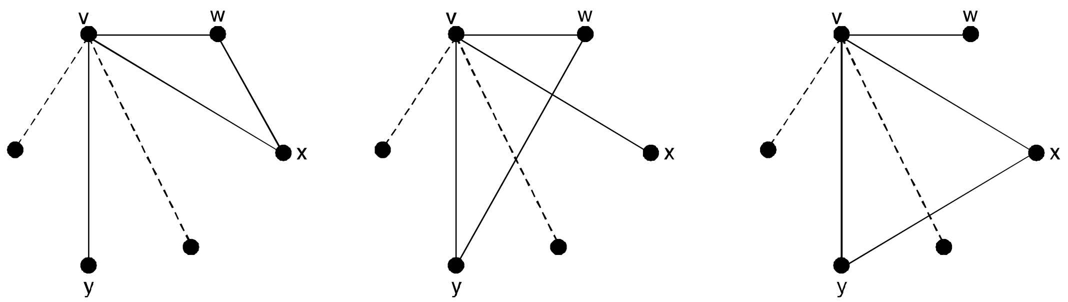


Fig. 3.9.

Finally, if no two of the people corresponding to the vertices *w*, *x* and *y* know each other, then *w*, *x* and *y* from a dotted triangle, as required (see figure 3.10).

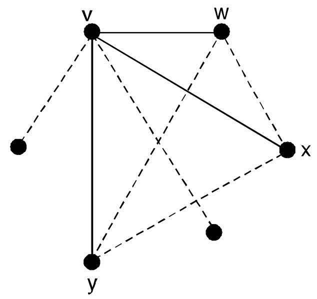


Fig. 3.10.

Problem 3.2. *Place the letters A, B, C, D, E, F, G, H into the eight circles in Figure (3.11), in such a way that no letter is adjacent to a letter that is next to it in the alphabet.*

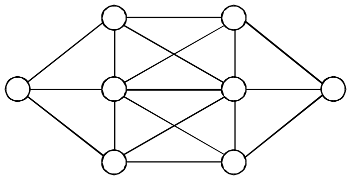


Fig. 3.11.

Solution. First note that trying all the possibilities is not a practical proposition, as there are 8 ! = 40320 ways of placing eight letters into eight circles.

Note that (*i*) the easiest letters to place are A and H, because each has only one letter to which it cannot be adjacent, namely, B and G, respectively.

(*ii*) the hardest circles to fill are those in the middle, as each is adjacent to six others.

This suggests that we place A and H in the middle circles. If we place A to the left of H, then the only possible positions for B and G are shown in Figure 3.12.

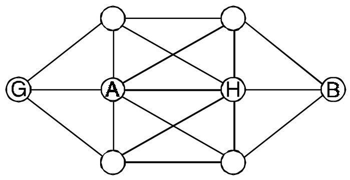


Fig. 3.12.

The letter C must now be placed on the left-hand side of the diagram, and F must be placed on the right-hand side.

It is then a simple matter to place the remaining letters, as shown in Figure 3.13.

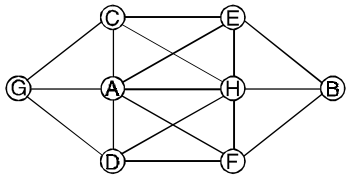


Fig. 3.13.

Problem 3.3. *Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw two such graphs.*

Solution. Suppose the graph with 6 vertices has e number of edges. Therefore by Handshaking lemma

$$\sum_{i=1}^{6} \deg(v_i) = 2e$$

$\Rightarrow \quad d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence the above equation,

$$(4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$\Rightarrow \quad 16 = 2e \quad \Rightarrow \quad e = 8.$

Hence the number of edges in a graph with 6 vertices with given condition is 8.

Two such graphs are shown below in Figure (3.14).

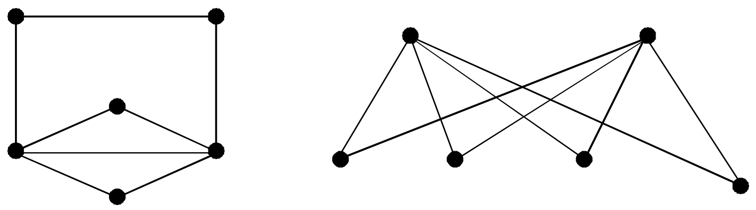


Fig. 3.14.

Problem 3.4. *How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree 2.*

Solution. Suppose these are P vertices in the graph with 6 degree. Also given the degree of each vertex is 2.

By handshaking lemma,

$$\sum_{i=1}^{P} \deg(v_i) = 2q = 2 \times 6$$

$\Rightarrow \quad d(v_1) + d(v_2) + ... + d(v_n) = 12$

$\Rightarrow \quad 2 + 2 + ... + 2 = 12$

$\Rightarrow \quad 2P = 12 \quad \Rightarrow \quad$ P = 6 vertices are needed.

Problem 3.5. *It is possible to construct a graph with 12 vertices such that 2 of the vertices have degree 3 and the remaining vertices have degree 4.*

Solution. Suppose it is possible to construct a graph with 12 vertices out of which 2 of them are having degree 3 and remaining vertices are having degree 4.

Hence by handshaking lemma,

$$\sum_{i=1}^{12} d(v_i) = 2e \text{ where } e \text{ is the number of edges}$$

According to given conditions

$$(2 \times 3) + (10 \times 4) = 2e$$

$\Rightarrow \quad 6 + 40 = 2e$

$\Rightarrow \quad 2e = 46 \qquad \Rightarrow \quad e = 23$

It is possible to construct a graph with 23 edges and 12 vertices which satisfy given conditions.

Problem 3.6. *It is possible to draw a simple graph with 4 vertices and 7 edges ? Justify.*

Solution. In a simple graph with P-vertices, the maximum number of edges will be $\frac{P(P-1)}{2}$.

Hence a simple graph with 4 vertices will have at most $\frac{4 \times 3}{2} = 6$ edges.

Therefore, the simple graph with 4 vertices cannot have 7 edges.

Hence such a graph does not exist.

Problem 3.7. *Show that the maximum degree of any vertex in a simple graph with P vertices is (P – 1).*

Solution. Let G be a simple graph with P-vertices. Consider any vertex v of G. Since the graph is simple (*i.e.,* without self loops and parallel edges), the vertex v can be adjacent to atmost remaining (P – 1) vertices.

Hence the maximum degree of any vertex in a simple graph with P vertices is (P – 1).

Problem 3.8. *Write down the vertex set and edge set of the following graphs shown in Figure 3.15(a) and 3.15(b).*

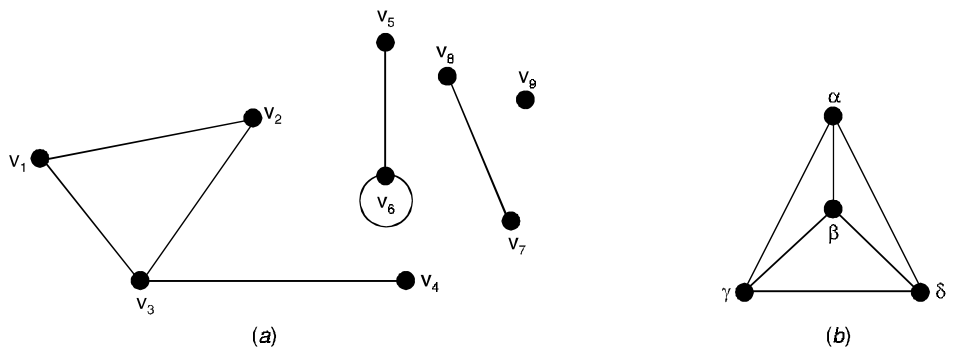


Fig. 3.15.

Solution. (*a*) V(G) = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$

E(G) = $\{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_5v_6, v_6v_6, v_7v_8\}$

(*b*) V(G) = $\{\alpha, \beta, \gamma, \delta\}$

E(G) = $\{\alpha\beta, \alpha\gamma, \alpha\delta, \beta\delta, \beta\gamma, \gamma\delta\}$.

Problem 3.9. *Show that the size of a simple graph of order n cannot exceed* $^nC_2$.

Solution. Let G be a graph of order *n*.

Let V be a vertex set of G.

Then cardinality of V is *n* and elements of E are distinct two elements subsets of V.

The number of ways we can choose two elements from a set V of *n* elements is $^nC_2$.

Thus, E may not have more than $^nC_2$ elements (edges).

Problem 3.10. *Find the degree sequence of the following graph.*

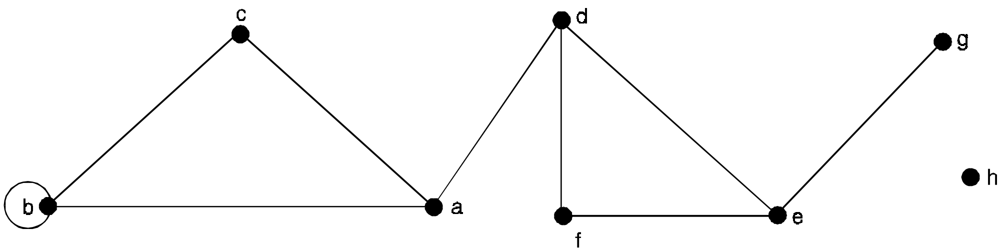


Fig. 3.16

Solution. $\deg_G(a) = 3$, $\deg_G(b) = 4$, $\deg_G(c) = 2$

$\deg_G(d) = 3$, $\deg_G(e) = 3$, $\deg_G(f) = 2$

$\deg_G(g) = 1$, $\deg_G(h) = 0$.

Therefore, the degree sequence of the graph is 0, 1, 2, 2, 3, 3, 4.

Problem 3.11. *Construct two graphs having same degree sequence.*

Solution. The following two graphs have the same degree sequence.

The degree sequence of the graphs is 2, 2, 2, 2, 2, 2.

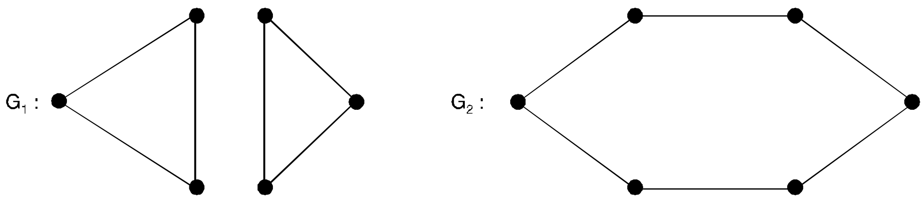


Fig. 3.17

Problem 3.12. *Show that there exists no simple graph corresponds to the following degree sequence :*

(i) 0, 2, 2, 3, 4 *(ii) 1, 1, 2, 3* *(iii) 2, 2, 3, 4, 5, 5* *(iv) 2, 2, 4, 6.*

Solution. (*i*) to (*iii*) :

There are odd number of odd degree vertices in the graph.

Hence there exists no graph corresponds to this degree sequence.

(*iv*) Number of vertices in the graph is four and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exceed one less than the number of vertices.

Problem 3.13. *Show that the total number of odd degree vertices of a (p, q)-graph is always even.*

Solution. Let $v_1, v_2 \ldots\ldots v_k$ be the odd degree vertices in G. Then, we have

$$\sum_{i=1}^{P} \deg_G (v_i) = 2q$$

i.e.,
$$\sum_{i=1}^{k} \deg_G (v_i) + \sum_{i=k+1}^{P} \deg_G (v_i) = 2q = \text{even number}$$

$$\Rightarrow \quad \sum_{i=1}^{k} \deg_G (v_i) = \text{even number} - \sum_{i=k+1}^{P} \deg_G (v_i)$$

$$\Rightarrow \quad \sum_{i=1}^{k} (\text{odd number}) = \text{even number} - \sum_{i=k+1}^{P} (\text{even number})$$

$$= \text{even number} - \text{even number}$$

$$= \text{even number.}$$

$\Rightarrow$ This implies that number of terms in the left-hand side of the equation is even.

Therefore, k is an even number.

Problem 3.14. *Show that the sequence 6, 6, 6, 6, 4, 3, 3, 0 is not graphical.*

Solution. To prove that the sequence is not graphical.

The given sequence is 6, 6, 6, 6, 4, 3, 3, 0

Resulting the sequence 5, 5, 5, 3, 2, 2, 0

Again consider the sequence 4, 4, 2, 1, 1, 0

Repeating the same 3, 1, 0, 0, 0

Since there exists no simple graph having one vertex of degree three and other vertex of degree one.

The last sequence is not graphical.

Hence the given sequence is also not graphical.

Problem 3.15. *Show that the following sequence is graphical. Also find a graph corresponding to the sequence 6, 5, 5, 4, 3, 3, 2, 2, 2.*

Solution. We can reduce the sequence as follows :

Given sequence	6, 5, 5, 4, 3, 3, 2, 2, 2
Reducing first 6 terms by 1 counting from second term	4, 4, 3, 2, 2, 1, 2, 2.
Writing in decreasing order	4, 4, 3, 2, 2, 2, 2, 1

Reducing first 4 terms by 1 counting from second	3, 2, 1, 1, 2, 2, 1
Writing in decending order	3, 2, 2, 2, 1, 1, 1
Reducing first 3 terms by 1, counting from second	1, 1, 1, 1, 1, 1

Sequence 1, 1, 1, 1, 1, 1 is graphical.

Hence the given sequence is also graphical.

The graph corresponding to the sequence 1, 1, 1, 1, 1, 1 is given below

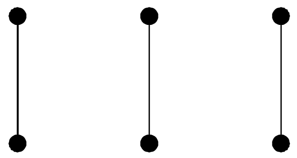


Fig. 3.18

To obtain a graph corresponding to the given sequence, add a vertex to each of the vertices whose degrees are $t_1 - 1$, $t_2 - 1$, ... $t_s - 1$.

And repeat the process.

Step 1 :

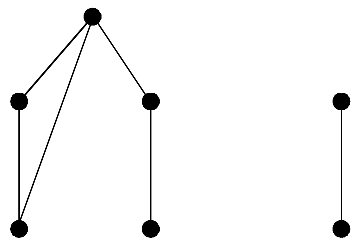


Fig. 3.19

Degree sequence of this graph is 3, 2, 2, 2, 1, 1, 1

Step 2 :

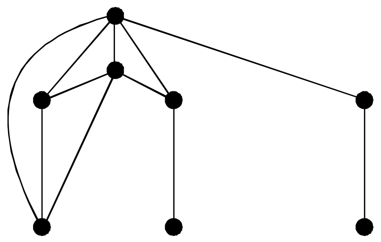


Fig. 3.20

Degree sequence of this graph is 4, 4, 3, 2, 2, 2, 2, 1.

Step 3 : Final graph

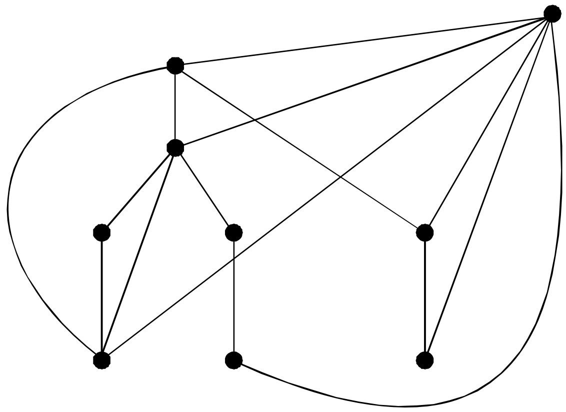


Fig. 3.21

Degree sequence of this graph is 6, 5, 5, 4, 3, 3, 2, 2, 2.

Problem 3.16. *Show that no simple graph has all degrees of its vertices are distinct.*

(*i.e., in a degree sequence of a graph atleast one number should repeat.*)

Solution. Let G be a graph of order *n*.

Then there are *n* terms in the degree sequence of G. If no number (integer) in the degree sequence repeats, then only possible case it is of the form

$$0, 1, 2, 3, 4, ..., n - 1$$

Since maximum degree cannot exceed $n - 1$. But the last vertex of degree $n - 1$ should be adjacent to every other vertex of G, since G is simple.

Thus minimum degree of every vertex is one.

A contradiction to the fact that the degree of one vertex is zero.

Problem 3.17. *Is there a simple graph with degree sequence (1, 1, 3, 3, 3, 4, 6, 7) ?*

Solution. Assume there is such a graph. Then the vertex of degree 7 is adjacent to all other vertices, so in particular it must be adjacent to both vertices of degree 1.

Hence, the vertex *v* of degree 6 cannot be adjacent to either of the two vertices of degree 1.

Problem 3.18. *Find the degree of each vertex of the following graph :*

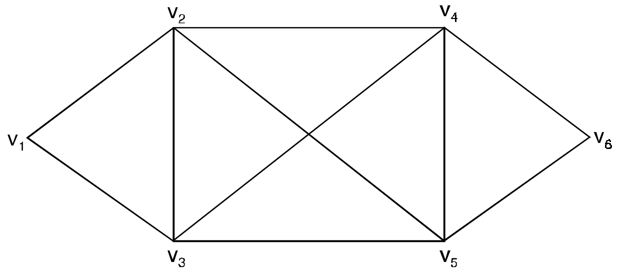


Fig. 3.22

Solution. It is an undirected graph. Then

deg (v_1) = 2, deg (v_2) = 4, deg (v_3) = 4

deg (v_4) = 4, deg (v_5) = 4, deg (v_6) = 2.

Problem 3.19. *Find the in degree out degree and of total degree of each vertex of the following graph.*

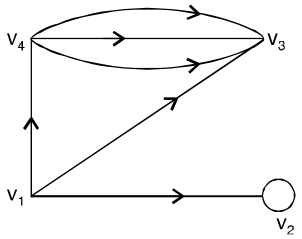


Fig. 3.23

Solution. It is a directed graph

in deg (v_1) = 0, out deg (v_1) = 3, total deg (v_1) = 4

in deg (v_2) = 2, out get (v_2) = 1, total deg (v_2) = 3

in deg (v_3) = 4, out deg (v_3) = 0, total deg (v_3) = 4

in deg (v_4) = 1, out deg (v_4) = 3, total deg (v_4) = 4.

Problem 3.20. *State which of the following graphs are simple ?*

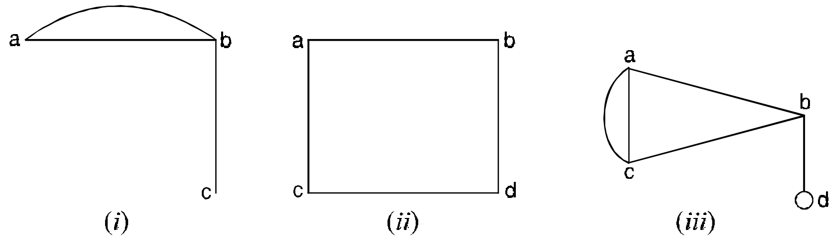


Fig. 3.24

Solution. (*i*) The graph is not a simple graph, since it contains parallel edge between two vertices *a* and *b*.

(*ii*) The graph is a simple graph, it does not contain loop and parallel edge.

(*iii*) The graph is not a simple graph, since it contains parallel edge and a loop.

Problem 3.21. *Draw the graphs of the chemical molecules of*

(i) Methane (CH_4) *(ii) Propane* (C_3H_8).

Solution. (*i*) (*ii*)

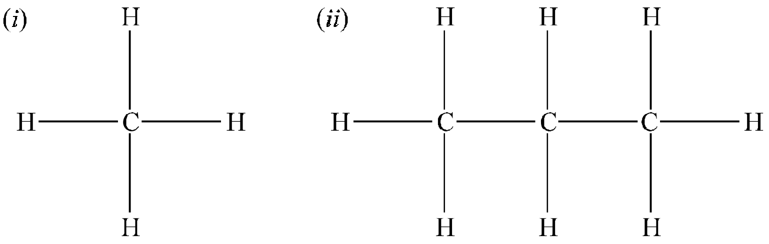


Fig. 3.25

Problem 3.22. *Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n – 1.*

Solution. Let v be a vertex of G, since G is simple, no multiple edges or loops are allowed in G.

Thus v can be adjacent to atmost all the remaining $n - 1$ vertices of G.

Hence v may have maximum degree $n - 1$ in G.

i.e., $$0 \leq \deg_G (v) \leq n - 1 \text{ for all } v \in V(G).$$

Problem 3.23. *Does there exists a simple graph with seven vertices having degrees (1, 3, 3, 4, 5, 6, 6) ?*

Solution. Suppose there exists a graph with seven vertices satisfying the given properties.

Since two vertices have degree 6, each of these two vertices is adjacent with every other vertex.

Hence the degree of each vertex is at least 2, so that the graph has no vertex of degree 1, which is a contradiction.

Hence there does not exist a simple graph with the given properties.

Problem 3.24. *Is there a simple graph corresponding to the following degree sequences ?*

(i) (1, 1, 2, 3) *(ii) (2, 2, 4, 6).*

Solution. (*i*) There are odd number (3) of odd degree vertices, 1, 1 and 3.

Hence there exist no graph corresponding to this degree sequence.

(*ii*) Number of vertices in the graph sequence is 4, and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exist on less than the number of vertices.

Problem 3.25. *Show that the maximum number of edges in a simple graph with n vertices is* $\frac{n(n-1)}{2}$.

Solution. By the handshaking theorem,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges with n vertices in the graph G.

$$\Rightarrow \quad d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 2e \qquad \ldots(1)$$

Since we know that the maximum degree of each vertex in the graph G can be $(n - 1)$.

Therefore, equation (1) reduces

$$(n-1) + (n-1) + \text{......} \text{ to } n \text{ terms} = 2e$$

$\Rightarrow \quad n(n-1) = 2e$

$\Rightarrow \quad e = \dfrac{n(n-1)}{2}.$

Hence the maximum number of edges in any simple graph with n vertices is $\dfrac{n(n-1)}{2}$.

Problem 3.26. *Consider the following graphs and determine the degree of each vertex :*

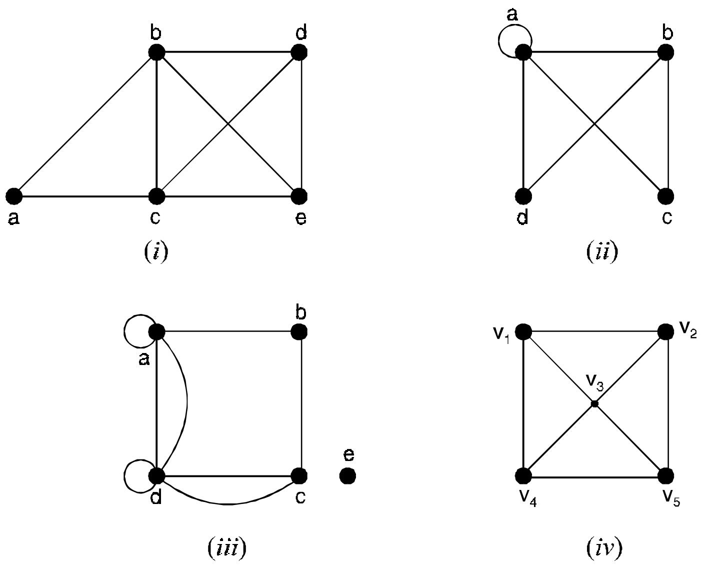


Fig. 3.26

Solution. (*i*) deg (*a*) = 2, deg (*b*) = 4, deg (*c*) = 4, deg (*d*) = 3, deg (*e*) = 3

(*ii*) deg (*a*) = 5, deg (*b*) = 2, deg (*c*) = 3, deg (*d*) = 6, deg (*e*) = 0

(*iii*) deg (*a*) = 5, deg (*b*) = 3, deg (*c*) = 2, deg (*d*) = 2,

(*iv*) Every vertex has degree 4.

Problem 3.27. *Find the in-degree and out-degree of each vertex of the following directed graphs :*

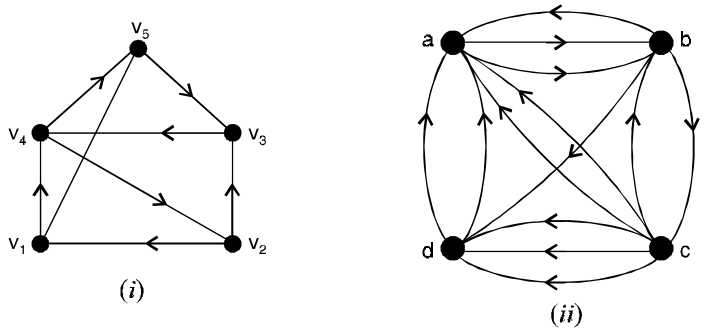


Fig. 3.27

Solution. (*i*) in-degree $v_1 = 2$, out-degree $v_1 = 1$

in-degree $v_2 = 2$, out-degree $v_2 = 2$

in-degree $v_3 = 2$, out-degree $v_3 = 1$

in-degree $v_4 = 2$, out-degree $v_4 = 2$

in-degree $v_5 = 0$, out-degree $v_5 = 3$

(*ii*) in-degree $a = 6$, out-degree $a = 1$

in-degree $b = 1$, out-degree $b = 5$

in-degree $c = 2$, out-degree $c = 5$

in-degree $d = 2$, out-degree $d = 2$.

Problem 3.28. *Draw a graph having the given properties or explain why no such graph exists.*

(i) Graph with four vertices of degree 1, 1, 2 and 3.

(ii) Graph with four vertices of degree 1, 1, 3 and 3

(iii) Simple graph with four vertices of degree 1, 1, 3 and 3

(iv) Graph with six vertices each of degree 3

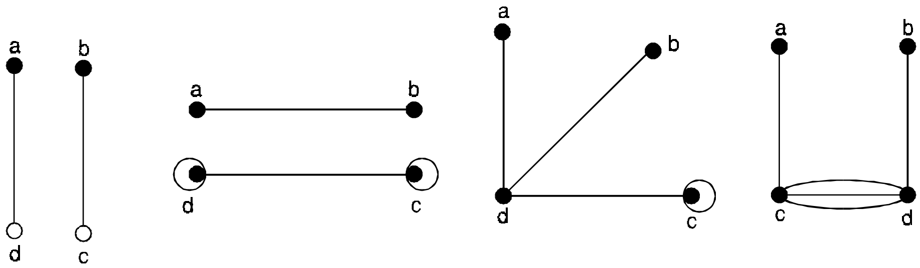
(v) Graph with six vertices and four edges

(vi) Graph with five vertices of degree 3, 3, 3, 3, 2

(vii) Graph with five vertices of degree 0, 1, 2, 2, 3.

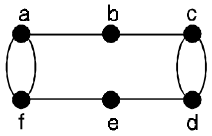
Solution. (*i*) No such graphs exists, total degree is odd.

(*ii*)

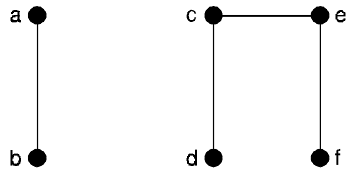


(*iii*) No simple graph.

(*iv*)



(*v*)



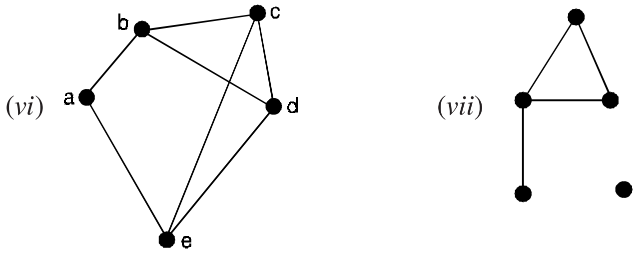


Fig. 3.28

Problem 3.29. *If the simple graph G has V vertices and e edges, how many edges does G′ (complement of G) have ?*

Solution. $\dfrac{v(v-1)}{2-e}$.

Problem 3.30. *Construct a 3-regular graph on 10 vertices.*

Solution. The following graphs are some examples of 3-regular graphs on 10 vertices.

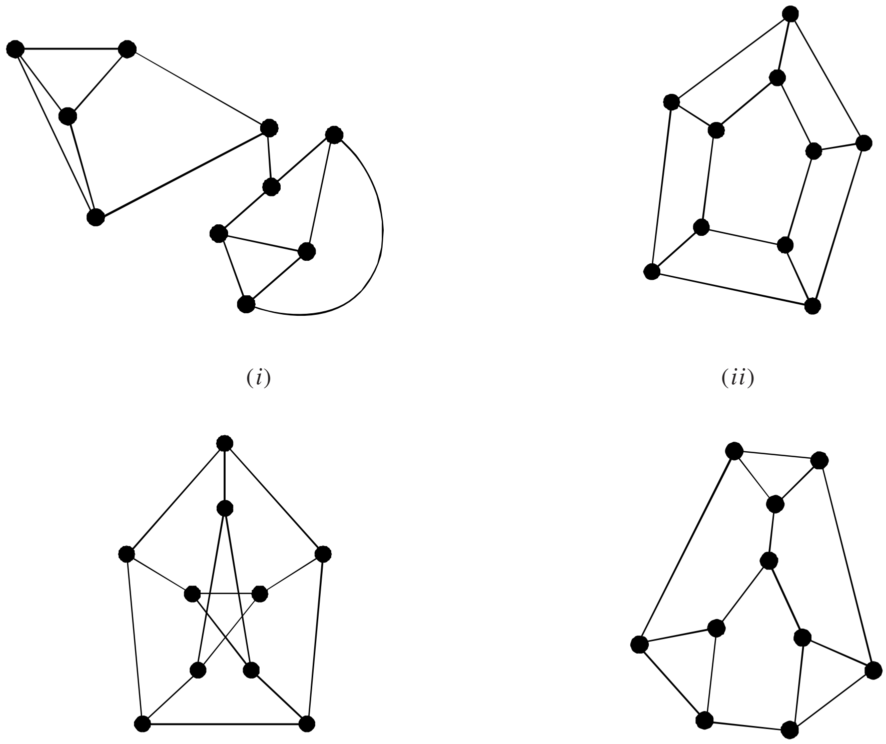


Fig. 3.29

Problem 3.31. *Does there exists a 4-regular graph on 6 vertices ? If so construct a graph.*

Solution. We have $q = \frac{P \times r}{2} = \frac{6 \times 4}{2} = 12$

Hence 4-regular graph on 6-vertices is possible and it contains 12 edges. One of the graph is shown below.

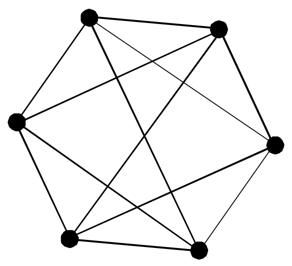


Fig. 3.30

Every 4-regular graph contains a 3-regular graph.

Problem 3.32. *What is the size of an r-regular (p, q)-graph.*

Solution. Since G is an r-regular graph.

By the definition of regularity of G.

We have $\deg_G (v_i) = r$ for all $v_i \in V(G)$

But $2q = \sum_{i=1}^{P} \deg_G (v_i)$

$$2q = \sum_{i=1}^{P} r = P \times r$$

$$\Rightarrow \quad q = \frac{P \times r}{2}.$$

Problem 3.33. *Does a 3-regular graph on 14 vertices exist ? What can you say on 17 vertices ?*

Solution. We have $q = \frac{P \times r}{2}$

given $r = 3$, $\quad P = 14$

Now $q = \frac{14 \times 3}{2} = 21$, is a positive integer.

Hence 3-regular graphs on 14 vertices exist.

Further, if $P = 17$, then $q = \frac{P \times r}{2} = \frac{17 \times 3}{2} = \frac{51}{2}$ is not a positive integer.

Hence no 3-regular graphs on 17 vertices exist.

3.6 TYPES OF GRAPHS

Some important types of graph are introduced here.

3.6.1. Null graph

A graph which contains only **isolated node,** is called a null graph.

i.e., the set of edges in a null graph is empty.

Null graph is denoted on n vertices by N_n

N_4 is shown in Figure (3.31), Note that each vertex of a null graph is isolated.

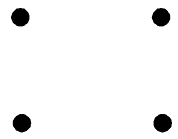


Fig. 3.31

3.6.2. Complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.

i.e., if G contains exactly one edge between each pair of distinct vertices.

A comple graph is usually denoted by $\mathbf{K}_n$. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges.

The graphs K_n for n = 1, 2, 3, 4, 5, 6 are shown in Figure 3.32.

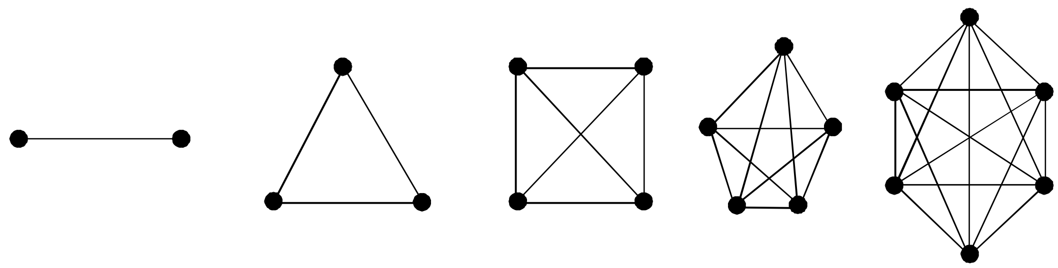


Fig. 3.32

3.6.3. Regular graph

A graph in which all vertices are of **equal degree,** is called a **regular graph.**

If the degree of each vertex is r, then the graph is called a regular **graph of degree r.**

Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n - 1$. Also, note that, if G has n vertices and is regular of degree r, then G has $\left(\frac{1}{2}\right) r\, n$ edges.

3.6.4. Cycles

The cycle C_n, $n \geq 3$, consists of n vertices v_1, v_2, ..., v_n and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cyles c_3, c_4, c_5 and c_6 are shown in Figure 3.33.

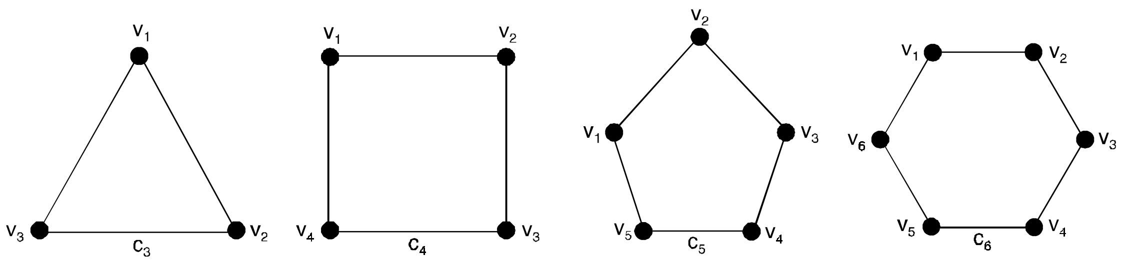


Fig. 3.33. Cycles C_3, C_4, C_5 and C_6.

3.6.5. Wheels

The wheel W_n is obtained when an additional vertex to the cycle c_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in c_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 3.34.

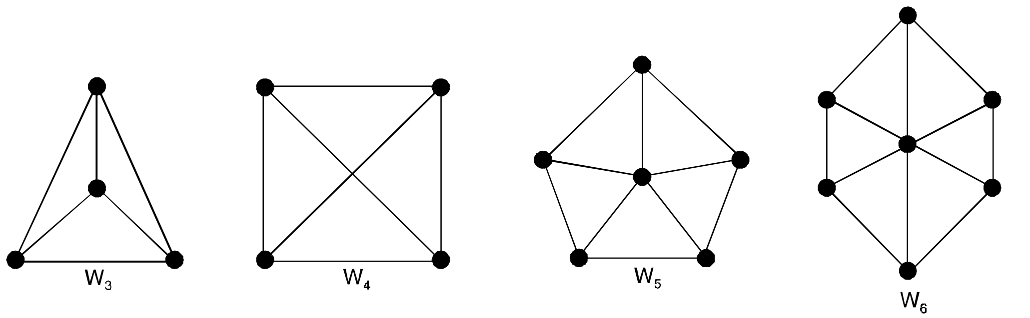


Figure 3.34. The wheels W_3, W_4, W_5 and W_6

3.6.6. Platonic graph

The graph formed by the vertices and edges of the five regular (platonic) solids—The tetrahedron, octahedron, cube, dodecahedron and icosahedron.

The graphs are shown in Figure 3.35.

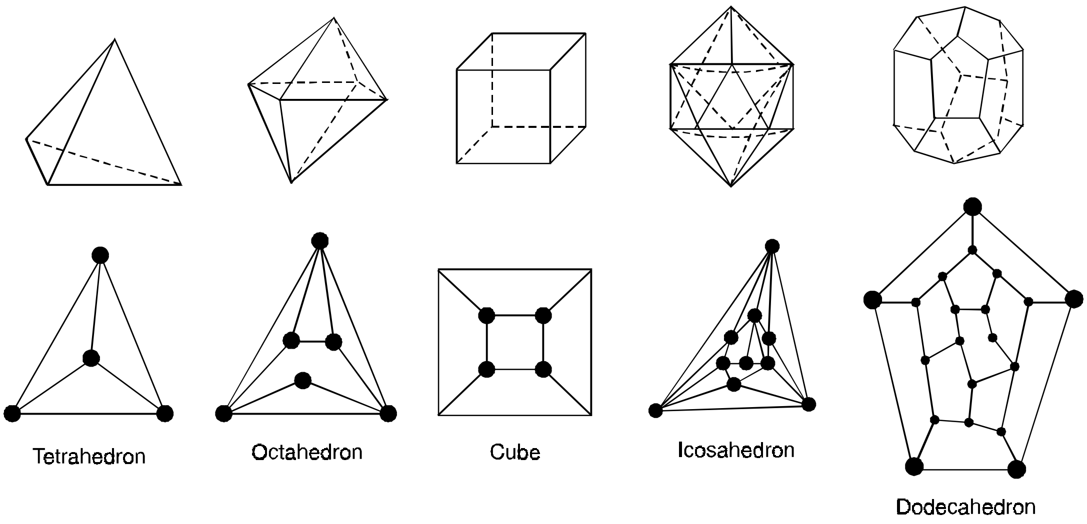


Fig. 3.35.

3.6.7. N-cube

The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in Figure 3.36. Thus Q_n has 2^n vertices and $n \cdot 2^{n-1}$ edges, and is regular of degree n.

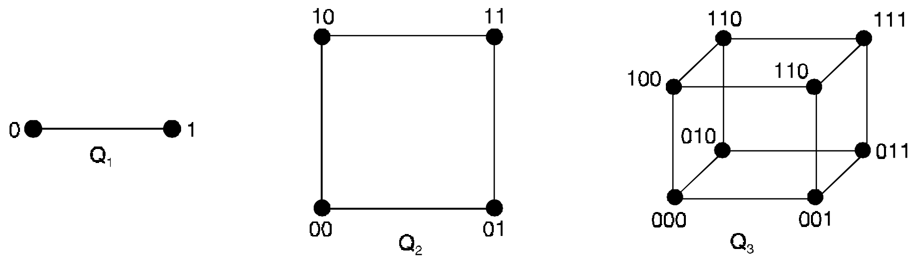


Fig. 3.36. The *n*-cube Q_n for n = 1, 2, 3.

Problem 3.34. *Determine whether the graphs shown is a simple graph, a multigraph, a pseudograph.*

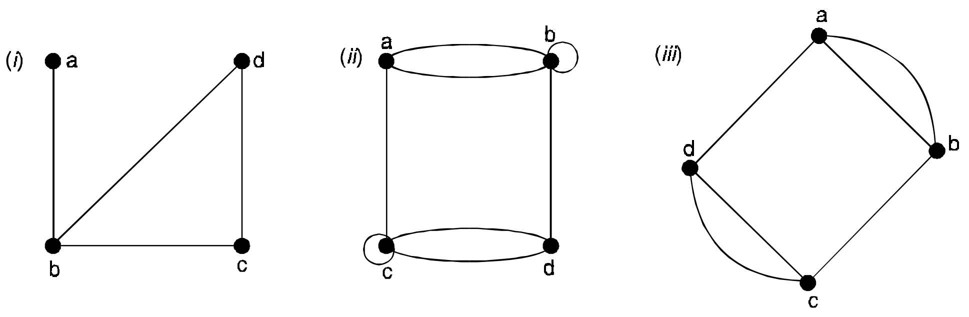


Figure 3.37.

Solution. (*i*) Simple graph

(*ii*) Pseudograph

(*iii*) Multigraph.

Problem 3.35. *Consider the following directed graph G : V(G) = {a, b, c, d, e, f, g}*

E(G) = {(a, a), (b, e), (a, e), (e, b), (g, c), (a, e), (d, f), (d, b), (g, g)}.

(*i*) *Identify any loops or parallel edges.*

(*ii*) *Are there any sources in G ?*

(*iii*) *Are there any sinks in G ?*

(*iv*) *Find the subgraph H of G determined by the vertex set V′ = {a, b, c, d}.*

Solution. (*i*) (*a*, *a*) and (*g*, *g*) are loops

(*a*, *a*) and (*a*, *e*) are parallel edges.

(*ii*) No sources

(*iii*) No sinks

(*iv*) V′ = {*a*, *b*, *c*, *d*}

E′ = {(*a*, *a*), (*d*, *b*)}

H = H(V′, E′).

Problem 3.36. *Consider the following graphs, determine the (i) vertex set and (ii) edge set.*

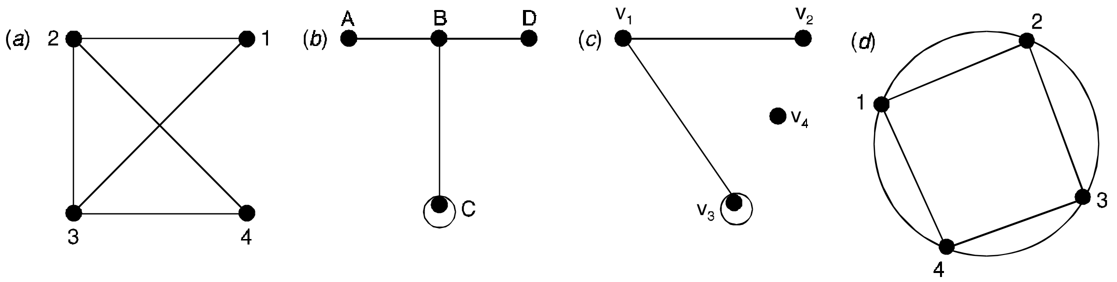


Figure 3.38.

Solution. (*a*) (*i*) Vertex set V = {1, 2, 3, 4},

(*ii*) Edge set E = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)}

(*b*) (*i*) Vertex set V = {A, B, C, D}

(*ii*) Edge set E = {(A, B), (B, C), (B, D), (C, C)}

(*c*) (*i*) Vertex set V = {v_1, v_2, v_3, v_4}

(*ii*) Edge set E = {(v_1, v_2), (v_1, v_3), (v_3, v_3)}

(*d*) (*i*) Vertex set V = {1, 2, 3, 4}

(*ii*) Edge set E = {(1, 2), (2, 3), (3, 4), (4, 1)}

All edges are double edges.

Problem 3.37. *How many vertices and how many edges do the following graphs have ?*

(*i*) K_n (*ii*) C_n (*iii*) W_n (*iv*) $K_{m,\, n}$ (*v*) Q_n.

Solution. (*i*) *n* vertices and $\dfrac{n(n-1)}{2}$ edges.

(*ii*) *n* vertices and *n* edges

(*iii*) $n + 1$ vertices and $2n$ edges

(*iv*) $m + n$ vertices and *mn* edges

(*v*) 2^n vertices and $n \,.\, 2^{n-1}$ edges.

Problem 3.38. *There are two different chemical molecules with formula C_4H_{10} (isobutane). Draw the graphs corresponding to these molecules.*

Solution.

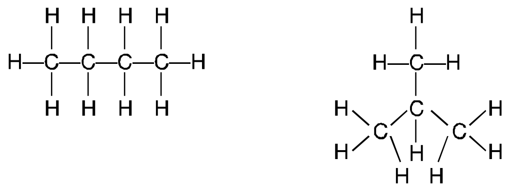


Fig. 3.39

Problem 3.39. *Draw all eight graphs with five vertices and seven or more edges.*

Solution.

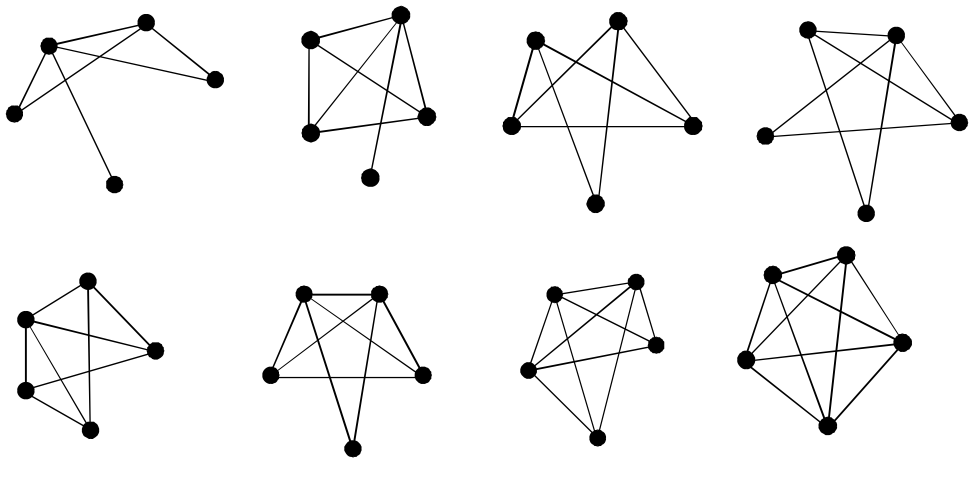


Fig. 3.39

Problem 3.40. *Draw all six graphs with five vertices and five edges.*

Solution.

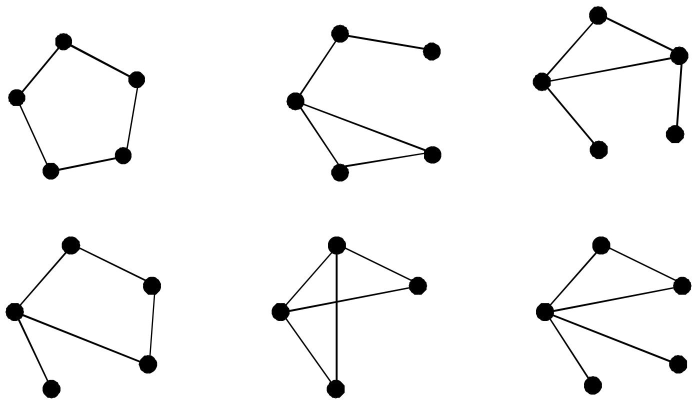


Fig. 3.40

3.7 SUBGRAPHS

A subgraph of G is a graph having all of its vertices and edges in G. If G_1 is a subgraph of G, then G is a super graph of G_1.

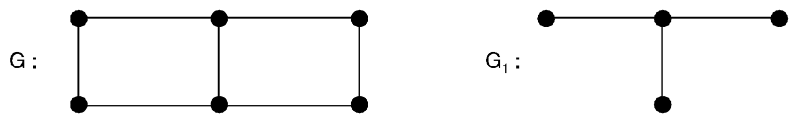


Fig. 3.41. G_1 is a subgraph of G.

In other words. If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G or G as a supergraph of H.

3.7.1. Spanning subgraph

A spanning subgraph is a subgraph containing all the vertices of G.

In other words, if $V(H) \subset V(G)$ and $E(H) \subseteq E(G)$ then H is a proper subgraph of G and if V(H) = V(G) then we say that H is a spanning subgraph of G.

A spanning subgraph need not contain all the edges in G.

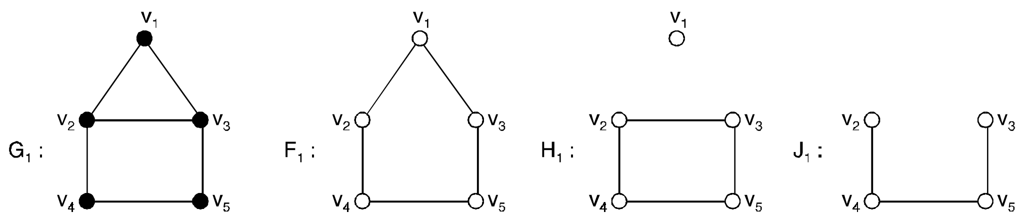


Fig. 3.42.

The graphs F_1 and H_1 of the above Fig. 3.42 are spanning subgraphs of G_1, but J_1 is not a spanning subgraph of G_1.

Since $V_1 \in V(G_1) - V(J_1)$. If E is a set of edges of a graph G, then G – E is a spanning subgraph of G obtained by deleting the edges in E from E(G).

In fact, H is a spanning subgraph of G if and only if H = G – E, where E = E(G) – E(H). If e is an edge of a graph G, then we write $G - e$ instead of $G - \{e\}$. For the graphs G_1, F_1 and H_1 of the Fig. 3.42, we have $F_1 = G_1 - v_2v_3$ and $H_1 = G_1 - \{v_1v_2, v_2v_3\}$.

3.7.2. Removal of a vertex and an edge

The removal of a vertex v_i from a graph G result in that subgraph $G - v_i$ of G containing of all vertices in G except v_i and all edges not incident with v_i. Thus $G - v_i$ is the maximal subgraph of G not containing v_i. On the otherhand, the removal of an edge x_j from G yields the spanning subgraph $G - x_j$ containing all edges of G except x_j.

Thus $G - x_j$ is the maximal subgraph of G not containing x_j.

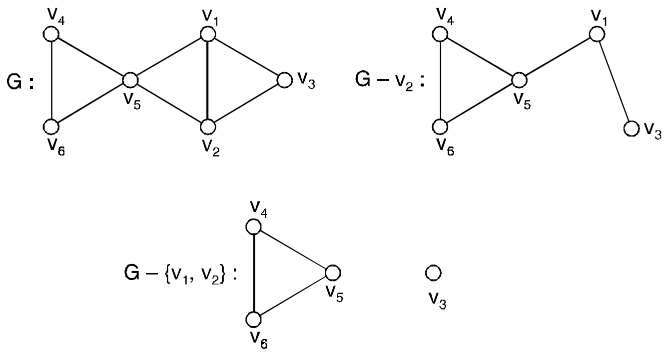


Fig. 3.43(*a*). Deleting vertices from a graph.

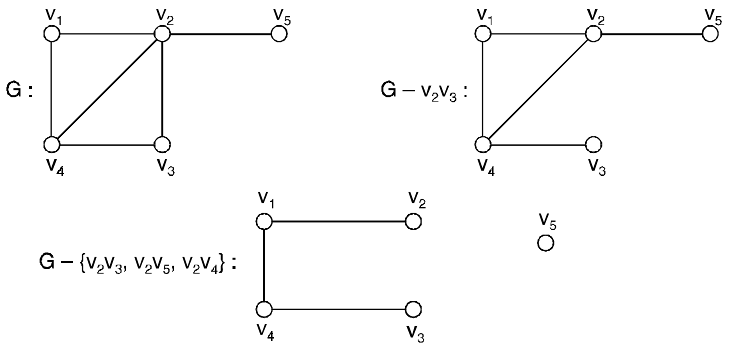
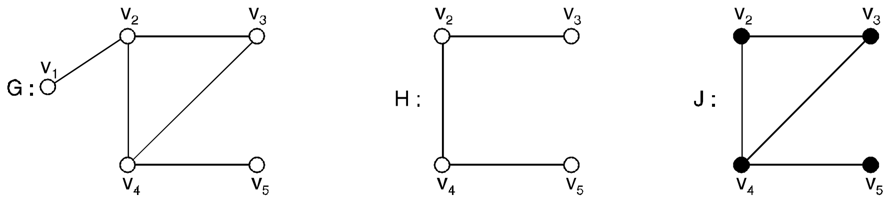


Fig. 3.43(*b*). Deleting edges from a graph.

3.7.3. Induced subgraph

For any set S of vertices of G, the vertex induced subgraph or simply an induced subgraph <S> is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in <S> if and only if they are adjacent in G.

In other words, if G is a graph with vertex set V and U is a subset of V then the subgraph G(U) of G whose vertex set is U and whose edge set comprises exactly the edges of E which join vertices in U is termed as induced subgraph of G.



Here H is not an induced subgraph since $v_4v_1 \in$ E(G), but $v_4v_3 \notin$ E(H).

On the otherhand the graph J is an induced subgraph of G. Thus every induced subgraph of a graph G is obtained by deleting a subset of vertices from G.

Note : Let | V | = m and | E | = n. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n.

Thus, number of subgraphs is equal to $(2^m - 1) \times 2^n$.

The number of spanning subgraphs is equal to 2^n.

3.8 GRAPHS ISOMORPHISM

Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f : v_1 \rightarrow v_1$ is called a graphs isomorphism if

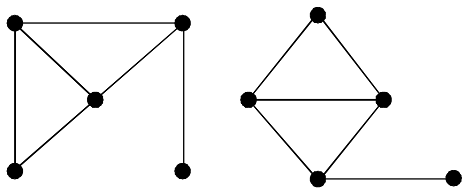
(i) f is one-to-one and onto.

(ii) for all $a, b \in v_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$ when such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \cong G_2$.

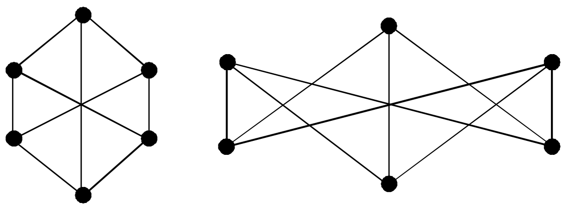
In other words, two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and between edges such that incidence relationship is preserve. Written as $G_1 \cong G_2$ or $G_1 = G_2$.

The necessary conditions for two graphs to be isomorphic are

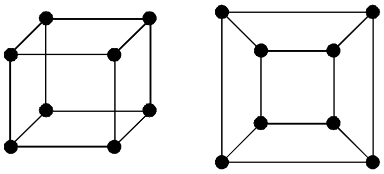
1. Both must have the **same number of vertices**
2. Both must have the **same number of edges**
3. Both must have **equal number of vertices** with the **same degree.**
4. They must have the same degree sequence and same cycle vector $(c_1, ..., c_n)$, where c_i is the number of cycles of length i.



(i)



(ii)



(*iii*)

Fig. 3.44(*i*), (*ii*) (*iii*) Isomorphic pair of graphs

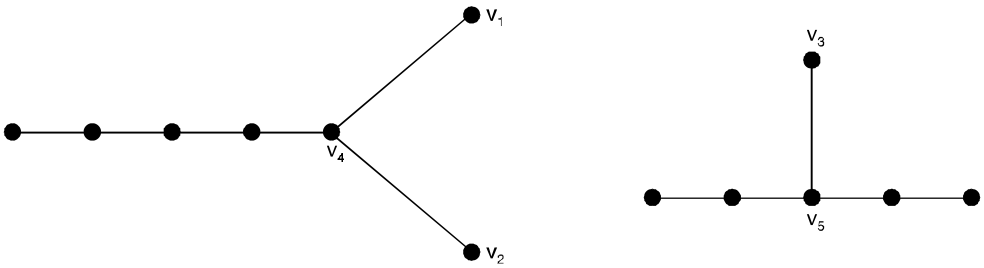


Fig. 3.45. Two graphs that are not isomorphic.

Problem 3.41. *Construct two edge-disjoint subgraphs and two vertex disjoint subgraphs of a graph G shown below*

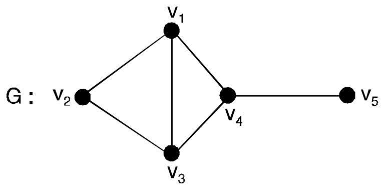


Fig. 3.46

Solution.

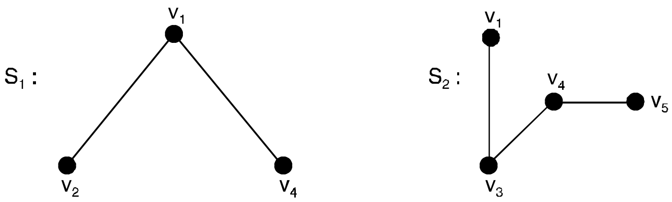


Fig. 3.47

The graphs S_1 and S_2 are edge-disjoint subgraphs of G.

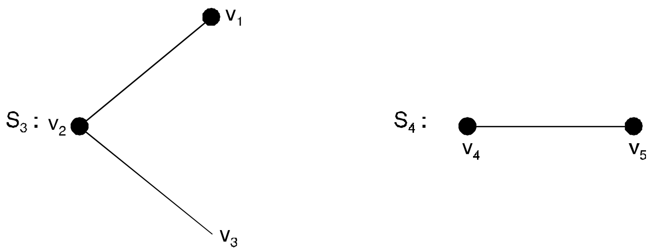


Fig. 3.48

S_3 and S_4 are vertex disjoint subgraphs of G which are also edge-disjoint subgraphs of G.

Problem 3.42. *Does there exist a proper subgraph S of G whose size is equal to the size of the graph ?*

Solution. Yes, consider the graph G shown in Figure below.

The graph S is a subgraph of G with $V(S) \subset E(G)$ and $E(S) = E(G)$.

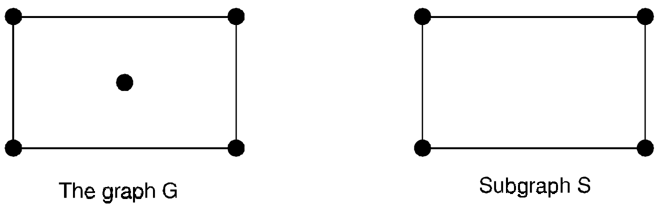


Fig. 3.49

Problem 3.43. *Write down all possible non-isomorphic subgraphs of the following graphs G. How many of they are spanning subgraphs ?*

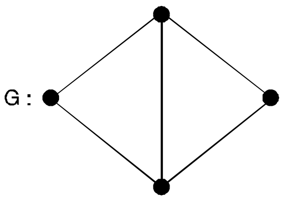
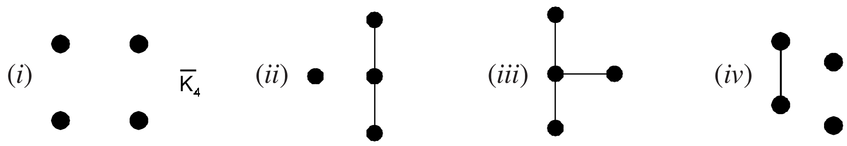


Fig. 3.50

Solution. Its possible all (non-isomorphic) subgraphs are



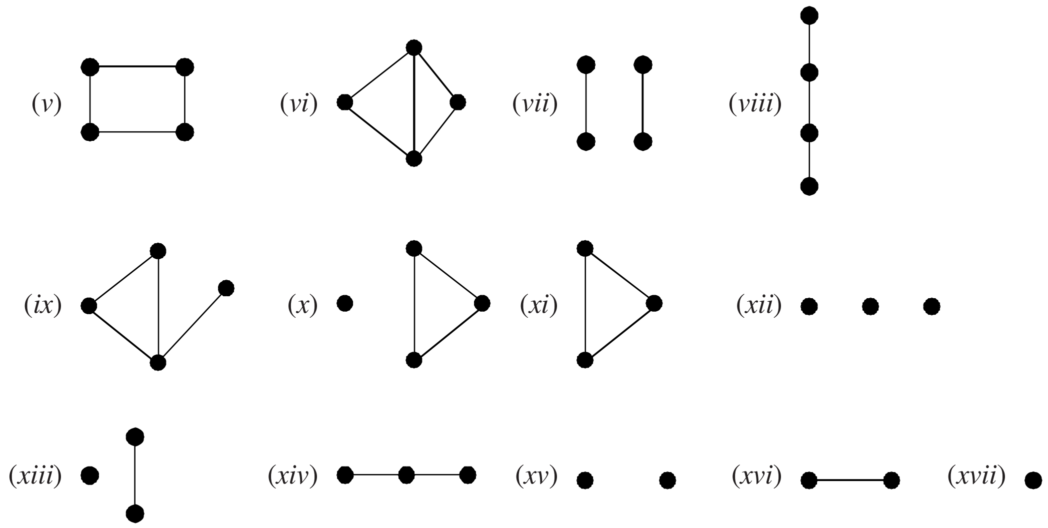


Fig. 3.51

of these graphs (*i*) to (*x*) are spanning subgraphs of G.

All the graphs except (*vi*) are proper subgraphs of G.

Problem 3.44. *Construct three non-isomorphic spanning subgraphs of the graph G shown below :*

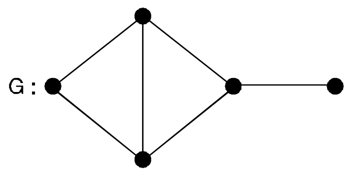


Fig. 3.52

Solution. Three non-isomorphic subgraphs are

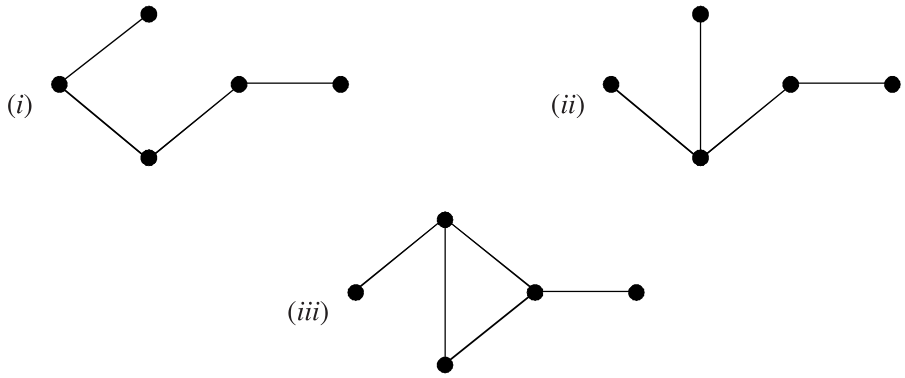


Fig. 3.53

Problem 3.45. *Find the total number of subgraphs and spanning subgraphs in K_6, L_5 and Q_3.*

Solution. In graph K_6, we have | V | = 6 and | E | = 15

Thus, total number of subgraph is

$$(2^6 - 1) \times 2^{15} = 63 \times 32768 = 2064384$$

The total number of spanning subgraph is : $2^{15} = 32768$.

In the linear graph L_5, we have | V | = 5 and | E | = 4

Thus, total number of subgraph is

$$(2^5 - 1) \times 2^4 = 31 \times 16 = 496.$$

The total number of spanning subgraph is : $2^4 = 16$.

In the 3-cube graph Q_3, we have | V | = 8 and | E | = 12

Thus, total number of subgraph is

$$(2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192$$

The total number of spanning subgraphs is

$$2^{12} = 4096.$$

Problem 3.46. *For the graph G shown below, draw the subgraphs*

(*i*) *G – e* (*ii*) *G – a* (*iii*) *G – b.*

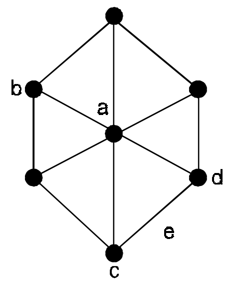


Fig. 3.54

Solution. (*i*) After deleting the edge *e* = (*c*, *d*) from the graph G, we get a subgraph G – *e* as shown below

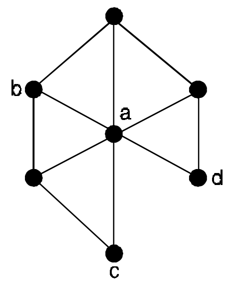


Fig. 3.55

(*ii*) After deleting the vertex *a* from the graph G, and all edges incident on this vertex, we set the subgraph G – *a* as shown below :

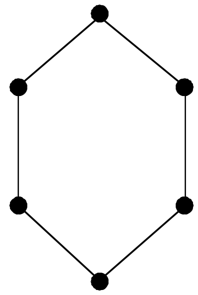


Fig. 3.56

(*iii*) The subgraph is obtained after deleting the vertex *b*.

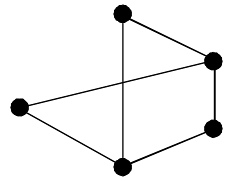


Fig. 3.57

Problem 3.47. *Consider the graph G(V, E) shown below, determine whether or not H(V_1, E_1) is a subgraph of G, where*

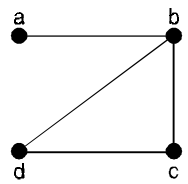


Fig. 3.58

(*i*) $V_1 = \{a, b, d\}$ $E_1 = \{(a, b), (a, d)\}$

(*ii*) $V_1 = \{a, b, c, d\}$ $E_1 = \{(b, c), (b, d)\}$

Solution. (*i*) H is not a subgraph because (*a*, *d*) is not an edge in G.

(*ii*) H is a subgraph because it satisfies condition for a subgraph of the given graph G.

Problem 3.48. *Find all possible non-isomorphic induced subgraphs of the following graph G corresponding to the three element subsets of the vertex set of G*

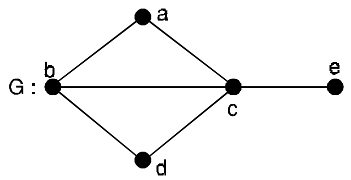
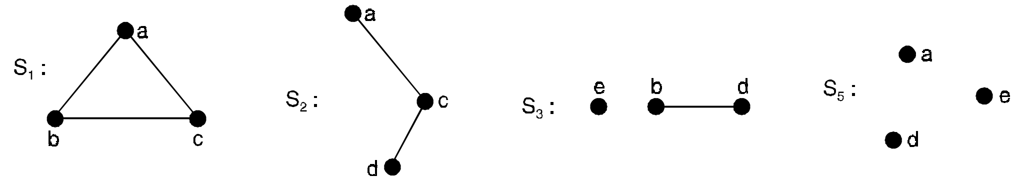


Fig. 3.59

Solution.



The subgraph S shown in Figure (3.60) of the above graph G shown in Figure 3.59 is not a induced subgraph of G.

For the edge (a, d) of G can be added to S. The graph obtained by adding this edge is again a subgraph of

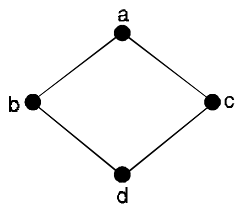


Fig. 3.60

Note : The graph G is itself a maximal subgraph of G.

Problem 3.49. *Show that the following graphs are isomorphic*

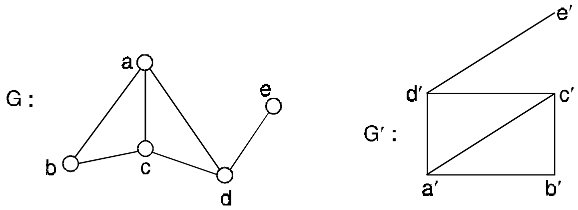


Fig. 3.61

Solution. Let $f : G \to G'$ be any function defined between two graphs degrees of the graph G and G′ are as follows :

deg (G)	deg (G′)
deg (a) = 3	deg (a') = 3
deg (b) = 2	deg (b') = 2
deg (c) = 3	deg (c') = 3
deg (d) = 3	deg (d') = 3
deg (e) = 1	deg (e') = 1

Each has 5-vertices and 6-edges.

$$d(a) = d(a') = 3$$

$$d(b) = d(b') = 2$$

$d(c) = d(c') = 3$

$d(d) = d(d') = 3$

$d(e) = d(e') = 1$

Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.

Therefore, the given two graphs are isomorphic.

Problem 3.50. *Show that the following graphs are isomorphic.*

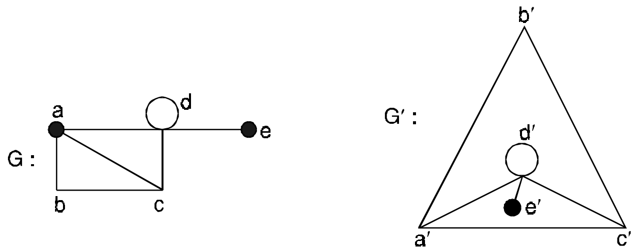


Fig. 3.62

Solution. Let $f : G \to G'$ be any function defined between two graphs degrees of the graphs G and G′ are as follows :

deg (G)	deg (G′)
deg (a) = 3	deg (a') = 3
deg (b) = 2	deg (b') = 2
deg (c) = 3	deg (c') = 3
deg (d) = 5	deg (d') = 5
deg (e) = 1	deg (e') = 1

Each has 5-vertices, 6-edges and 1-circuit.

$\deg(a) = \deg(a') = 3$

$\deg(b) = \deg(b') = 2$

$\deg(c) = \deg(c') = 3$

$\deg(d) = \deg(d') = 5$

$\deg(e) = \deg(e') = 1$

Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.

Therefore, the given two graphs G and G′ are isomorphic.

Problem 3.51. *Are the 2-graphs, is given below, is isomorphic ? Give a reason.*

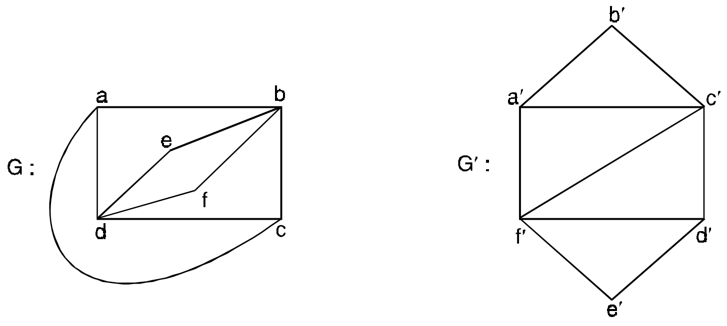


Fig. 3.63

Solution. Let us enumerate the degree of the vertices

Vertices of degree 4 : $b - f'$

$d - c'$

Vertices of degree 3 : $a - a'$

$c - d'$

Vertices of degree 2 : $e - b'$

$f - e'$

Now the vertices of degree 3, in G are a and c and they are adjacent in G′, while these are a' and d' which are not adjacent in G′.

Hence the 2-graphs are not isomorphic.

Problem 3.52. *Show that the two graphs shown in Figure are isomorphic.*

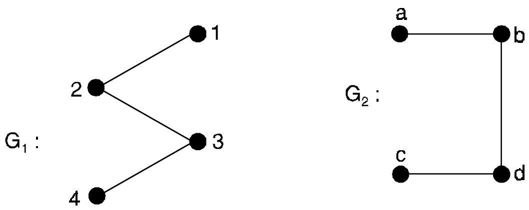


Fig. 3.64

Solution. Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$

$E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ and $E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}$

Define a function $f : V(G_1) \to V(G_2)$ as

$f(1) = a, f(2) = b, f(3) = d,$ and $f(4) = c$

f is clearly one-one and onto, hence an isomorphism.

Further, $\{1, 2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2, 3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3, 4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

and $\{1, 3\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

$\{1, 4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

$\{2, 4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$.

Hence f preserves adjacency as well as non-adjacency of the vertices.

Therefore, G_1 and G_2 are isomorphic.

Problem 3.53. *For each pair of graphs shown, either label the graphs so as to exhibit an isomorphism or explain why the graphs are not isomorphic.*

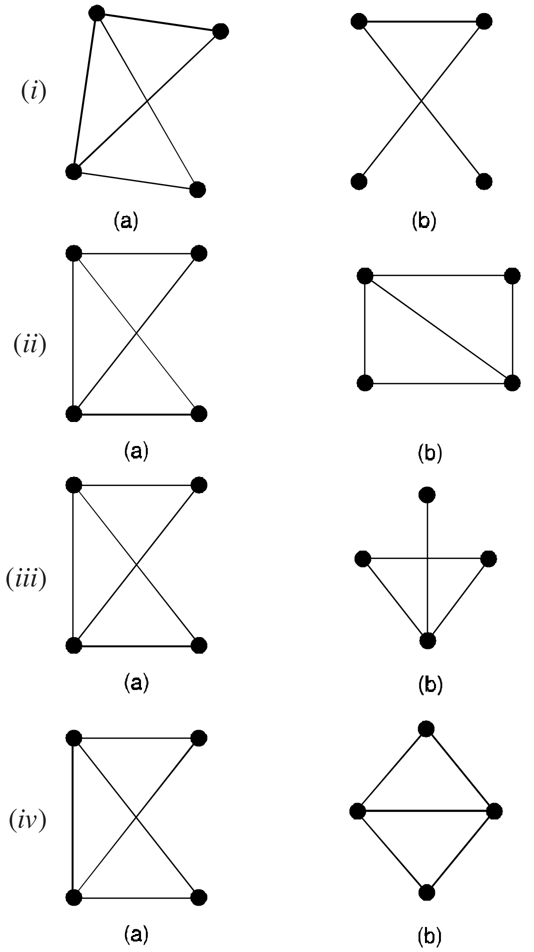


Fig. 3.65

Solution. (*i*) The graphs are not isomorphic because (*a*) has 5-edges and (*b*) has 4-edges.

(*ii*) The graphs are isomorphic, as shown by the labelling

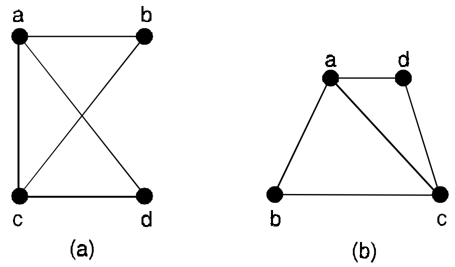


Fig. 3.66

(*iii*) The graphs are not isomorphic because (*b*) has a vertex of degree 1 and (*a*) does not have.

(*iv*) The graphs are isomorphic, as shown by the labelling

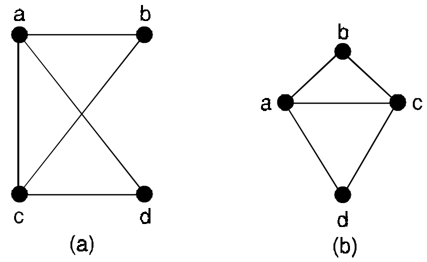


Fig. 3.67

Problem 3.54. *Whether the following pair of non-directed graphs in figure (3.68) are isomorphic or not ? Justify your answer ?*

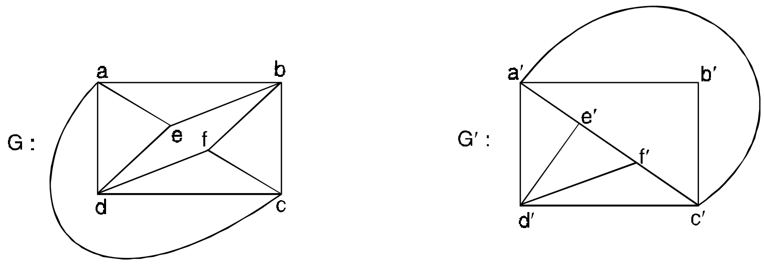


Fig. 3.68

Solution. Here, G′ has vertex b' of degree 2, while G has no vertex of degree 2.

Hence, they are not isomorphic.

Problem 3.55. *How many different non-isomorphic trees are possible for a graph of order 4 ? Draw all of them.*

Solution. The sum of the degrees of the 4-vertices equals

$$2(e) = 2(n - 1) = 2n - 2 = 8 - 2 = 6$$

Hence, the degree of 4-vertices are (2, 2, 1, 1) or (3, 1, 1, 1), they are drawn as shown in Figure below

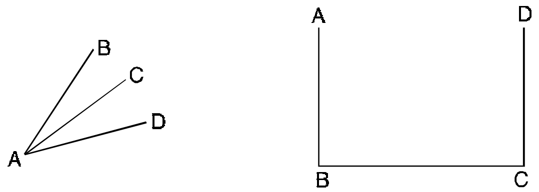


Fig. 3.69

Problem 3.56. *Draw a cycle graph which is isomorphic to its complement.*

Solution. First we draw G and the complement of G denoted G′, by drawing edges between vertices which are non-adjacent in G.

The vertices in G′ are labelled so as to corresponds to those of G as follows :

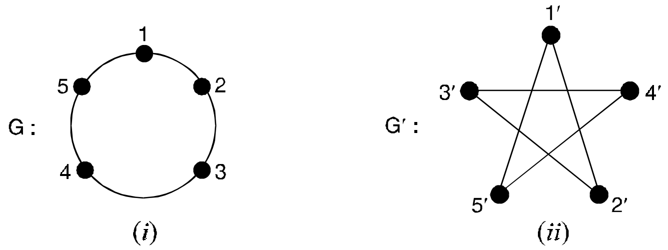


Fig. 3.70

From Figure (3.70)

Vertices in G	*Vertices in G′*
1	1′
2	2′
3	3′
4	4′
5	5′

This labelling ensures that 5′ and 2′ are adjacent to 1′ in G′, while 5 and 2 are adjacent to 1 in G, 3′ and 1′ are adjacent to 2′ in G′, while 3 and 1 are adjacent to 2 in G.

Also $d(i') = d(i)$ for all i.

Hence G and G′ are isomorphic.

Problem 3.57. *If a simple graph with n-vertices is isomorphic with its complement, how many vertices will that have ? Draw the corresponding graph.*

Solution. If e is the number of edges of G and $\bar{e}$ the number of edges in the complement $\overline{G}$, then

$e = \bar{e} = \dfrac{n(n+1)}{4}$. Hence n or $n + 1$ must be divisible by 4.

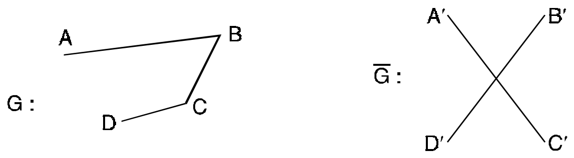


Fig. 3.71

Problem 3.58. *Determine whether the following pairs of graphs are isomorphic. If the graphs are not isomorphic, give an invariant that the graphs do not share.*

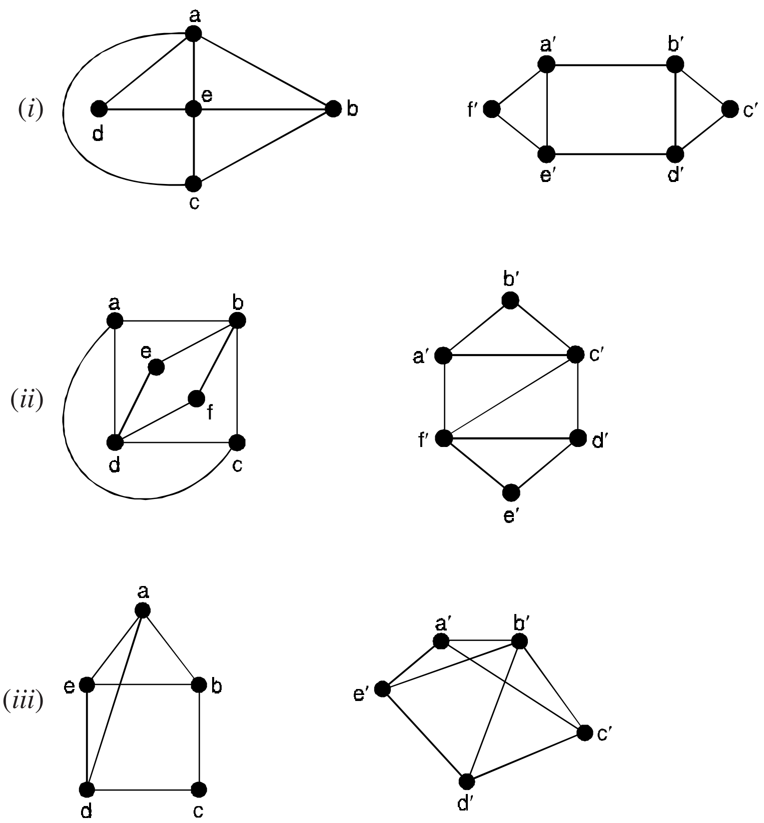


Fig. 3.72

Solution. (*i*) Non isomorphic, they do not have the same number of vertices.

(*ii*) Non isomorphic, vertices of degree 3 are adjacent in one graph, non adjacent in the other.

(*iii*) Non isomorphic, one has a vertex of degree 2 but other does not.

Problem 3.59. *Find whether the following pairs of graphs are isomorphic or not.*

(*i*)

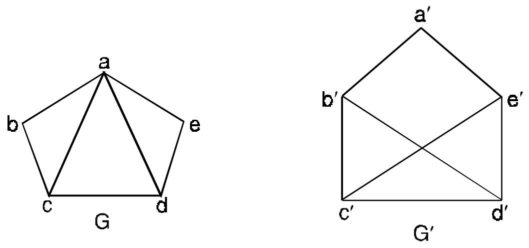


Fig. 3.73

(*ii*)

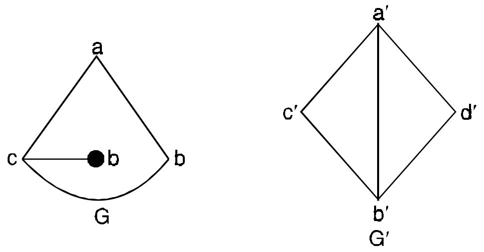


Fig. 3.74

Solution. (*i*) Not isomorphic.

G has 2 nodes *b* and *e* of degree 2 while G′ has one node *a*′ of degree 2.

(*ii*) Not isomorphic.

G has 4 edges, and G′ has edges.

Problem 3.60. *If a graph G of n vertices is isomorphic to its complement $\overline{G}$, show that n or (n – 1) must be a multiple of 4.*

Solution. Since G ≈ $\overline{G}$, both of G and $\overline{G}$ have the same number of edges.

Also, the total number of edges in G and $\overline{G}$ taken together must be equal to the number of edges in K_n.

Since K_n has $\frac{n(n-1)}{2}$ edges, it follows that each of G and $\overline{G}$ has $\frac{n(n-1)}{4}$ edges.

Thus, $\frac{n(n-1)}{4}$ must be a positive integer, as such, *n* or (*n* – 1) must be a multiple of 4.

Problem 3.61. *Consider two graphs G_1 and G_2 as shown below, show that the graphs G_1 and G_2 are isomorphic.*

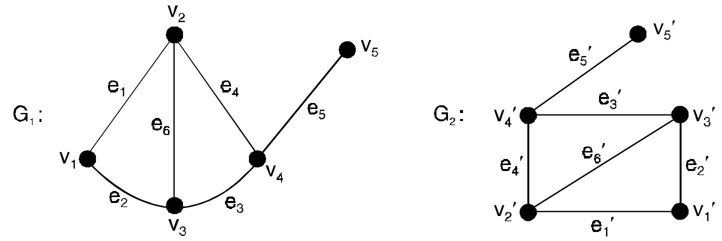


Fig. 3.75

Solution. The correspondence between the graphs is as follows :

The vertices (v_1, v_2, v_3, v_4, v_5) in G_1 correspond to (v_1', v_2', v_3', v_4', v_5') respectively in G_2.

The edges (e_1, e_2, e_3, e_4, e_5, e_6) in G_1 correspond to (e_1', e_2', e_3', e_4', e_5', e_6') respectively in G_2.

Here the incidence property is preserved.

Therefore the graphs G_1 and G_2 are isomorphic to each other.

Problem 3.62. *Draw all non-isomorphic graphs on 2 and 3 vertices.*

Solution. All non-isomorphic graphs on 2 vertices are

All non-isomorphic graphs on 3 vertices are

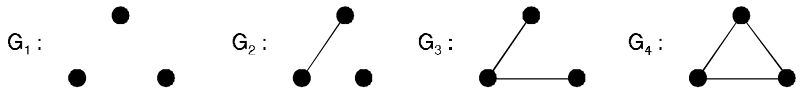


Fig. 3.76

Problem 3.63. *Show that the following graphs are isomorphic.*

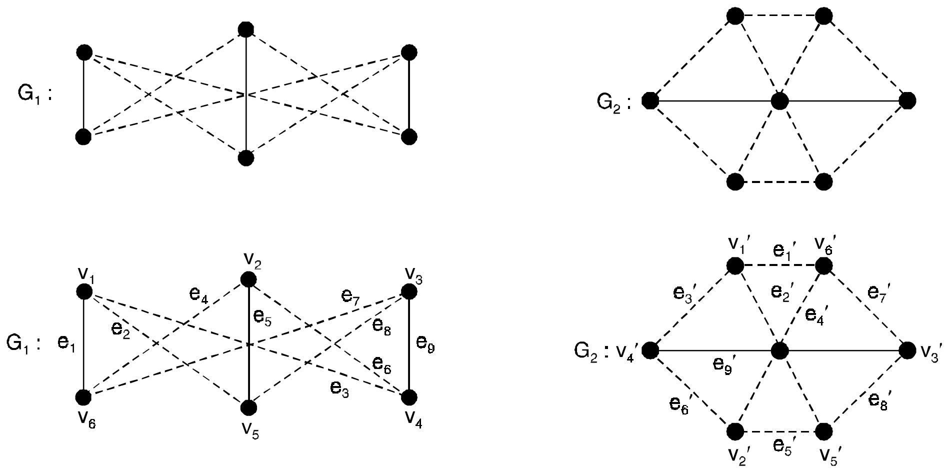


Fig. 3.77

Solution. There is one-to-one correspondence between vertices and one-to-one correspodence between edges. Further incidence property is preserved.

Therefore G_1 is isomorphic to G_2,

Problem 3.64. *Determine whether the following graphs are isomorphic or not*

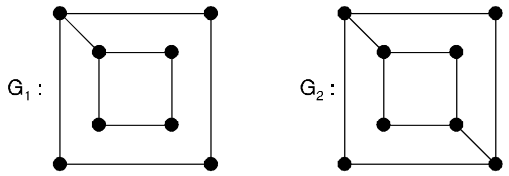


Fig. 3.78

Solution. Here both the graphs G_1 and G_2 contains 8 vertices and 10 edges.

The number of vertices of degree 2 in both the graphs are four.

Also the number of vertices of degree 3 in both the graphs are four.

For adjacency, consider the vertex of degree 3 in G_1. It is adjacent to two vertices of degree 3 and one vertex of degree 2.

But in G_2 there does not exist any vertex of degree 3, which is adjacent to two vertices of degree 3 and one vertex of degree 2.

i.e., adjacency is not preserved.

Hence, given graphs are not isomorphic.

Problem 3.65. *Show that the following graphs are isomorphic.*

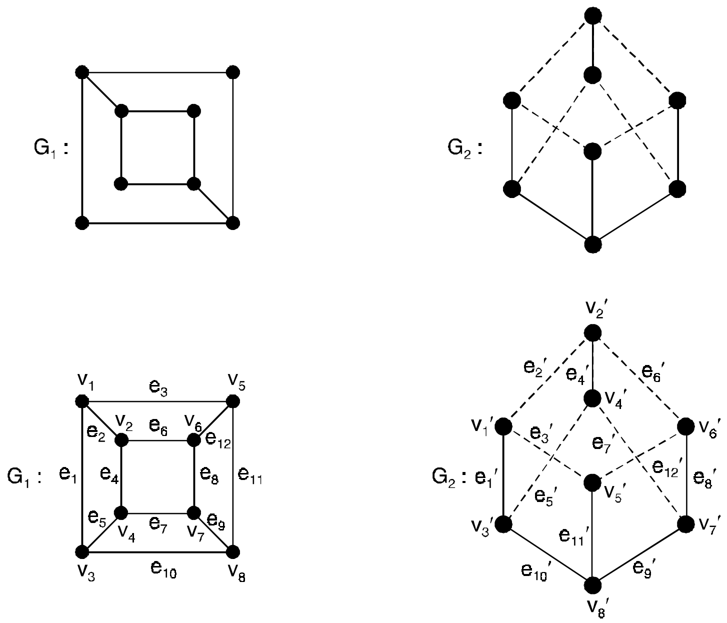


Fig. 3.79

Solution. There are one-to-one correspondence between the vertices as well as between edges. Further, the incidence property is preserved.

Therefore, G_1 is isomorphic to G_2.

Problem 3.66. *Establish a one-one correspondence between the vertices and edges to show that the following graphs are isomorphic.*

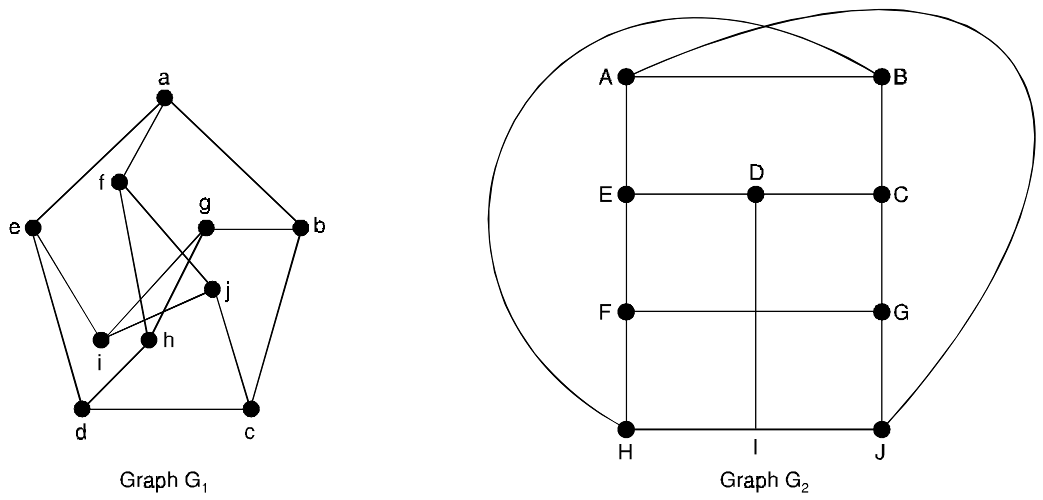


Fig. 3.80

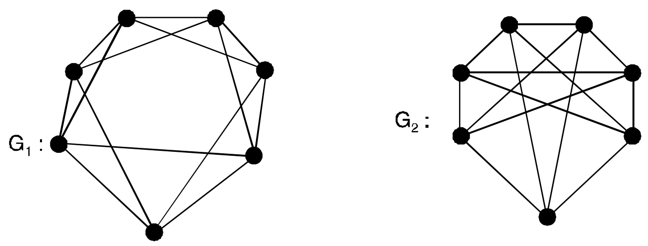
Solution. Define $\phi : V(G_1) \to V(G_2)$ by $\phi(a) = A$, $\phi(b) = B$

$\phi(c) = C$, $\phi(d) = D$, $\phi(e) = E$

$\phi(f) = J$, $\phi(g) = H$, $\phi(h) = I$

$\phi(i) = F$, $\phi(j) = G$.

Problem 3.67. *Show that the following graphs are isomorphic.*



Solution. We first label the vertices of the graph as follows :

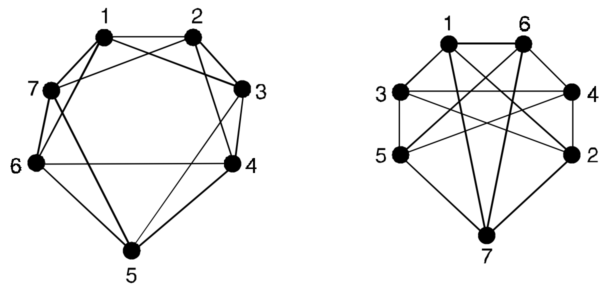


Fig. 3.81

Define an isomorphism $\phi : V(G_1) \to V(G_2)$ by $\phi(i) = i$, we observe that ϕ preserves the adjacency and non-adjacency of the vertices.

Hence G_1 and G_2 are isomorphic to each other.

3.9 OPERATIONS OF GRAPHS

3.9.1. Union

Given two graphs G_1 and G_2, their union will be a graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

and
$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

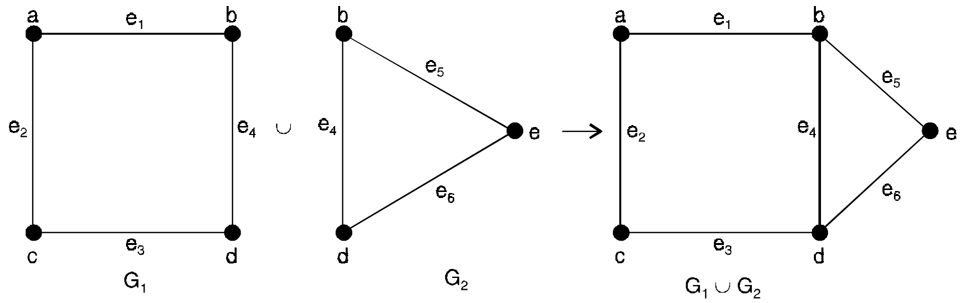


Fig. 3.82

3.9.2. Intersection

Given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

and
$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

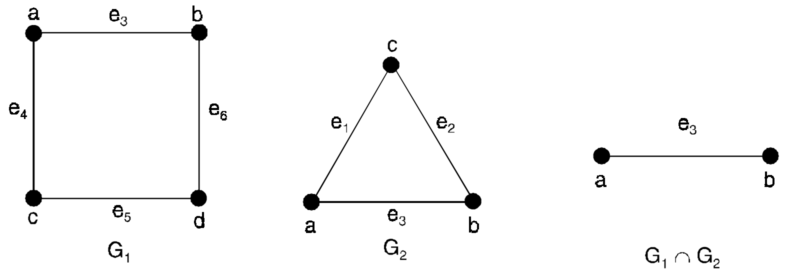


Fig. 3.83

3.9.3. Sum of two graphs

If the graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \phi$, then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting those edges, which are in G_1 and in G_2 and the edges obtained, by joining each vertex of G_1 to each vertex of G_2.

For example,

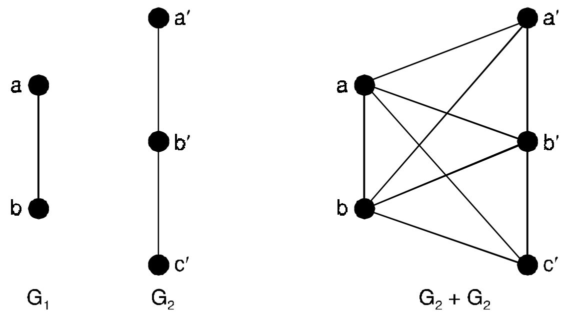


Fig. 3.84

3.9.4. Ring sum

Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two graphs. Then the ring sum of G_1 and G_2, denoted by $G_1 \oplus G_2$ is defined as the graph G such that :

(*i*) $V(G) = V(G_1) \cup V(G_2)$

(*ii*) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$

i.e., the edges that either in G_1 or G_2 but not in both. The ring sum of two graphs G_1 and G_2 is shown below.

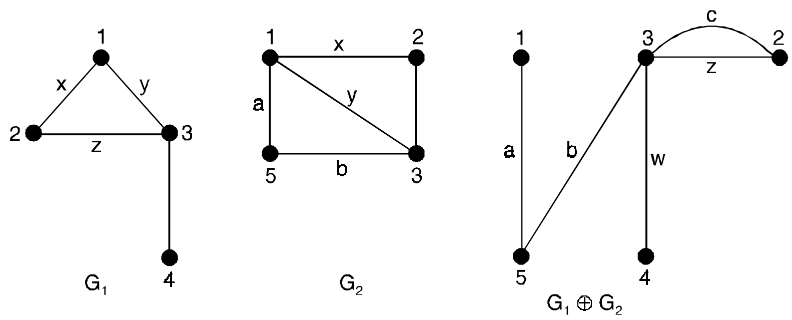


Fig. 3.85

3.9.5. Product of graphs

To define the product $G_1 \times G_2$ of two graphs consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adj. $v_2]$ or $[u_2 = v_1$ and u_1 adj. $v_1]$

For example,

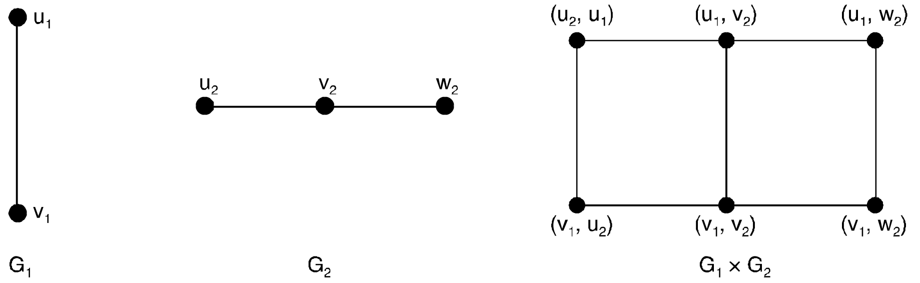


Fig. 3.86. The product of two graphs.

3.9.6. Composition

The composition $G = G_1[G_2]$ also has $V = V_1 \times V_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $(u_1$ adj. $v_1)$ or $(u_1 = v_1$ and u_2 adj. $v_1)$

For the graphs G_1 and G_2 of Figure. 3.86(a), both compositions $G_1[G_2]$ and $G_2[G_1]$ are shown in Figure 3.87.

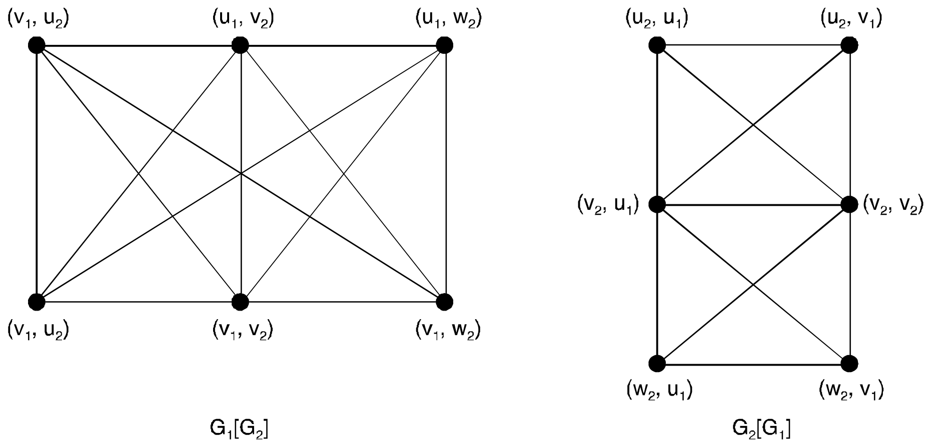


Fig. 3.87. Two compositions of graphs

3.9.7. Complement

The complement G' of G is defined as a simple graph with the same vertex set as G and where two vertices u and v adjacent only when they are not adjacent in G.

For example,

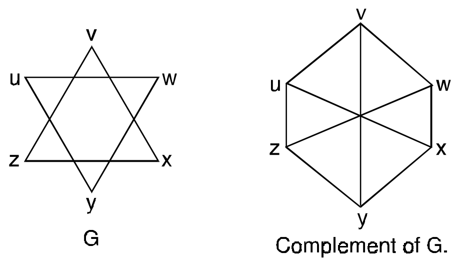


Fig. 3.88

A graph G is self-complementary if it is isomorphic to its complement.

For example, the graphs

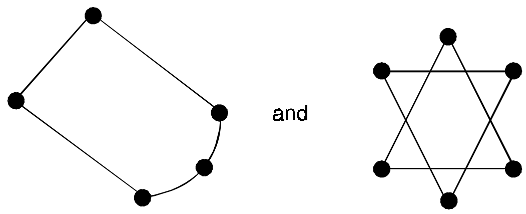


Fig. 3.89

Self-complementary. The other self-complementary graph with five vertices is

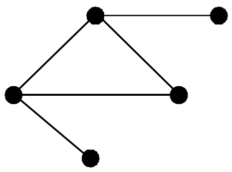


Fig. 3.90

3.9.8. Fusion

A pair of vertices v_1 and v_2 in graph G is said to be 'fused' if these two vertices are replaced by a single new vertex v such that every edge that was adjacent to either v_1 or v_2 or both is adjacent v.

Thus we observe that the fusion of two vertices does not alter the number of edges of graph but reduced the vertices by one.

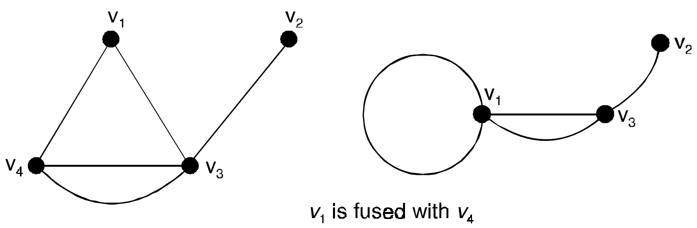


Fig. 3.91

Problem 3.68. *For any graph G with six points, G or $\overline{G}$ contains a triangle.*

Solution. Let v be a point of a graph G with six points. Since v is adjacent either in G or in $\overline{G}$ to the other five points of G.

We can assume without loss of generality that there are three points u_1, u_2, u_3 adjacent to v in G.

If any two of these points are adjacent, then they are two points of a triangle whose third point is v.

If no two of them are adjacent in G, then u_1, u_2 and u_3 are the points of a triangle in $\overline{G}$.

Problem 3.69. *Prove that at any party with six people, there are three mutual acquaintances or three mutual nonacquaintances.*

Solution. This situation may be represented by a graph G with six points standing for people, in which adjacency indicates acquaintance.

Then the problem is to demonstrate that G has three mutually adjacent points or three mutually nonadjacent ones.

The complement $\overline{G}$ of a graph G also has V(G) as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

In Figure 3.92, G has no triangles, while $\overline{G}$ consists of exactly two triangles.

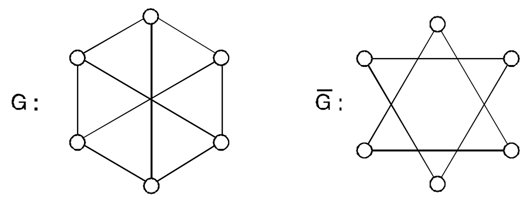


Fig. 3.92. A graph and its complement

In figure 3.93 : A self-complementary graph is isomorphic with its complement.

The complete graph K_p has every pair of its P points adjacent. Since V is not empty, $P \geq 1$.

Thus K_P has $\binom{P}{2}$ lines and is regular of degree $P - 1$.

As we have seen, K_3 is called a triangle. The graphs $\overline{K}_P$ are totally disconnected, and are regular of degree 0.

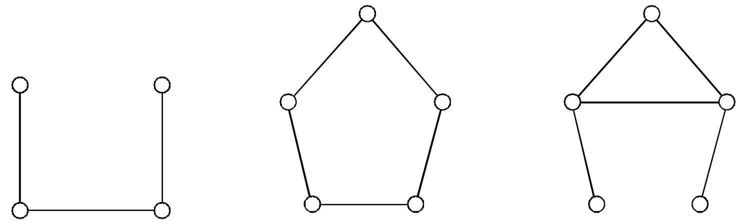


Figure 3.93. The smallest nontrivial self-complementary graphs.

Theorem 3.3. *The maximum number of lines among all P point graphs with no triangles is* $\left[\dfrac{P^2}{4}\right]$.

Proof. The statement is obvious for small values of P. An inductive proof may be given separately for odd P and for even P.

Suppose the statement is true for all even $P \le 2n$.

We then prove it for $P = 2n + 2$

Thus, let G be a graph with $P = 2n + 2$ points and no triangles.

Since G is not totally disconnected, there are adjacent points u and v.

The subgraph $G' = G - \{u, v\}$ has $2n$ points and no triangles, so that by the inductive hypothesis G′ has at most $\left[\dfrac{4n^2}{4}\right] = n^2$ lines.

There can be no point W such that u and v are both adjacent to W, for then u, v and w would be points of a triangle in G.

Thus if u is adjacent to K points of G′, v can be adjacent to at most $2n - K$ points.

Then G has at most

$$n^2 + K + (2n - K) + 1 = n^2 + 2n + 1 = \frac{P^2}{4} = \left[\frac{P^2}{4}\right] \text{ lines.}$$

Theorem 3.4. *Every graph is an intersection graph.*

Proof. For each point v_i of G

Let S_i be the union of $\{v_i\}$ with the set of lines incident with v_i.

Then it is immediate that G is isomorphic with Ω (F) where $F = \{S_i\}$.

Note : The intersection number ω' (G) of a given graph G is the minimum number of elements in a set S such that G is an intersection on S.

Corollary (1)

If G is connected and $P \ge 3$, then $\omega(G) \le q$.

Proof. In this case, the points can be omitted from the sets S_i used in the proof of the theorem, so that $S = X(G)$.

Corollary (2)

If G has P_0 isolated points and no K_2 components, then $\omega(G) \le q + P_0$.

Theorem 3.5. *Let G be a connected graph with P > 3 points. Then ω(G) = q if and only if G has no triangles.*

Proof. We first prove the sufficiency.

To show that $\omega(G) \ge q$ for any connected G with atleast 4 points having no triangles.

By definition of the intersection number, G is isomorphic with an intersection graph Ω(F) on a set S with | S | = ω(G).

For each point v_i of G, let S_i be the corresponding set.

Because G has no triangles, no element of S can belong to more than two of the sets S_i, and $S_i \cap S_j \ne \phi$ if and only if v_iv_j as a line of G.

Thus we can form a 1 – 1 correspondence between the lines of G and those elements of S which belong to exactly two sets S_i.

Therefore $\omega(G) = | S | \ge q$ so that $\omega(G) = q$.

To prove necessity :

Let $\omega(G) = q$ and assume that G has a triangle then let G_1 be a maximal triangle-free spanning subgraph of G. $\omega(G_1) = q_1 = | X(G_1) |$.

Suppose that $G_1 = \Omega(F)$, where F is a family of subsets of some set S with cardinality q_1.

Let x be a line of G not in G_1 and consider $G_2 = G_1 + x$. Since G_1 is a maximal triangle-free, G_2 must have some triangle, say u_1, u_2, u_3 where $x = u_1u_3$.

Denote by S_1, S_2, S_3 the subsets of S corresponding to u_1, u_2, u_3. Now if u_2 is adjacent to only u_1 and u_3 in G_1, replace S_2 by a singleton chosen from $S_1 \cap S_2$ and add that element to S_3.

Otherwise, replace S_3 by the union of S_3 and any element in $S_1 \cap S_2$.

In either case this gives a famly F′ of distinct subsets of S such that $G_2 = \Omega(F')$.

Thus $\omega(G_2) \le q_1$ while $| X(G_2) | = q_1 + 1$

If $G_2 \cong G$ there is nothing to prove.

But if $G_2 \ne G$, then let $| X(G) | - | X(G_2) | = q_0$

It follows that G is an intersection graph on a set with $q_1 + q_0$ elements.

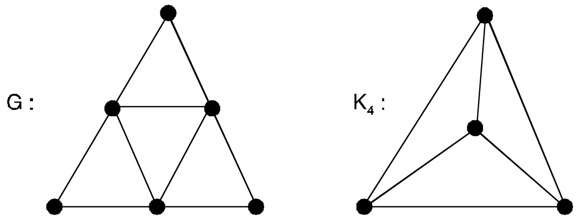
However, $q_1 + q_0 = q - 1$

Thus $\omega(G) < q$

Hence the proof.

Theorem 3.6. *For any graph G with P ≥ 4 points,* $\omega(G) \le \left[\frac{P^2}{4}\right]$.

Theorem 3.7. *A graph G is a clique graph if and only if it contains a family F of complete subgraphs, whose union in G, such that whenever every pair of such complete graphs in some subfamily F ′ have a non empty intersection, the intersection of all the members of F ′ is non empty.*



A graph and its clique graph.

Fig. 3.94

3.10 CONNECTED AND DISCONNECTED GRAPHS

A graph G is said to be a **connected** if every pair of vertices in G are connected. Otherwise, G is called a **disconnected** graph. Two vertices in G are said to be connected if there is at least one path from one vertex to the other.

In other words, a graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path.

A graph is **connected** if we can reach any vertex from any other vertex by travelling along the edges and disconnected otherwise.

For example, the graphs in Figure 3.95(*a, b, c, d, e*) are connected whereas the graphs in Figure 3.96(*a, b, c*) are disconnected.

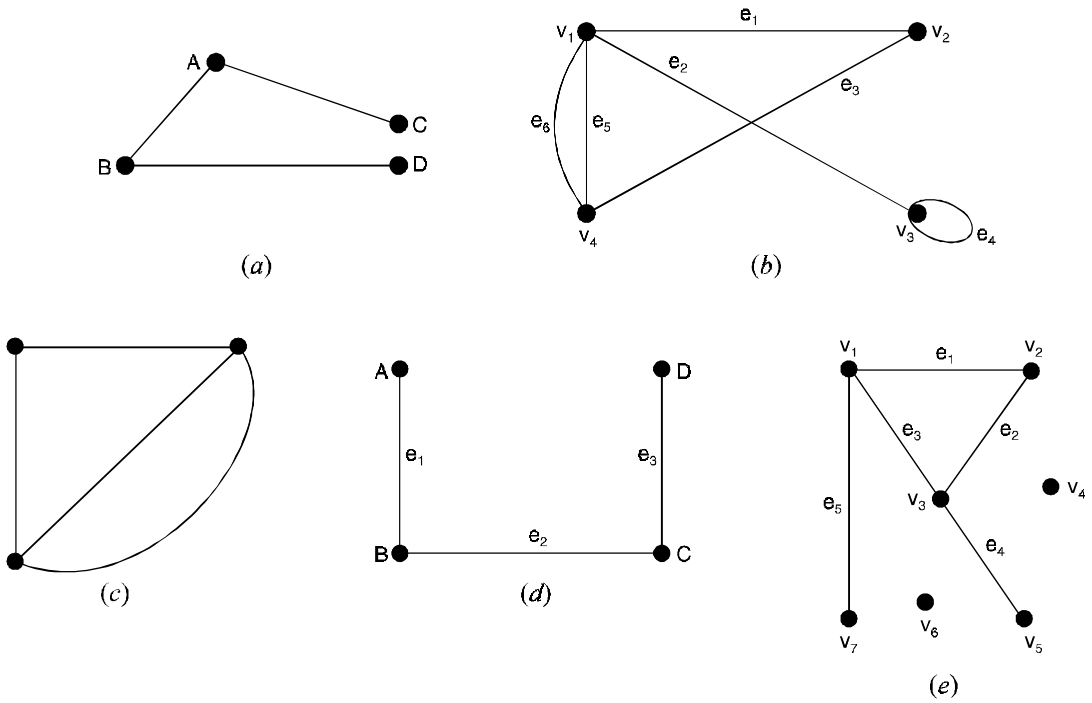


Fig. 3.95.

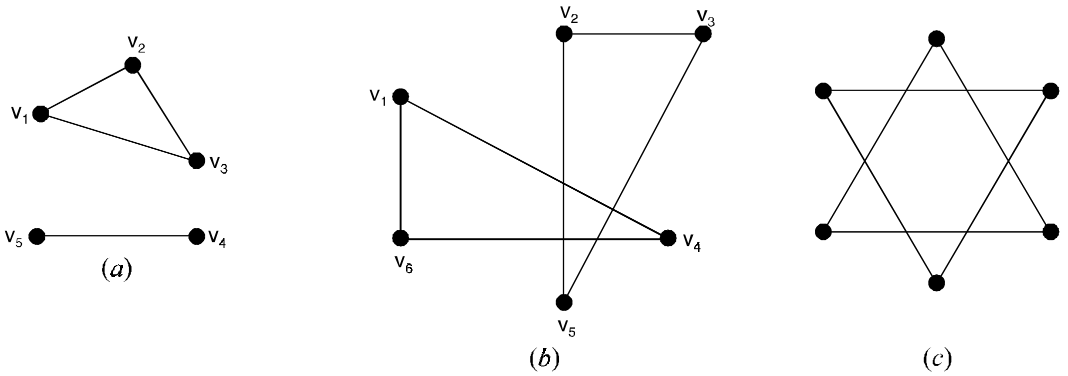


Fig. 3.96

A complete graph is always connected, also, a null graph of more than one vertex is disconnected (see Fig. 3.97). All paths and circuits in a graph G are connected subgraphs of G.

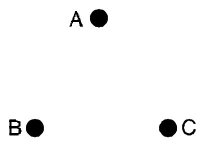


Fig. 3.97

Every graph G consists of one or more connected graphs, each such connected graph is a subgraph of G and is called a component of G. A connected graph has only one component and a disconnected graph has two or more components.

For example, the graphs in Figure 3.96(*a, b*) have two components each.

3.10.1. Path graphs and cycle graphs

A connected graph that is 2-regular is called a cycle graph. Denote the cycle graph of *n* vertices by Γ_n. A circuit in a graph, if it exists, is a cycle subgraph of the graph.

The graph obtained from Γ_n by removing an edge is called the path graph of *n* vertices, it is denoted by P_n.

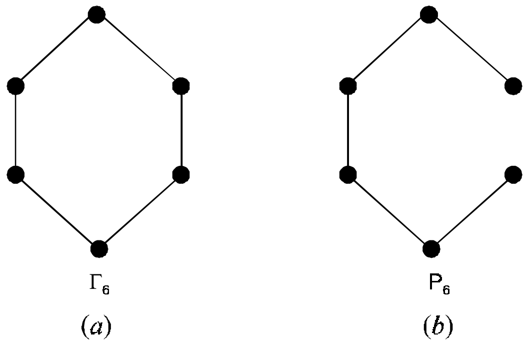


Fig. 3.98

The graphs Γ_6 and P_6 are shown in Figure 3.98(*a*) and 3.98(*b*) respectively.

3.10.2. Rank and nullity

For a graph G with n vertices, m edges and k components we define the rank of G and is denoted by $\rho(G)$ and the nullity of G is denoted by $\mu(G)$ as follows.

$$\rho(G) = \text{Rank of } G = n - k$$

$$\mu(G) = \text{Nullity of } G = m - \rho(G) = m - n + k$$

If G is connected, then we have

$$\rho(G) = n - 1 \text{ and } \mu(G) = m - n + 1.$$

Problem 3.70. *Prove that a simple graph with n vertices must be connected if it has more than* $\frac{(n-1)(n-2)}{2}$ *edges.*

Solution. Consider a simple graph on n vertices.

Choose $n - 1$ vertices $v_1, v_2, ..., v_{n-1}$ of G.

We have maximum ${}^{n-1}C_2 = \frac{(n-1)(n-2)}{2}$ number of edges only can be drawn between these vertices.

Thus if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge should be drawn between the nth vertex v_n to some vertex v_i, $1 \le i \le n - 1$ of G.

Hence G must be connected.

Problem 3.71. *Show that if a and b are the only two odd degree vertices of a graph G, then a and b are connected in G.*

Solution. If G is connected, nothing to prove.

Let G be disconnected.

If possible assume that a and b are not connected.

Then a and b lie in the different components of G.

Hence the component of G containing a (similarly containing b) contains only one odd degree vertex a, which is not possible as each component of G is itself a connected graph and in a graph number of odd degree vertices should be even.

Therefore a and b lie in the same component of G.

Hence they are connected.

Problem 3.72. *Prove that a connected graph G remains connected after removing an edge e from G if and only if e lie in some circuit in G.*

Solution. If an edge e lies in a circuit C of the graph G then between the end vertices of e, there exist atleast two paths in G.

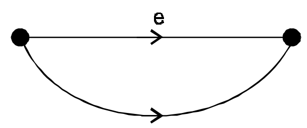


Fig. 3.99

Hence removal of such an edge *e* from the connected graph G will not effect the connectivity of G.

Conversely, if *e* does not lies in any circuit of G then removal of *e* disconnects the end vertices of *e*.

Hence G is disconnected.

Problem 3.73. *If G_1 and G_2 are (edge) decomposition of a connected graph G, then prove that $V(G_1) \cap V(G_2) \neq \phi$.*

Solution. If $V(G_1) \cap V(G_2) = \phi$ then $V(G_1)$ and $V(G_2)$ are the vertex partition of V(G) (there exists no edges left in G to include between vertex of $V(G_1)$ and $V(G_2)$ as G_1 and G_2 are edge partition of G).

Hence, G is disconnected, a contradiction to the fact that G is connected.

Problem 3.74. *Which of the graphs below are connected :*

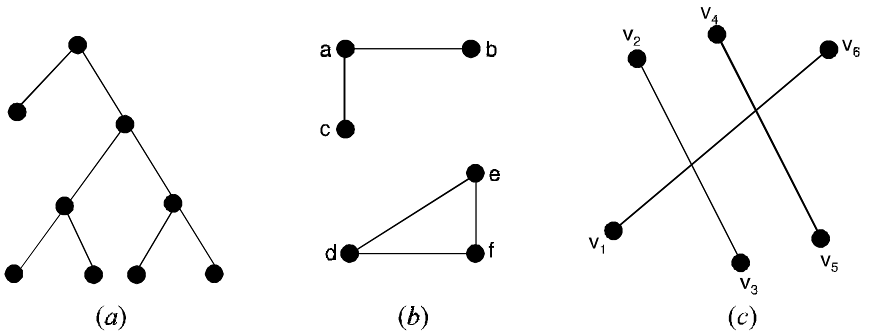


Fig. 3.100

Solution. The graph shown in Figure 3.100(*a*) is connected graph since for every pair of distinct vertices there is a path between them.

The graph shown in Figure 3.100(*b*) is not connected since there is no path in the graph between vertices *b* and *d*.

The graph shown in Figure 3.100(*c*) is not connected. In drawing a graph two edges may cross at a point which is not a vertex. The graph can be redrawn as :

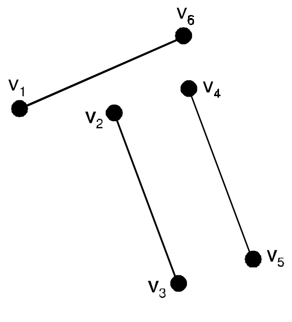


Fig. 3.101

Theorem 3.8. *If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.*

Proof. Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd.

From theorem, which holds for every graph and therefore for every component of a diconnected graph,

No graph can have an odd number of odd vertices.

Therefore, in graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Theorem 3.9. *A simple graph with n vertices and k components cannot have more than* $\frac{(n-k)(n-k+1)}{2}$ *edges.*

Proof. Let n_i = the number of vertices in component i,

$$1 \le i \le k, \qquad \text{then } \sum_{i=1}^{k} n_i = n.$$

A component with n_i vertices will have the maximum possible number of edges when it is complete.

That is, it will contain $\frac{1}{2} n_i(n_i - 1)$ edges.

Hence the maximum number of edges is :

$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i$$

$$\le \frac{1}{2} [n^2 - (k-1)(2n-k)] - \frac{1}{2} n$$

$$= \frac{1}{2} [n^2 - 2nk + k^2 + n - k]$$

$$= \frac{1}{2} (n-k)(n-k+1).$$

Corollary :

If $m > \frac{1}{2} (n-1)(n-2)$ then a simple graph with n vertices and m edges is connected.

Proof. Suppose the graph is disconnected. Then it has at least two components, therefore by theorem.

$$m \le \frac{1}{2} (n-k)(n-k+1) \text{ for } k \ge 2$$

$$\le \frac{1}{2} (n-2)(n-1)$$

This contradicts the assumption that $m > \frac{1}{2}(n-1)(n-2)$.

Therefore, the graph should be connected.

Theorem 3.10. *A graph G is disconnected if and only if its vertex set V can be partitioned into two subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in the subset V_1 and the other in the subset V_2.*

Proof. Let G be disconnected. Then we have by the definition that there exists a vertex x in G and a vertex y in G such that there is no path between x and y in G

Let $V_1 = \{Z \in V : z$ is connected to $x\}$. Then V_1 is the set of all vertices of G which are connected to x.

Let $V_2 = V - V_1$. Then $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

Hence V_1 and V_2 are the partition of V(G). Let a be any vertex of V_1.

To prove that 'a' is not adjacent to any vertex of V_2.

If possible let $b \in V_2$ such that $ab \in E(G)$. Then $a \in V_1$ there exist a path P_1 : from x to a.

This path can be extended to the path $P_2 = P_1, ab, b$.

P_2 is a path from x to b in G.

Therefore x and b are connected. This implies that $b \in V_1$ which is contradiction to the fact $V_1 \cap V_2 = \phi$.

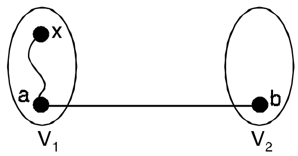


Fig. 3.102

Conversely, let us assume that V can be partitioned into two subsets V_1 and V_2 such that no vertex of V_1 is adjacent to a vertex of V_2.

Let x be any vertex in V_1 and y be any vertex in V_2.

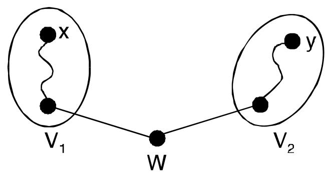


Fig. 3.103

To prove that G is disconnected, if possible, suppose G is connected. Then x and y are connected.

Therefore, there exists a path between x and y in G. But this path is possible only through a vertex W in G which is not either in V_1 or V_2.

Hence $V_1 \cup V_2 \neq V$, a contradiction.

Theorem 3.11. *Show that a simple (p, q)-graph is connected then $P \leq q + 1$.*

Proof. The proof is by induction on the number of edges in G. If G has only one or two edges then the theorem is true. Assume that the theorem is true for each graph with fewer than n edges.

Let G be given connected (p, q) graph.

Case (*i*) : G contains a circuit.

Let S be a graph obtained by G by removing an edge from a circuit of G. Then S is a connected graph having $q - 1$ edges. The number of vertices of S and G are same, hence by inductive hypothesis $p \leq q - 1 + 1$.

Thus $p \leq q$, hence certainly $p \leq q + 1$.

Case (*ii*) : G does not contain a circuit.

Let p be a longest path in G. Let a and b be the end vertices of the path. The vertex a must be of degree 1, otherwise the path could be made longer, or there would be a circuit in G.

Remove the vertex a and the edge incident with the vertex a.

Let H be the graph so obtained. Then H contains exactly one vertex and one edge less than that of G.

Further H is connected, hence by inductive hypothesis $p - 1 \leq (q - 1) + 1$.

Hence $p \leq q + 1$.

Problem 3.75. *Prove that a connected graph G remains connected after removing an edge e from G if and only if e belongs to some circuit in G.*

Solution. Suppose e belongs to some circuit C in G. Then the end vertices of e, say, A and B are joined by atleast two paths, one of which is e and the other $C - e$.

Hence the removal of e from G will not affect the connectivity of G ; even after the removal of e the end vertices of e. (*i.e.,* A and B) remain connected.

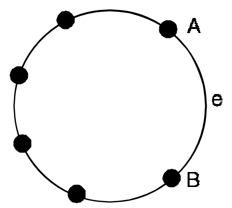


Fig. 3.104

Conversely, suppose e does not belong to any circuit in G. Then the end vertices of e are connected by atmost one path.

Hence the removal of e from G disconnects these end points. This means that $G - e$ is a disconnected graph.

Thus, if e does not belong to any circuit in G then $G - e$ is disconnected.

This is equivalent to saying that if $G - e$ is connected then e belongs to some circuit in G.

Problem 3.76. *Let G be a disconnected graph with n vertices where n is even. If G has two components each of which is complete, prove that G has a minimum of* $\frac{n(n-2)}{4}$ *edges.*

Solution. Let x be the number of vertices in one of the components.

Then the other component has $n - x$ number of vertices since both components are complete graphs, the number of edges they have are $\frac{x(x-1)}{2}$ and $\frac{(n-x)(n-x-1)}{2}$ respectively.

Therefore, the total number of edges in G is

$$m = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

$$= x^2 - nx + \frac{n}{2}(n-1)$$

$\Rightarrow \quad m' = 2x - n,\ m'' = 2 > 0, \qquad \left(m' = \frac{dm}{dx} \text{ and } m'' \frac{d^2m}{dx^2} \right)$

Therefore, m is minimum when $2x - n = 0$

$$\Rightarrow \quad x = \frac{n}{2}$$

$$\text{Min. } m = \left(\frac{n}{2}\right)^2 - n\left(\frac{n}{2}\right) + \frac{n}{2}(n-1)$$

$$= \frac{n(n-2)}{4}.$$

Problem 3.77. *Find the rank and nullity of the complete graph k_n.*

Solution. k_n is a connected graph with n vertices and

$$m = \frac{n(n-1)}{2} \text{ edges.}$$

Therefore, by the definitions of rank and nullity, we have

Rank of $k_n = n - 1$

$$\text{Nullity of } k_n = m - n + 1 = \frac{1}{2}n(n-1) - n + 1$$

$$= \frac{1}{2}(n-1)(n-2).$$

3.11 WALKS, PATHS AND CIRCUITS

3.11.1. Walk

A **walk** is defined as a finite **alternative sequence** of vertices and edges, of the form

$$v_i e_j,\ v_{i+1}\ e_{j+1},\ v_{i+2},\ \ldots\ldots,\ e_k v_m$$

which **begins** and **ends** with **vertices,** such that

(*i*) each edge in the sequence is incident on the vertices preceding and following it in the sequence.

(*ii*) no edge appears more than once in the sequence, such a sequence is called a walk or a trial in G.

For example, in the graph shown in Figure 3.105, the sequences

$v_2e_4v_6e_5v_4e_3v_3$ and $v_1e_8v_2e_4v_6e_6v_5e_7v_5$ are walks.

Note that in the first of these, each vertex and each edge appears only once whereas in the second each edge appears only once but the vertex v_5 appears twice.

These walks may be denoted simply as $v_2v_6v_4v_3$ and $v_7v_2v_6v_5v_5$ respectively.

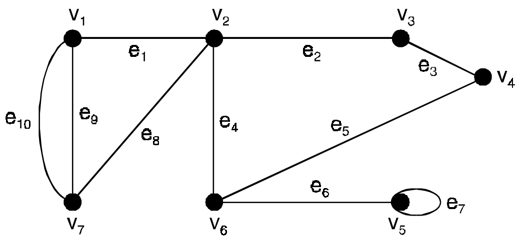


Fig. 3.105

The vertex with which a walk begins is called the initial vertex and the vertex with which a walk ends is called the final vertex of the walk. The initial vertex and the final vertex are together called terminal vertices. Non-terminal vertices of a walk are called its internal vertices.

A walk having u as the initial vertex and v as the final vertex is called a walk from u to v or briefly a $\boldsymbol{u - v}$ **walk.** A walk that begins and ends at the same vertex is called a **closed walk. In other words,** a closed walk is a walk in which the terminal vertices are coincident.

A walk that is not closed is called an **open walk.**

In other words, an open walk is a walk that begins and ends at two different vertices.

For example, in the graph shown in Figure 3.105.

$v_1e_9v_7e_8v_2e_1v_1$ is a closed walk and $v_5e_7v_5e_6v_6e_5v_4$ is an open walk.

3.11.2. Path

In a walk, a vertex can appear more than once. An open walk in which no vertex appears more than once is called a **simple path** or a **path.**

For example, in the graph shown in Figure 3.105.

$v_6e_5v_4e_3v_3e_2v_2$ is a path whereas $v_5e_7v_5e_6v_6$ is an open walk but not a path.

3.11.3. Circuit

A closed walk with atleast one edge in which no vertex except the terminal vertices appears more than once is called **a circuit** or a cycle.

For example, in the graph shown in Figure 3.105,

$v_1e_1v_2e_8v_7e_9v_1$ and $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ are circuits.

But $v_1e_9v_7e_8v_2e_4v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is a closed walk but not a circuit.

Note : (*i*) In walks, path and circuit, no edge can appears more than once.

(*ii*) A vertex can appear more than once in a walk but not in a path.

(*iii*) A path is an open walk, but an open walk need not be a path.

(*iv*) A circuit is a closed walk, but a closed walk need not be a circuit.

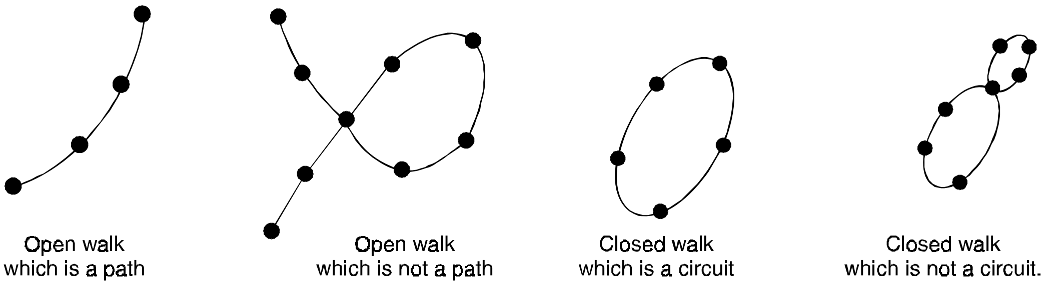


Fig. 3.106

3.11.4. Length

The number of edges in a walk is called its length. Since paths and circuits are walks, it follows that the length of a path is the number of edges in the path and the length of a circuit is the number of edges in the circuit.

A circuit or cycle of length *k*, (with *k* edges) is called a *k*-circuit or a *k*-cycle. A *k*-circuit is called odd or even according as *k* is odd or even. A 3-cycle is called a triangle.

For example, in the graph shown in Figure 3.105,

The length of the open walk $v_6e_6v_5e_7v_5$ is 2

The length of the closed walk $v_1e_9v_7e_8v_2e_1v_1$ is 3

The length of the circuit $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ is 4

The length of the path $v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is 4

The circuit $v_1e_1v_2e_8v_7e_{10}v_1$ is a triangle.

Note : (*i*) A self-loop is a 1-cycle.

(*ii*) A pair of parallel edges form a cycle of length 2.

(*iii*) The edges in a 2-cycle are parallel edges.

Problem 3.78. *Write down all possible*

(*i*) *paths from v_1 to v_8* (*ii*) *Circuits of G and* (*iii*) *trails of length three.*

in G from v_3 to v_5 of the graph shown in Figure (3.107).

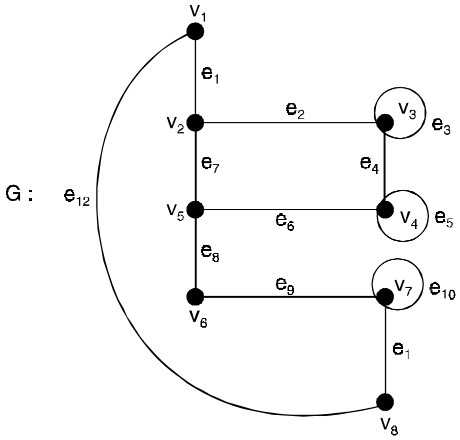


Fig. 3.107.

Solution.

(*i*) $P_1 : v_1e_{12}v_8$, $l(P_1) = 1$

$P_2 : v_1e_1v_2e_7v_5e_8v_6e_9v_7e_{11}v_8$, $l(P_2) = 5$

$P_3 : v_1e_1v_2e_2v_3e_4v_4e_6v_5e_8v_6e_9v_7e_{11}v_8$, $l(P_3) = 7$

These are the only possible paths from v_1 to v_8 in G.

(*ii*) $C_1 : v_1e_1v_2e_7v_5e_8v_6e_9v_7e_{11}v_8e_{12}v_1$, $l(C_1) = 6$

$C_2 : v_1e_1v_2e_2v_3e_4v_4e_6v_5e_8v_6e_9v_7e_{11}v_8e_{12}v_1$, $l(C_2) = 8$

$C_3 : v_2e_2v_3e_4v_4e_6v_5e_7v_2$, $l(C_3) = 4$

$C_4 : v_3e_3v_3$, $l(C_4) = 1$

$C_5 : v_4e_5v_4$, $l(C_5) = 1$

$C_6 : v_7e_7v_{10}$, $l(C_6) = 1$

These are the only possible circuits of G.

$W_1 : v_3e_3v_3e_2v_2e_7v_5$, $l(W_1) = 3$

$W_2 : v_3e_3v_3e_4v_4e_6v_5$, $l(W_2) = 3$

$W_3 : v_3e_4v_4e_5v_4e_6v_5$, $l(W_3) = 3$.

These are the only possible trails of length three from v_3 to v_5.

Problem 3.79. *In the graph below, determine whether the following are paths, simple paths, trails, circuits or simple circuits,*

(*i*) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ (*ii*) $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$

(*iii*) v_2 (*iv*) $v_5v_2v_3v_4v_4v_4v_5$.

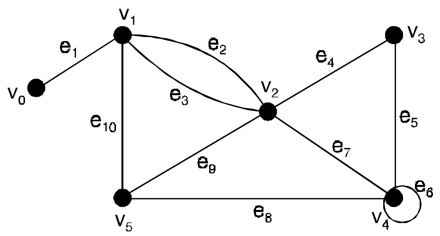


Fig. 3.108

Solution. (*i*) The sequence has a repeated vertex v_1 but does not have a repeated edge so it is a trail. It is not cycle or circuit.

(*ii*) The sequence has a repeated vertex v_2 and repeated edge e_9. Hence it is a path. It is not cycle or circuit.

(*iii*) It has no repeated edge, no repeated vertex, starts and ends at same vertex. Hence it is a simple circuit.

(*iv*) It is a circuit since it has no repeated edge, starts and ends at same vertex. It is not a simple circuit since vertex v_4 is repeated.

Problem 3.80. *In a graph (directed or undirected) with n vertices, if there is a path from vertex u to vertex v then the path cannot be of length greater than (n – 1).*

Solution. Let $\pi : u, v_1, v_2, v_3, ... v_k, v$ be the sequence of vertices in a path u and v.

If there are m edges in the path then there are $(m + 1)$ vertices in the sequence.

If $m < n$, then the theorem is proved by default. Otherwise, if $m \geq n$ then there exists a vertex v_j in the path such that it appears more than once in the sequence

$$(u, v_1, ..., v_j ......, v_j , ... v_k , v).$$

Deleting the sequence of vertices that leads back to the node v_j, all the cycles in the path can be removed.

The process when completed yields a path with all distinct nodes. Since there are n nodes in the graph, there cannot be more than n distinct nodes and hence $n - 1$ edges.

Problem 3.81. *For the graph shown in Figure, indicate the nature of the following sequences of vertices*

(a) $v_1v_2v_3v_2$ *(b)* $v_4v_1v_2v_3v_4v_5$ *(c)* $v_1v_2v_3v_4v_5$

(d) $v_1v_2v_3v_4v_1$ *(e)* $v_6v_5v_4v_3v_2v_1v_4v_6$

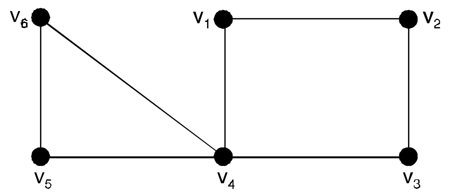


Fig. 3.109

Solution. (*a*) Not a walk

(*b*) Open walk but not a path

(*c*) Open walk which is a path

(*d*) Closed walk which is a circuit

(*e*) Closed walk which is not a circuit.

Theorem 3.12. *Let G = (V, E) be an undirected graph, with a, b ∈ V, a ≠ b. If there exists a trail (in G) from a to b, then there is a path (in G) from a to b.*

Proof. Since there is an trail from a to b.

We select one of shortest length, say$\{a, x_1\}, \{x_1, x_2\}, ..., \{x_n, b\}$.

If this trail is not a path, we have the situation $\{a, x_1\}, \{x_1, x_2\}, ..., \{x_{k-1}, x_k\}, \{x_k, x_{k+1}\}, \{x_{k+1}, x_{k+2}\}, ..., \{x_{m-1}, x_m\}, (x_m, x_{m+1}\}, ..., \{x_n, b\}$,

where $k < m$ and $x_k = x_m$, possibly with $k = 0$ and $a(= x_0) = x_m$, or $m = n + 1$ and $x_k = b\ (= x_{n+1})$

But then we have a contradiction, because

$\{a,x_1\}, \{x_1, x_2\}, ..., \{x_{k-1}, x_k\}, \{x_m, x_{m+1}\}, ..., \{x_n, b\}$ is a shortest trail from a to b.

Problem 3.82. *Let G = (V, E) be a loop-free connected undirected graph, and let {a, b} be an edge of G. Prove that {a, b} is part of a cycle if and only if its removal (the vertices a and b are left) does not disconnect G.*

Solution. If $\{a, b\}$ is not part of a cycle, then its removal disconnects a and b (and G).

If not, there is a path P from a to b, and P together with $\{a, b\}$ provides a cycle containing $\{a, b\}$.

Conversely, if the removal of $\{a, b\}$ from G disconnects G, there exist $x, y \in$ V such that the only path P from x to y contains $e = \{a, b\}$. If e were part of a cycle C, then the edges in $(P - \{e\}) \cup (C - \{e\})$ would contain a second path connecting x to y.

Theorem 3.13. *In a graph G, every u – v path contains a simple u – v path.*

Proof. If a path is a closed path, then it certainly contains the trivial path.

Assume, then, that P is an open $u - v$ path.

We complete the proof by induction on the length n of P.

If P has length one, then P is itself a simple path.

Suppose that all open $u - v$ paths of length k. Where $1 \le k \le n$, contains a simple $u - v$ path. Now suppose that P is the open $u - v$ path

$\{v_0, v_1\}$, ... $\{v_n, v_{n+1}\}$, where $u = v_0$ and $v = v_{n+1}$ of course, it may be that P has repeated vertices, but if not, then P is a simple $u - v$ path.

If, on the other hand, there are repeated vertices in P.

Let i and j be distinct positive integers where $i < j$ and $v_i = v_j$.

If the closed path $v_i - v_j$ is removed from P, an open path P′ is obtained having length $\le n$, since at least the edge $\{v_i, v_{i+1}\}$ was deleted from P.

Thus, by the inductive hypothesis, P′ contains a simple $u - v$ path and, thus, so does P.

Problem 3.83. *Find all circuits in the graph shown below :*

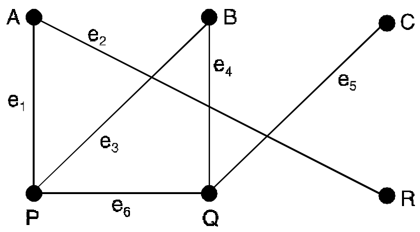


Fig. 3.110

Solution. There are no circuits beginning and ending with the vertices A, C and R.

The circuits beginning and ending with the vertices

B, P, Q are $Be_3Pe_6Qe_4B$, $Pe_6Qe_4Be_3P$, $Qe_4Be_3Pe_6Q$

But all of these represent one and the same circuit.

Thus, there is only one circuit in the graph.

Problem 3.84. *Consider the graph shown in Figure, find all paths from vertex A to vertex R. Also, indicate their lengths.*

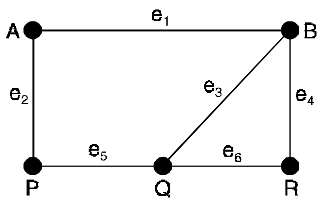


Fig. 3.111

Solution. There are four paths from A to R.

These are Ae_1Be_4R, $Ae_1Be_3Qe_6R$, $Ae_2Pe_5Qe_6R$, $Ae_2Pe_5Qe_3Be_4R$.

These paths contain, two, three and four edges.

Their lengths are two, three, three and four respectively.

Problem 3.85. *Prove the following :*

(*a*) *A path with n vertices is of length n – 1*

(*b*) *If a circuit has n vertices, it has n edges*

(*c*) *The degree of every vertex in a circuit is two.*

Solution. (*a*) In a path, every vertex except the last is followed by precisely one edge.

Therefore, if a path has n vertices, it must have $n - 1$ edges. Its length is $n - 1$.

(*b*) In a circuit, every vertex is followed by precisely one edge.

Therefore, if a circuit has n vertices, it must have n edges.

(*c*) In a circuit, exactly two edges are incident on every vertex.

Therefore, the degree of every vertex in a circuit is two.

Problem 3.86. *If G is a simple graph in which every vertex has degree at least k, prove that G contains a path of length at least k. Deduce that if* $k \geq 2$ *then G also contains a circuit of length at least* $k + 1$.

Solution. Consider a path P in G, which has a maximum number of vertices. Let u be an end vertex of P. Then every neighbour of u belongs to P. Since u has at least k neighbours and since G is simple, P must have at least k vertices other than u.

Thus, P is a path of length at least k

If $k \geq 2$ then P and the edge from u to its farthest neighbour v constitute a circuit of length at least $k + 1$.

3.12 EULERIAN GRAPHS

3.12.1. Euler path

A path in a graph G is called Euler path if it includes every edges exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

3.12.2. Euler circuit

An Euler path that is circuit is called Euler circuit. A graph which has a Eulerian circuit is called an Eulerian graph.

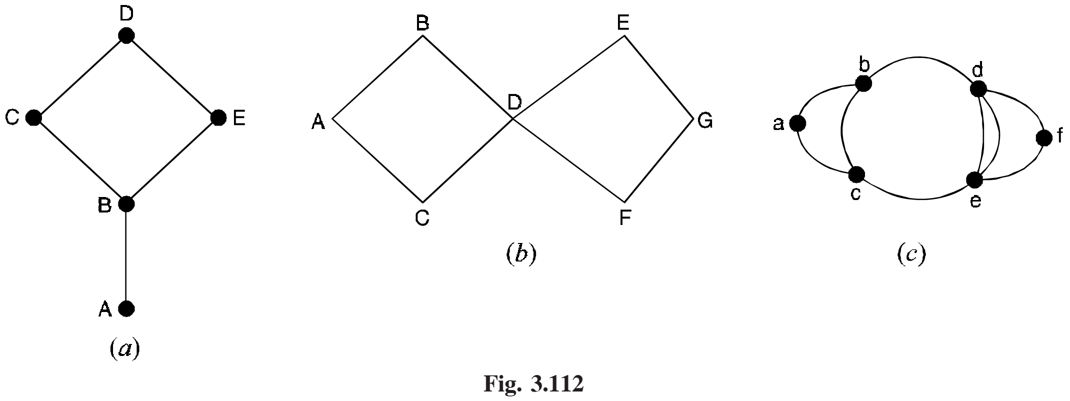


Fig. 3.112

The graph of Figure 3.112(*a*) has an Euler path but no Euler circuit. Note that two vertices A and B are of odd degrees 1 and 3 respectively. That means AB can be used to either arrive at vertex A or leave vertex A but not for both.

Thus an Euler path can be found if we start either from vertex A or from B.

ABCDEB and BCDEBA are two Euler paths. Starting from any vertex no Euler circuit can be found.

The graph of Figure 3.112(*b*) has both Euler circuit and Euler path. ABDEGFDCA is an Euler path and circuit. Note that all vertices of even degree.

No Euler path and circuit is possible in Figure 3.112(*c*).

Note that all vertices are not even degree and more than two vertices are of odd degree.

The existence of Euler path and circuit depends on the degree of vertices.

Note : To determine whether a graph G has an Euler circuit, we note the following points :

(*i*) List the degree of all vertices in the graph.

(*ii*) If any value is zero, the graph is not connected and hence it cannot have Euler path or Euler circuit.

(*iii*) If all the degrees are even, then G has both Euler path and Euler circuit.

(*iv*) If exactly two vertices are odd degree, then G has Euler path but no Euler circuit.

Theorem 3.14. *The following statements are equivalent for a connected graph G :*

(*i*) *G is Eulerian*

(*ii*) *Every point of G has even degree*

(*iii*) *The set of lines of G be partitioned into cycles.*

Proof. (*i*) implies (*ii*)

Let T be an Eulerian trail in G.

Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

(*ii*) implies (*iii*)

Since G is connected and non trivial, every point has degree at least 2, so G contains a cycle Z.

The removal of the lines of Z results in a spanning subgraph G_1 in which every point still has even degree.

If G_1 has no lines, then (*iii*) already holds ; otherwise, repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even, etc.

When a totally disconnected graph G_n is obtained, we have a partition of the lines of G into *n* cycles.

(*iii*) implies (*i*)

Let Z_1 be one of the cycles of this partition.

If G consists only of this cycle, then G is obviously Eulerian.

Otherwise, there is another cycle Z_2 with a point *v* in common with Z_1.

The walk beginning at *v* and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the lines of these two cycles.

By continuing this process, we can construct a closed trail containing all lines of G.

Hence G is Eulerian.

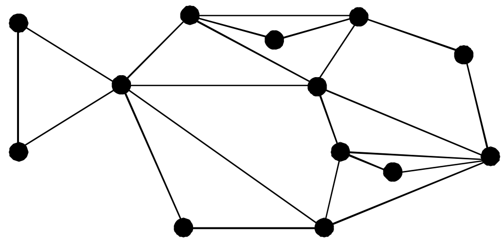


Fig. 3.113. An Eulerian graph.

For example, the connected graph of Figure 3.113 in which every point has even degree has an Eulerian trail, and the set of lines can be partitioned into cycles.

Corollary (1) :

Let G be a connected graph with exactly $2n$ odd points, $n \geq 1$, then the set of lines of G can be partitioned into n open trails.

Corollary (2) :

Let G be a connected graph with exactly two odd points. Then G has an open trail containing all the points and lines of G (which begins at one of the odd points and ends at the other).

Problem 3.87. *A non empty connected graph G is Eulerian if and only if its vertices are all of even degree.*

Proof. Let G be Eulerian.

Then G has an Eulerian trail which begins and ends at u, say.

If we travel along the trail then each time we visit a vertex we use two edges, one in and one out.

This is also true for the start vertex because we also ends there.

Since an Eulerian trial uses every edge once, each occurrence of v represents a contribution of 2 to its degree.

Thus deg(v) is even.

Conversely, suppose that G is connected and every vertex is even.

We construct an Eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding an edge after the other.

If T_1 is not closed at any step, say T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1.

Hence we can extend T_1 by another edge incident on v.

Thus we can continue to extend T_1 until T_1 returns to its initial vertex u.

i.e., until T_1 is closed.

If T_1 includes all the edges of G then T_1 is an Eulerian trail.

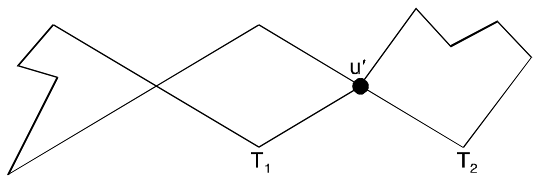


Fig. 3.114

Suppose T_1 does not include all edges of G.

Consider the graph H obtained by deleting all edges of T_1 from G.

H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex.

Since G is connected, there is an edge e' of H which has an end point u' in T_1.

We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree.

We can continue to extent T_2 until T_2 returns to u' as shown in Figure.

We can clearly put T_1 and T_2 together to form a larger closed trail in G.

We continue this process until all the edges of G are used.

We finally obtain an Eulerian trail, and so G is Eulerian.

Theorem 3.15. *A connected graph G has an Eulerian trail if and only if it has at most two odd vertices.*

i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof. Suppose G has an Eulerian trail which is not closed. Since each vertex in the middle of the trail is associated with two edges and since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and the other vertices must be even.

Conversely, suppose that G is connected with atmost two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

The leaves us to treat the case when G has two odd vertices (G cannot have just one odd vertex since in any graph there is an even number of vertices with odd degree).

Corollary (1) :

A directed multigraph G has an Euler path if and only if it is unilaterally connected and the in degree of each vertex is equal to its out degree with the possible exception of two vertices, for which it may be that the in degree of is larger than its out degree and the in degree of the other is oneless than its out degree.

Corollary (2) :

A directed multigraph G has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its out degree.

Problem 3.88. *Show that the graph shown in Figure has no Eulerian circuit but has a Eulerian trail.*

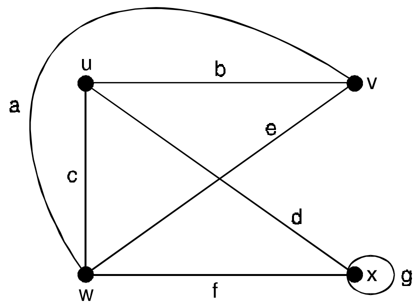


Fig. 3.115

Solution. Here deg (u) = deg (v) = 3 and deg (w) = 4, deg (x) = 4

Since u and v have only two vertices of odd degree, the graph shown in Figure, does not contain Eulerian circuit, but the path.

$b - a - c - d - g - f - e$ is an Eulerian path.

Problem 3.89. *Let G be a graph of Figure. Verify that G has an Eulerian circuit.*

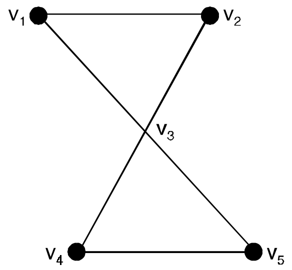


Fig. 3.116

Solution. We observe that G is connected and all the vertices are having even degree

$$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2.$$

Thus G has a Eulerian circuit.

By inspection, we find the Eulerian circuit

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1.$$

Problem 3.90. *Show that the graphs in Figure below contain no Eulerian circuit.*

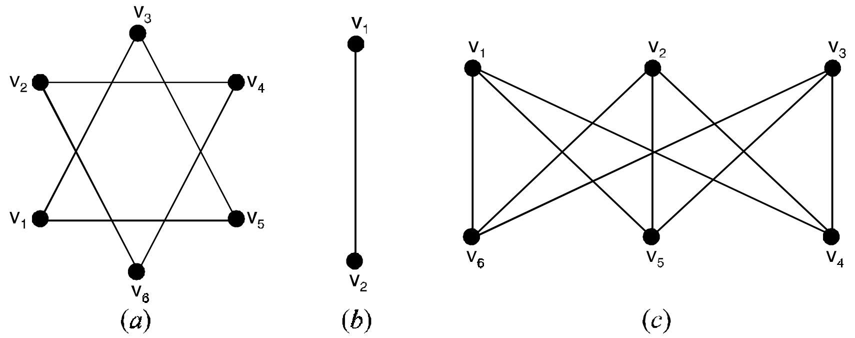


Fig. 3.117

Solution. The graph shown in Figure 3.117(a) does not contain Eulerian circuit since it is not connected.

The graph shown in Figure 3.117(b) is connected but vertices v_1 and v_2 are of degree 1.

Hence it does not contain Eulerian circuit.

All the vertices of the graph shown in Figure 3.117(c) are of degree 3.

Hence it does not contain Eulerian circuit.

Problem 3.91. *Which of the following graphs have Eulerian trail and Eulerian circuit.*

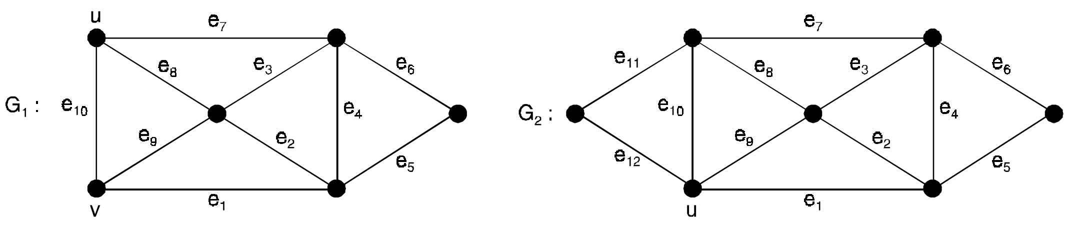


Fig. 3.118

Solution. In G_1 an Eulerian trail from u to v is given by the sequence of edges $e_1, e_2, ..., e_{10}$.

While in G_2 an Eulerian cycle (circuit) from u to v is given by $e_1, e_2, ... e_{11}, e_{12}$.

Problem 3.92. *Show that a connected graph with exactly two odd vertices is a unicursal graph.*

Solution. Suppose A and B be the only two odd vertices in a connected graph G.

Join these vertices by an edge e.

Then A and B become even vertices.

Since all other vertices in G are of even degree, the graph $G \cup e$ is an Eulerian graph.

Therefore, it has an Euler line which must include. The open walk got by deleting e from this Euler line is a semi-Euler line in G.

Hence G is a unicursal graph.

Problem 3.93. (*i*) *Is there is an Euler graph with even number of vertices and odd number of edges ?*

(*ii*) *Is there an Euler graph with odd number of vertices and even number of edges ?*

Solution. (*i*) Yes. Suppose C is a circuit with even number of vertices.

Let v be one of these vertices.

Consider a circuit C′ with odd number of vertices passing through v such that C and C′ have no edge in common.

The closed walk q that consists of the edges of C and C′ is an Eulerian graph of the desired type.

(*ii*) Yes, in (*i*), suppose C and C′ are circuits with odd number of vertices.

Then q is an Eulerian graph of the desired type.

Problem 3.94. *Find all positive integers n such that the complete graph k_n is Eulerian.*

Solution. In the complete graph k_n, the degree of every vertex is $n - 1$.

Therefore, k_n is Eulerian if and only if $n - 1$ is even, *i.e.,* if and only if n is odd.

Problem 3.95. *Which of the undirected graph in Figure have an Euler circuit ? Of those that do not, which have an Euler path ?*

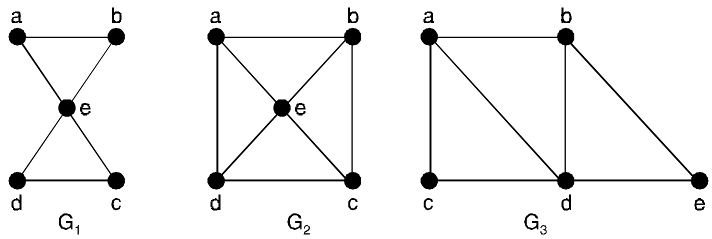


Fig. 3.119. The undirected graphs G_1, G_2 and G_3.

Solution. The graph G_1 has an Euler circuit.

For example, *a, e, c, d, e, b, a.* Neither of the graphs G_2 or G_3 has an Euler circuit. However, G_3 has an Euler path, namely *a, c, d, e, b, d, a, b.*

G_2 does not have an Euler path.

Problem 3.96. *Which of the directed graphs in Figure have an Euler circuit ? Of those that do not, which have an Euler path ?*

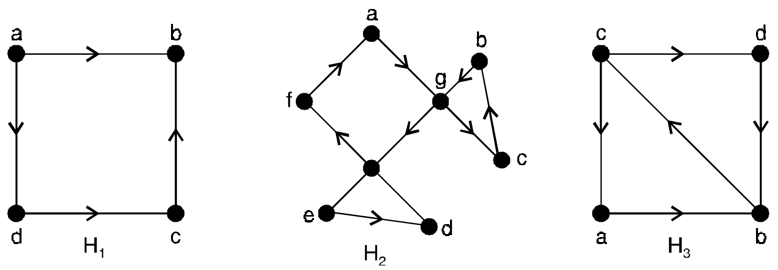


Fig. 3.120. The directed graphs H_1, H_2, H_3

Solution. The graph H_2 has an Euler circuit, for example *a, g, c, b, g, e, d, f, a.* Neither H_1 nor H_3 has an Euler circuit. H_3 has an Euler path, namely *e, a, b, c, d, b* but H_1 does not.

Problem 3.97. *Which graphs shown in Figure have an Euler path ?*

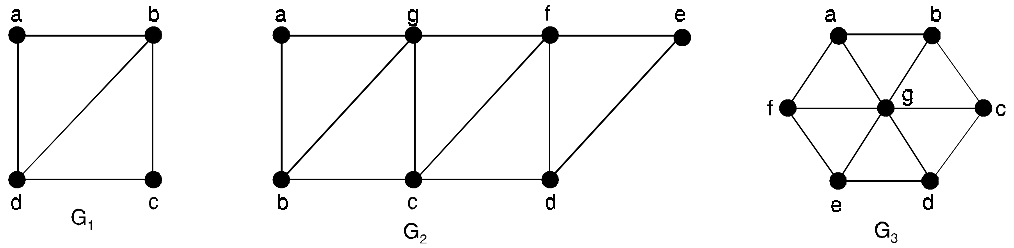


Fig. 3.121. Three undirected graphs.

Solution. G_1 contains exactly two vertices of odd degree, namely, *b* and *d*.

Hence it has an Euler path that must have *b* and *d* as its end points. One such Euler path is *d, a, b, c, d, b.* Similarly, G_2 has exactly two vertices of odd degree, namely, *b* and *d*. So it has an Euler path that must have *b* and *d* as end points. One such Euler path is *b, a, g, f, e, d, c, g, b, c, f, d.*

G_3 has no Euler path since it has six vertices of odd.

Problem 3.98. *If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.*

Solution. If G has any loops or multiple edges, the result is trivial.

Suppose that G is a simple graph.

Let v be any vertex of G.

We construct a walk $v \to v_1 \to v_2 \to ......$ inductively by choosing v_1 to be any vertex adjacent to v and for each $i > 1$.

Choosing v_{i+1} to be any vertex adjacent to v_i except v_{i-1}, the existence of such a vertex is guaranteed by our hypothesis.

Since G has only finitely many vertices, we must eventually choose a vertex that has been chosen before.

If v_k is the first such vertex, then that part of the walk lying between the two occurrences of v_k is the required cycle.

Theorem 3.16. *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Proof. Suppose that P is an Eulerian trail of G. Whenever P passes through a vertex, there is a contradiction of 2 towards the degree of that vertex.

Since each edge occurs exactly once in P, each vertex must have even degree.

The proof is by induction on the number of edges of G.

Suppose that the degree of each vertex is even.

Since G is connected, each vertex has degree at least 2 and so by lemma, G contains a cycle C.

If C contains every edge of G, the proof is complete.

If not, we remove from G the edges of C to form a new, possibly disconnected, graph H with fewer edges that G and in which each vertex still has even degree.

By the induction hypothesis, each component of H has an Eulerian trail.

Since each component of H has at least one vertex in common with C, by connectedness, we obtain the required Eulerian trail of G by following the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H and so on.

The whole process terminates when we return to the initial vertex (see Figure below)

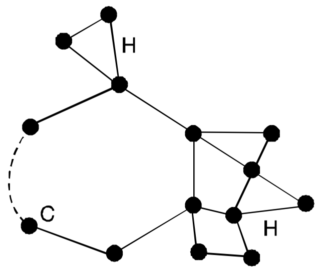


Fig. 3.122

Corollary (1) :

A connected graph is Eulerian if and only if its set of edges can be split up into disjoint cycles.

Corollary (2) :

A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Theorem 3.17. *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trail of G.*

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following results :

(*i*) *erase the edges as they are traversed, and if any isolated vertices result, erase them too ;*

(*ii*) *at each stage, use a bridge only if there is no alternative.*

Proof. We show first that the construction can be carried out at each stage.

Suppose that we have just reached a vertex v.

If $v \neq u$ then the subgraph H that remains is connected and contains only two vertices of odd degree u and v.

To show that the construction can be carried out, we must show that the removal of the next edge does not disconnected H or equivalently, that v is incident with atmost one bridge.

But if this is not the case, then there exists a bridge vw such that the component K of H – vw containing w does not contain u (see Figure, below).

Since the vertex w has odd degree in K, some other vertex of K must also have odd degree, giving the required contradiction.

If $v = u$, the proof is almost identical, as long as there are still edges incident with u.

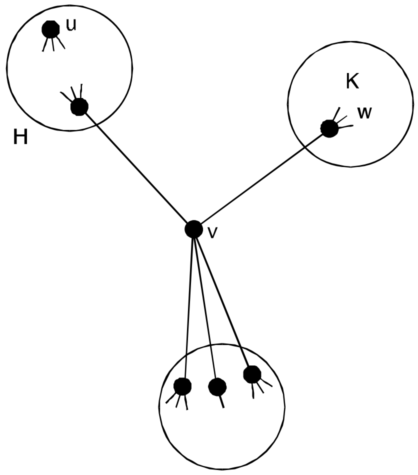


Fig. 3.123

It remains only to show that this construction always yields a complete Eulerian trail.

But this is clear, since there can be no edges of G remaining untraversed when the last edge incident to u is used, since otherwise the removal of some earlier edge adjacent to one of these edges would have disconnected the graph, contradicting (*ii*).

Theorem 3.18. *(a) If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.*

(b) If G is connected and has exactly two vertices of odd degree, there is an Euler path in G. Any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof. (*a*) Let v_1, v_2, v_3 be vertices of odd degree.

Any possible Euler path must leave (or arrive at) each of v_1, v_2, v_3 with no way to return (or leave) since each of these vertices has odd degree.

One vertex of these three vertices may be the beginning of the Euler path and another the end, but this leaves the third vertex at one end of an untraveled edge.

Thus there is no Euler path.

(*b*) Let u and v be the two vertices of odd degree. Adding the edge $\{u, v\}$ to G produces a connected graph G′ all of whose vertices has even degree. There is an Euler circuit π' in G′.

Omitting $\{u, v\}$ from π' produces an Euler path that begins at u (or v) and ends at v (or u).

Theorem 3.19. *(a) If a graph G has a vertex of odd degree, there can be no Euler circuit in G.*

(b) If G is a connected graph and every vertex has even degree, then there is an Euler circuit in G.

Proof. (*b*) Suppose that there are connected graphs where every vertex has even degree, but there is no Euler circuit. Choose such a G with the smallest number of edges.

G must have more than one vertex since, if there were only one vertex of even degree, there is clearly in Euler circuit. We show first that G must have atleast one circuit. If v is a fixed vertex of G, then since G is connected and has more than one vertex, there must be an edge between v and some other vertex v_1.

This is a simple path (of length 1) and so simple paths exists. Let π_0 be a simple path in G having the longest possible length, and let its vertex sequence be $v_1, v_2, ... v_s$. Since v_s has even degree and π_0 uses only one edge that has v_s as a vertex, there must be an edge e not in π_0 that also has v_s as a vertex.

If the other vertex of e is not one of the v_i, then we could construct a simple path longer than π_0. Which is a contradiction.

Thus e has some v_i as its other vertex, and therefore we can construct a circuit.

$$v_i, v_{i+1}, ... v_s, v_i \text{ in G.}$$

Since we know that G has circuits, we may choose a circuit π in G that has the longest possible length. Since we assumed that G has no Euler circuits, π cannot contain all the edges of G.

Let G_1 be the graph formed from G by deleting all edges in π (but not vertices).

Since π is a circuit, deleting its edges will reduce the degree of every vertex by 0 or 2, so G_1 is also a graph with all vertices of even degree.

The graph G_1 may not be connected, but we can choose a largest connected component (piece) and call this graph G_2 (G_2 may be G_1).

Now G_2 has fewer edges than G, and so (because of the way G was chosen), G_2 must have an Euler path π'.

If π' passes through all the vertices on G, then π and π' clearly have vertices in common.

If not, then these must be an edge in G between some vertex v' in π', and some vertex v not in π'.

Otherwise we could not get from vertices in π' to the other vertices in G, and G would not be connected.

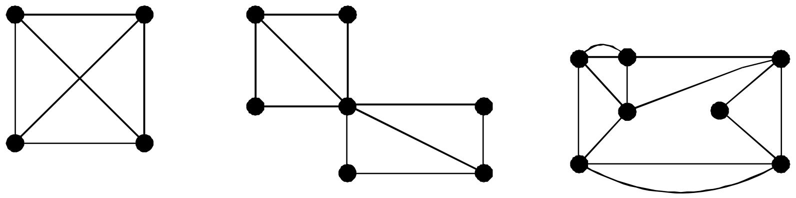
Since e is not in π', it must have been deleted when G_1 was created from G, and so must be an edge in π.

Then v' is also in the vertex sequence of π, and so in any case π and π' have atleast one vertex v' in common. We can then construct a circuit in G that is longer than π by combining π and π' at v'.

This is a contradiction, since π was chosen to be the longest possible circuit in G.

Hence the existence of the graph G always produces a contradiction, and so no such graph is possible.

Problem 3.99. *Which of the graphs in Figure (a), (b), (c) have an Euler circuit, an Euler path but not an Euler circuit, or neither ?*



(*a*) (*b*) (*c*)

Fig. 3.124

Solution. (*i*) In Figure 3.124(*a*), each of the four vertices has degree 3 ; thus, there is neither an Euler circuit nor an Euler path.

(*ii*) The graph in Figure 3.124(*b*) has exactly two vertices of odd degree. There is no Euler circuit, but there must be an Euler path.

(*iii*) In Figure 3.124(*c*), every vertex has even degree ; thus the graph must have an Euler circuit.

3.13 FLEURY'S ALGORITHM

Let G = (V, E) be a connected graph with each vertex of even degree.

Step 1. Select an edge e_1 that is not a bridge in G.

Let its vertices be v_1, v_2.

Let π be specified by $V_\pi : v_1, v_2$ and $E_\pi : e_1$.

Remove e_1 from E and v_1 and v_2 from V to create G_1.

Step 2. Suppose that $V_\pi : v_1, v_2, ... v_k$ and $E_\pi : e_1, e_2, ...e_{k-1}$ have been constructed so far, and that all of these edges and vertices have been removed from v and E to form G_{k-1}.

Since v_k has even degree, and e_{k-1} ends there, there must be an edge e_k in G_{k-1} that also has v_k as a vertex.

If there is more than one such edge, select one that is not a bridge for G_{k-1}.

Denote the vertex of e_k other than v_k by v_{k+1}, and extend V_π and E_π to $V_\pi : v_1, v_2, ..., v_k, v_{k+1}$ and $E_\pi : e_1, e_2, ..., e_{k-1}, e_k$.

Step 3. Repeat step 2 until no edges remain in E.

End of algorithm.

Problem 3.100. *Use Fleury's algorithm to construct an Euler circuit for the graph in Figure 3.125.*

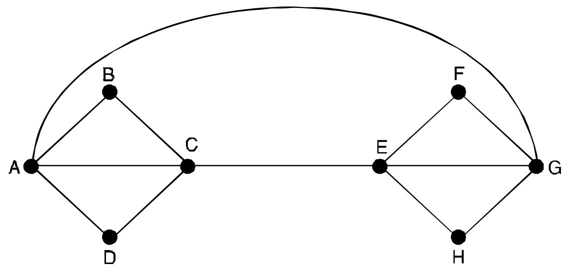


Fig. 3.125

Solution. According to step 1, we may begin anywhere.

Arbitrarily choose vertex A. We summarize the results of applying step 2 repeatedly in Table.

Current Path	*Next Edge*	*Reasoning*
π : A	{A, B}	No edge from A is a bridge. Choose any one.
π : A, B	{B, C}	Only one edge from B remains.
π : A, B, C	{C, A}	No edges from C is a bridge. Choose any one.
π : A, B, C, A	{A, D}	No edges from A is a bridge. Choose any one.
π : A, B, C, A, D	{D, C}	Only one edge from D remains.
π : A, B, C, A, D, C	{C, E}	Only one edge from C remains.
π : A, B, C, A, D, C, E	{E, G}	No edge from E is a bridge. Choose any one.
π : A, B, C, A, D, C, E, G	{G, F}	{A, G} is a bridge. Choose {G, F} or {G, H}.
π : A, B, C, A, D, C, E, G, F	{F, E}	Only one edge from F remains.
π : A, B, C, A, D, C, E, G, F, E	{E, H}	Only one edge from E remains.
π : A, B, C, A, D, C, E, G, F, E, H	{H, G}	Only one edge from H remains
π : A, B, C, A, D, C, E, G, F, E, H, G	{G, A}	Only one edge from G remains.
π : A, B, C, A, D, C, E, G, F, E, H, G, A		

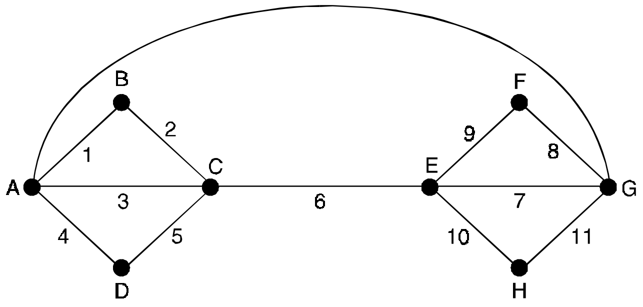


Fig. 3.126

The edges in Figure 3.126 have been numbered in the order of their choice in applying step 2.

In several places, other choices could have been made.

In general, if a graph has an Euler circuit, it is likely to have several different Euler circuits.

Problem 3.101. *Using Fleury's algorithm, find Euler circuit in the graph of Figure.*

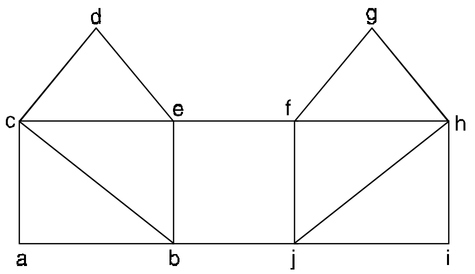


Fig. 3.127

Solution. The degrees of all the vertices are even. There exists an Euler circuit in it.

Current Path	*Next Edge*	*Remark*
$\pi : a$	$\{a, j\}$	No edge from a is a bridge choose (a, j). Add j to π and remove (a, j) from E.
$\pi : aj$	$\{j, f\}$	No edge from j is a bridge. Choose (j, f). Add f to π and remove (j, f) from E.
$\pi : ajf$	$\{f, g\}$	(f, e) is a bridge and (f, g) is not a bridge. Other option (f, h)
$\pi : ajfg$	$\{g, h\}$	(g, h) is the only edge.
$\pi : ajfgh$	$\{h, i\}$	(h, i) is the other option
$\pi : ajfghi$	$\{i, j\}$	(i, j) is the only edge.
$\pi : ajfghij$	$\{j, h\}$	(j, h) is the only edge.
$\pi : ajfghijh$	$\{h, f\}$	(h, f) is the only edge
$\pi : ajfghijhf$	$\{f, e\}$	(f, e) is the only edge
$\pi : ajfghijhfe$	$\{e, d\}$	Other options are (e, c), (e, a)
$\pi : ajfghijhfed$	$\{d, c\}$	(d, c) is the only option.
$\pi : ajfghijhfedc$	$\{c, b\}$	Other options are (c, e), (c, a).
$\pi : ajfghijkfedcb$	$\{b, a\}$	(b, a) is the only option.
$\pi : ajfghijkfedcba$	$\{a, c\}$	Other options are (a, e)
$\pi : ajfghijkfedcbac$	$\{c, e\}$	(c, e) is the only option.
$\pi : ajfghijkfedcbace$	$\{e, a\}$	(e, a) is the only option.
$\pi : ajfghijkfedcbacea$		No edge is remaining in E.

This is the Euler circuit.

Problem 3.102. *Using Fleury's algorithm, find Euler circuit in the graph of Figure.*

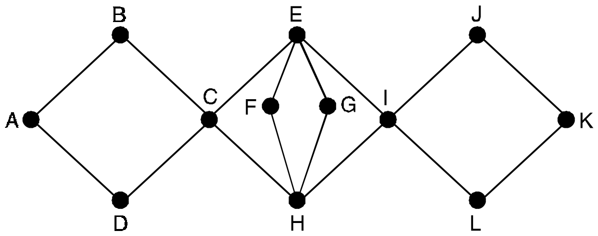


Fig. 3.128

Solution. The degree spectrum of the graph is (2, 2, 4, 2, 4, 2, 2, 4, 4, 2, 2, 2) considering the node from A to L in alphabetical order. Since all values are even there exists an Euler circuit in it. The process is summarized in the following table. The start node is A.

S.No.	*Current path*	*Next Edge Considered*	*Remark*
1.	π : A	{A, B}	We select (A, B). Add B to π and remove (A, B) from E.
2.	π : AB	{B, C}	It is the only option. Remove (B, C) from E and B from V. Add C to π.
3.	π : ABC	{C, E}	(C, D) cannot be selected, as it is a bridge. Add E to π and remove (C, E) from E.
4.	π : ABCE	{E, F}	Other options are there.
5.	π : ABCEF	{F, H}	Other option is (H, I). We cannot select
6.	π : ABCEFH	{H, G}	(H, C), as it is a bridge.
7.	π : ABCEFHG	{G, E}	As in Sl. No. 2
8.	π : ABCEFHGE	{E, I}	As in Sl. No. 2
9.	π : ABCEFHGEI	{I, J}	Other options are also there. Edge (I, H) is a bridge.
10.	π : AFCEFHGEIJ	{J, K}	As in Sl. No. 2.
11.	π : ABCEFHGEIJK	{K, L}	As in Sl. No. 2
12.	π : ABCEFHGEIJKL	{L, I}	As in Sl. No. 2
13.	π : ABCEFHGEIJKLI	{I, H}	As in Sl. No. 2
14.	π : ABCEFHGEIJKLIH	{H, C}	As in Sl. No. 2
15.	π : ABCEFHGEIJKLIHC	{C, D}	As in Sl. No. 2
16.	π : ABCEFHGEIJKLIHCD	{D, A}	As in Sl. No. 2
17.	π : ABCEFHGEIJKLIHCDA		This is the Euler cycle

3.14 HAMILTONIAN GRAPHS

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vertices of a graph appear exactly once.

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that **appears twice** is known as **Hamiltonian circuit.**

A graph G is called a **Hamiltonian graph,** if it contains a Hamiltonian circuit.

A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visists each vertex exactly once but may repeat edges. While there is a criterion for determining whether or not a graph contains an Eulerian circuit, a similar criterion does not exist for Hamiltonian ciruits.

In the following figures, hamiltonian path and cycles are shown :

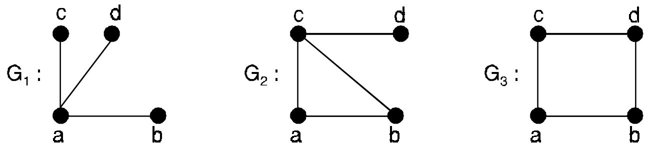


Fig. 3.129

The graph G_1 has no hamiltonian path (and so hamiltonian cycle), G_2 has hamiltonian path *abcd* but no hamiltonian cycle, while G_3 has hamiltonian cycle *abdca*.

The cycle C_n with n distinct (and n edges) is hamiltonian. Moreover given hamiltonian graph G then if G′ is a subgraph obtained by adding in new edges between vertices of G, G′ will also be hamiltonian. Since any hamiltonian cycle in G will also be hamiltonian cycle in G′. In particular k_n, the complete graph on n vertices, in such a supergraph of a cycle, k_n is hamiltonian.

A simple graph G is called maximal non-hamiltonian if it is not hamiltonian but the addition to it any edge connecting two non-adjacent vertices forms a hamiltonian graph. The graph G_2 is a maximal non-hamiltonian since the addition of an edge *bd* gives hamiltonian graph G_3.

3.14.1. Dirac's theorem (3.20)

Let G be a graph of order $p \geq 3$. If deg $v \geq \dfrac{p}{2}$ for every vertex v of G, then G is hamiltonian.

Proof. If $p = 3$, then the condition on G implies that $G \cong k_3$ and hence G is hamiltonian.

We may assume, therefore, that $p \geq 4$.

Let P : $v_1, v_2, \ldots\ldots. v_n$ be a longest path in G (see Figure 3.130). Then every neighbour of v_1 and every neighbour of v_n is on P.

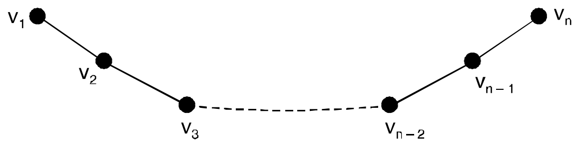


Fig. 3.130

Otherwise, there would be a longer path than P.

Consequently, $n \geq 1 + \frac{p}{2}$.

There must be some vertex v_i where $2 \leq i \leq n$, such that v_1 is adjacent to v_i and v_n is adjacent to v_{i-1}.

If this were not the case, then whenever v_1 is adjacent to a vertex v_i, the vertex v_n is not adjacent to v_{i-1}.

Since atleast $\frac{p}{2}$ of $p - 1$ vertices different from v_n are not adjacent to v_n.

Hence, deg $v_n \leq (P - 1) - \frac{p}{2} < \frac{p}{2}$, which contradicts the fact that deg $v_n \geq \frac{p}{2}$.

Therefore as we claimed, there must be a vertex v_i adjacent to v_1 and v_{i-1} is adjacent to v_n (see Figure 3.131).

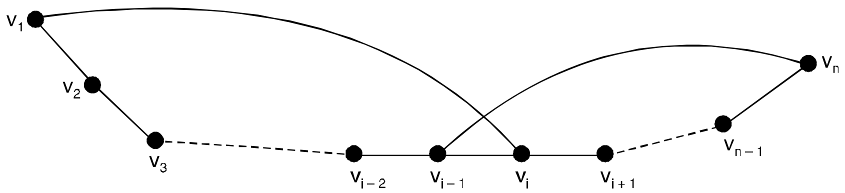


Fig. 3.131

We now see that G has cycle C : $v_1, v_i, v_{i+1}, \ldots\ldots v_{n-1}, v_n, v_{i-1}, v_{i-2}, \ldots\ldots, v_2, v_1$ that contains all the vertices of P.

If C contains all the vertices of G (if $n = p$) then C is a hamiltonian cycle, and the proof.

Otherwise, there is some vertex u of G that is not on C.

By hypothesis, deg $u \geq \frac{p}{2}$. Since P contains at least $1 + \frac{p}{2}$ vertices, there are fewer than $\frac{p}{2}$ vertices not on C ; so u must be adjacent to a vertex v that lies on C.

However, the edge uv plus the cycle C contain a path whose length is greater than that of P, which is impossible.

Thus C contains all vertices of G and G is hamiltonian.

Hence the proof.

Corollary :

Let G be a graph with p-vertices. If deg $v \geq \frac{p-1}{2}$ for every vertex v of G then G contains a hamiltonian path.

Proof. If $p = 1$ then $G \cong k_1$ and G contains a (trivial) Hamiltonian path.

Suppose then that $p \geq 2$ and define $H = G + k_1$.

Let v denote the vertex of H that is not in C.

Since H has vertex $p + 1$, it follows that deg $v \geq p$.

Moreover, for every vertex u of G,

$$\deg_H u = \deg_G u + 1 \geq \frac{p-1}{2} + 1 = \frac{p+1}{2} = \frac{|V(H)|}{2}.$$

By Dirac's theorem, H contains a hamiltonian cycle C. By removing the vertex v from C, we obtain a hamiltonian path in G.

Hence the proof.

Theorem 3.21. *If G is a connected graph of order three or more which is not hamiltonian, then the length k of a longest path of G satisfies $k \geq 2\delta(G)$.*

Proof. Let $p : u_0, u_1, ..., u_k$ be a longest path in G.

Since P is longest path, each of u_0 and u_k is adjacent only two vertices of P.

If $u_0u_i \in E(G)$, $1 \leq i \leq k$, then $u_{i-1} u_k \notin E(G)$ for otherwise the cycle C : $u_0, u_1, u_2, ..., u_{i-1}, u_k, u_{k-1}, u_{k-2}, ..., u_i, u_0$ of length $k + 1$ is present in G.

The cycle C cannot contain all vertices of G, since G is not Hamiltonian.

Therefore, there exists a vertex w not on C adjacent with a vertex of C, however this implies G contains a path of length $k + 1$, which is impossible.

Hence for each vertex of $\{u_1, u_2, ..., u_k\}$ adjacent to u_0 there is a vertex of $\{u_0, u_1, ..., u_{k-1}\}$ not adjacent with u_k.

Thus deg $u_k \leq k -$ deg u_0 so that

$$k \geq \deg u_0 + \deg u_k \geq 2\delta(G).$$

Hence the proof.

Problem 3.103. *Let G be a simple graph with n vertices and let u and v be an edge. Then G is hamiltonian if and only if G + uv is hamiltonian.*

Solution. Suppose G is hamiltonian. Then the super graph G + uv must also be hamiltonian.

Conversely, suppose that G + uv is hamiltonian.

Then if G is not hamiltonian.

i.e., if G is a graph with $p \geq 3$ vertices such that for all non adjacent vertices u and v, deg u + deg $v \geq p$.

We obtain the inequality deg u + deg $v < n$.

However by hypothesis, deg u + deg $v \geq n$.

Hence G must be hamiltonian.

This completes the proof.

3.14.2. Ore's theorem (3.22)

If G is a group with $p \geq 3$ vertices such that for all non adjacent vertices u and v, deg u + deg $v \geq p$, then G is hamiltonian.

Proof. Let k denotes the number of vertices of G whose degree does not exceed n,

$$\text{where} \quad 1 \leq n \leq \frac{p}{2}$$

These k vertices induce a subgraph H which is complete, for if any two vertices of H were not adjacent, there would exist two non adjacent vertices, the sum of whose degree is less than p.

This implies that $k \le n + 1$. However $k \ne n + 1$, for otherwise each vertex of H is adjacent only two vertices of H, and if $u \in V(H)$ and $v \in V(G) - V(H)$, then $\deg u + \deg v \le n + (p - n - 2) = p - 2$, which is a contradiction.

Further $k \ne n$; otherwise each vertex of H is adjacent to at most one vertex of G not in H.

However, since $k = n < \frac{p}{2}$, there exists a vertex $w \in V(G) - V(H)$ adjacent to no vertex of H. Then if $u \in V(H)$, $\deg u + \deg \omega \le n + (p - n - 1) = p - 1$, which again a contradiction.

Therefore $k < n$, which implies that G satisfies the condition, so that G is Hamiltonian.

Hence the proof.

Problem 3.104. *Let G be a simple graph with n vertices and m edges where m is at least 3. If* $m \ge \frac{1}{2}(n - 1)(n - 2) + 2$.

Prove that G is Hamiltonian. Is the converse true ?

Solution. Let u and v be any two non-adjacent vertices in G.

Let x and y be their respective degrees.

If we delete u, v from G, we get a subgraph with $n - 2$ vertices.

If this subgraph has q edges then $q \le \frac{1}{2}(n - 2)(n - 3)$.

Since u and v are non-adjacent, $m = q + x + y$

Thus, $x + y = m - q \ge \left\{\frac{1}{2}(n-1)(n-2) + 2\right\} - \left\{\frac{1}{2}(n-2)(n-3)\right\} = n.$

Therefore, the graph is Hamiltonian.

The converse of the result just proved is not always true.

Because, a 2-regular graph with 5-vertices (see Figure below) is Hamiltonian but the inequality does not hold.

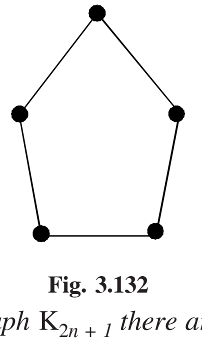


Fig. 3.132

Theorem 3.23. *In a complete graph* K_{2n+1} *there are n edge disjoint Hamiltonian cycles.*

Proof. We first label the vertices of K_{2n+1} as $v_1, v_2, \ldots\ldots v_{2n+1}$ then we construct n paths $P_1, P_2, \ldots\ldots P_n$ on the vertices $v_1, v_2, \ldots\ldots, v_{2n}$ as follows :

$P_i = v_i v_{i-1} v_{i+1} v_{i-2}, \ldots\ldots, v_{i+n-1}, v_{i-n}, \quad 1 \le i \le n.$

We note that the jth vertex of P_i is v_k where $k = i + (-1)^{j+1}\left(\frac{j}{2}\right)$, and all subscripts are taken as the integers 1, 2, ..., $2n$ (mod $2n$).

The Hamiltonian cycle C_1 is got by joining v_{2n+1} to the end vertices of P_i.

The Figure below illustrates the construction of Hamiltonian cycles in k_7.

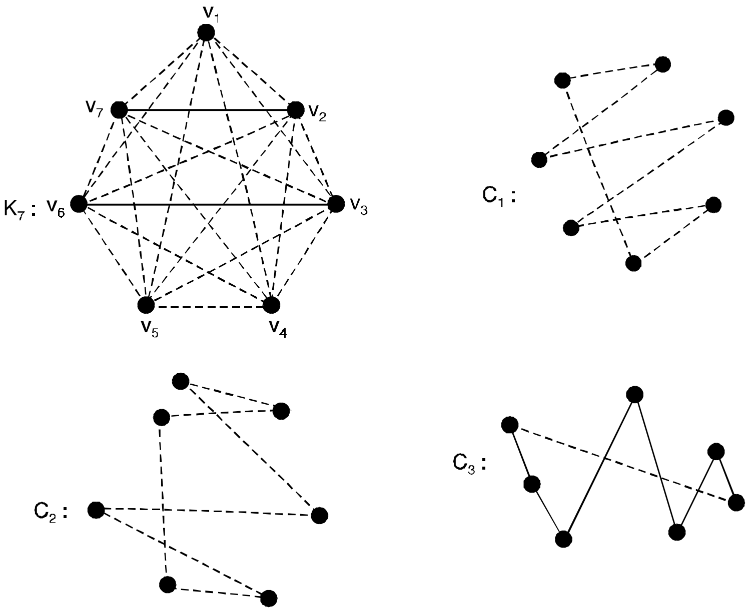


Fig. 3.133

It is still an open problem to find a convenient method to determine which graphs are Hamiltonian.

A graph G in which every edge is assigned a real number is called a weighted graph. The real number associated with an edge is called its weight, and the sum of the weights of the edges of G is called the weight of G.

Problem 3.105. *Which of the graphs given in Figure below is Hamiltonian circuit. Give the circuits on the graphs that contain them.*

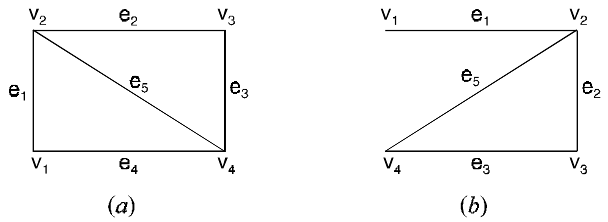


Fig. 3.134

Solution. The graph shown in Figure 3.134(*a*) has Hamiltonian circuit given by $v_1e_1v_2e_2v_3e_3v_4e_4v_1$.

Note that all vertices appear in this a circuit but not all edges.

The edge e_5 is not used in the circuit.

The graph shown in Figure 3.134(*b*) does not contain circuit since every circuit containing every vertex must contain the e_1 twice.

But the graph does have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

Problem 3.106. *Give an example of a graph which is Hamiltonian but not Eulerian and vice-versa.*

Solution. The following graph shown in Figure below is Hamiltonian but non-Eulerian.

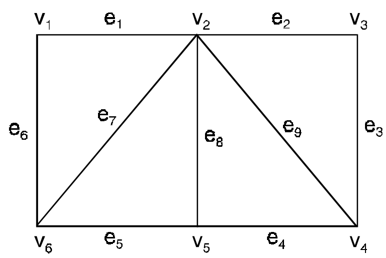


Fig. 3.135

The graph contains a Hamiltonian circuit $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_1$.

Since the degree of each vertex is not *n* even the graph is non-Eulerian.

The graph shown in Figure below is Eulerian but not Hamiltonian.

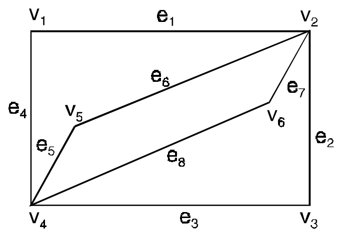


Fig. 3.136

The graph is Eulerian since the degree of each vertex is even.

It does not contain Hamiltonian circuit.

This can be seen by noting any circuit containing every vertex must contain a vertex twice except starting vertex and ending vertex.

Problem 3.107. *Show that any k-regular simple graph with 2k – 1 vertices is Hamiltonian.*

Solution. In a *k*-regular graph, the degree of every vertex is *k* and $k > k - \frac{1}{2} = \frac{1}{2}(2k-1) = \frac{n}{2}$.

Where $n = 2k - 1$ is the number of vertices. Therefore, the graph considered is Hamiltonian.

Problem 3.108. *Prove that the complete graph k_n, $n \geq 3$ is a Hamiltonian graph.*

Solution. In k_n,the degree of every vertex is $n - 1$. If $n > 2$, we have $n - 2 > 0$ or $2n - 2 > n$ or $n - 1 > \frac{n}{2}$.

Thus, in k_n, where $n > 2$, the degree of every vertex is greater than $\frac{n}{2}$.

Hence k_n is Hamiltonian.

Theorem 3.24. *Let G be a simple graph on n vertices. If the sum of degrees of each pair of vertices in G is at least n – 1, then there exists a Hamiltonian path in G.*

Proof. We first prove that G is connected.

If not, then G contains at least two components say G_1 and G_2.

Let n_1 and n_2 be the number of vertices of G in the components G_1 and G_2.

Then $n_1 + n_2 \leq n$, the degree of a vertex x of G that is in the component G_1 is atmost $n_1 - 1$ and the degree of a vertex y of G that is in the component G_2 is atmost $n_2 - 1$.

Hence the sum of degrees of the vertices x and y of G is at most $(n_1 + n_2) - 2 \leq n - 2 < n - 1$, a contradiction.

Now we show the existence of the Hamiltonian path, by construction. The construction is as follows :

Step 1. Choose a vertex a of G.

Step 2. Starting from 'a' construct a path P in G.

Step 3. If P is a Hamiltonian path stop, otherwise go to step 4.

Step 4. Extend the path on both the ends to the maximum (make P be a maximal path).

That is if x is a vertex of G adjacent to the end vertex of the path P and not in P, then includes the vertex to P with the corresponding edge and repeats the process. Call the path so obtained as P.

Step 5. If P is a Hamiltonian path then stop. Otherwise, we observe that there exists a vertex x in G that is not in P and adjacent to a vertex y in P (but y is not an end vertex of P).

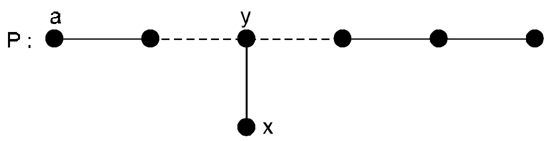


Fig. 3.137

Step 6. Since P is maximal, no vertices of G which are not in P adjacent to the end vertex P.

The end vertices are adjacent to only those vertices in P.

Let P : $a = a_1, a_2, ..., a_k$. Then $k < n$ (otherwise, process stops at step 5).

If a_1 is adjacent to a_k, then obtain a circuit C by join a_1 and a_k, go to step 8. Otherwise, go to step 7 with the following observation.

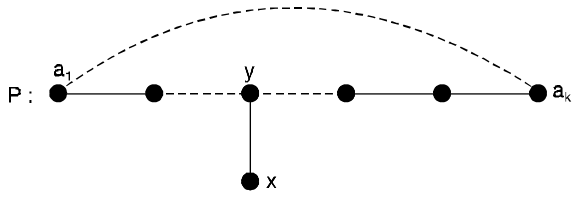


Fig. 3.138

We observe that there exist i, $1 \leq i \leq k$, such that $a_1 a_{i+1}$ and $a_i a_k$ are the edges in G.

If not, then a_1 is not adjacent to any vertex a_{j+1} in P, which is adjacent to a_k.

But the vertices adjacent to a_k are only the vertices of P (follows by the construction of P), it follows that, if degree of a_k is m, then there are m vertices which are not adjacent to a_1 in P.

Thus, there are at most $k - m - 1$ vertices of P (since a_1 is not adjacent to a_1).

Hence degree of a_1 + degree of $a_k \leq (k - m - 1) + m = k - 1 < n - 1$, a contradiction to the assumption made in the statement of the theorem.

Step 7. Construct a circuit C by deleting an edge $a_i a_{i+1}$ in P and joining the edges $a_1 a_{i+1}$ and $a_i a_k$ to P.

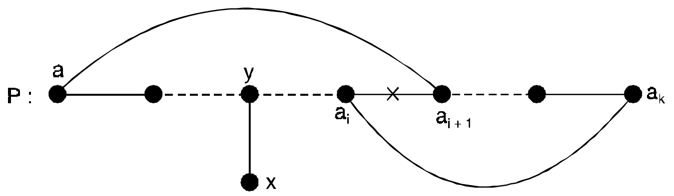


Fig. 3.139

Step 8. Join the edge between the vertex x of G and the vertex y in P (the vertices x and y are those vertices which are observed in step 5) to the circuit C. And delete an edge yz incident with y in C.

Step 9. Step 8 yields a path between the vertex x and the vertex z. This path contains one more vertex than the path P so far we have in our hand (*i.e.,* obtained in step 4) call this path as P and go to step 4.

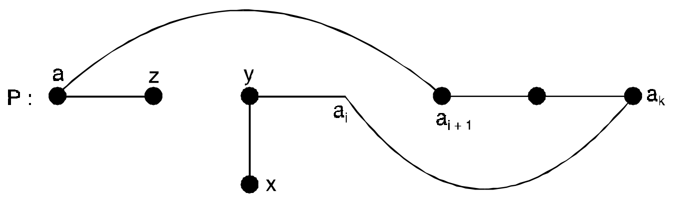


Fig. 3.140

Finally, we note that the process terminates as in each time we are getting a path on one more vertex (that is not in the earlier path) than the earlier path. Moreover, the final output is the desired Hamiltonian path.

Hence the proof.

Theorem 3.25. *In a complete graph with n-vertices there are* $\frac{n-1}{2}$ *edge-disjoint hamiltonian circuits, if n is an odd number* ≥ 3.

Proof : A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges, and a hamiltonian circuit consists of n edges.

Therefore, the number of edge-disjoint hamiltonian circuits in G cannot exceed $\frac{(n-1)}{2}$.

This implies there are $\frac{n-1}{2}$ edge-disjoint hamiltonian circuits, when n is odd it can be shown as by keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by

$$\frac{360}{(n-1)}, \frac{2.360}{(n-1)}, \frac{3.360}{(n-1)}, \ldots\ldots, \frac{n-3}{2} \cdot \frac{360}{(n-1)} \text{ degrees.}$$

At each rotation we get a hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have $\frac{n-3}{2}$ new hamiltonian circuits, all edges disjoint from one and also edge disjoint among themselves.

Hence the proof.

Problem 3.109. *Which of the simple graphs in Figure have a Hamilton circuit or, if not, a Hamilton path ?*

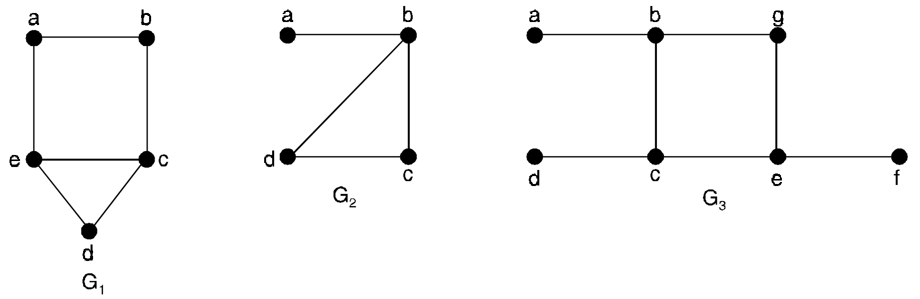


Fig. 3.141. Three simple graphs.

Solution. G_1 has a Hamilton circuit : a, b, c, d, e, a.

There is no Hamilton circuit in G_2, but G_2 does have a Hamilton path, namely a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, since any path containing all vertices must contain one of the edges $\{a, b\}$ $\{e, f\}$ and $\{c, d\}$ more than once.

Problem 3.110. *Show that neither graph displayed in Figure has a Hamilton circuit.*

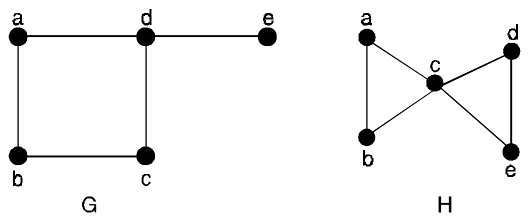


Fig. 3.142. Two graphs that do not have a Hamilton circuit.

Solution. There is no Hamilton circuit in G since G has a vertex of degree one, namely, *e*. Now consider H. Since the degrees of the vertices *a*, *b*, *d* and *e* are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with C, which is impossible.

Problem 3.111. *Find the minimum Hamiltonian circuit starting from node E in the graph of the Figure.*

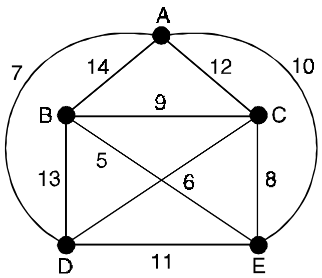


Fig. 3.143

Solution. We have to start with the node E. Closest node to E is the node B. Move to B. Now closest node to B is C move to C, extend path up to C and drop node B and all edges from it, from the graph. From C move to D.

From D, move to A and then to E back.

Finally, we have only node E left in the graph.

Thus, we have a Hamiltonian circuit in the graph, which is π : EBCDAE.

The total minimum of this circuit is :

$$EB + BC + CD + DA + EA = 5 + 9 + 6 + 7 + 10 = 37.$$

Problem 3.112. *At a committee meeting of 10 people, every member of the committee has previously sat next to at most four other members. Show that the members may be seated round a circular table in such a way that no one is next to some one they have previously sat beside.*

Solution. Consider a graph with 10 vertices representing the 10 members.

Let two vertices be joined by an edge if the corresponding members have not previously sat next to each other.

Since any member has not previously sat next to at least five members, the degree of every vertex is at least five.

Therefore, the graph has a Hamiltonian circuit. This circuit provides a seating arrangement of the desired type.

Problem 3.113. *Find three distinct Hamiltonian cycles in the following graph. Also find their weights.*

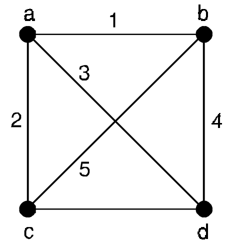


Fig. 3.144

Solution. The cycles C_1, C_2 and C_3 are three distinct Hamiltonian cycles.

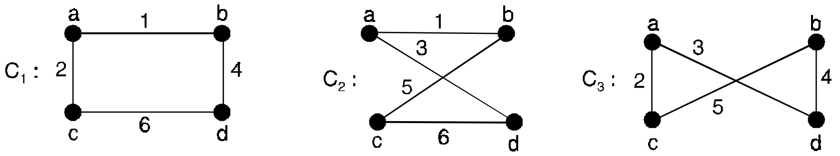


Fig. 3.145

Weight of the cycle $C_1 = 1 + 4 + 6 + 2 = 13$.

Weight of the cycle $C_2 = 1 + 5 + 6 + 3 = 15$

Weight of the third cycle $C_3 = 3 + 4 + 5 + 2 = 14$

Hence the first cycle is of minimum weight.

Theorem 3.26. *A complete graph k_{2n} has a decomposition into n Hamiltonian paths.*

Proof. Consider a complete graph k_{2n}.

Now join a vertex x into K_{2n} and the edges $xv_i \; \forall \; i, \quad 1 \leq i \leq 2n$.

Then the graph G′ so obtained in K_{2n+1}.

Hence G′ can be decomposed into n Hamiltonian cycles.

Removal of the vertex x from each of these cycles we get n edge disjoint Hamiltonian paths which are the required decomposition of K_{2n}.

Theorem 3.27. *Let G be a connected graph with n vertices, n > 2, and no loops or multiple edges. G has a Hamiltonian circuit if for any two vertices u and v of G that are not adjacent, the degree of u plus the degree of v is greater than or equal to n.*

Corollary : G has a Hamiltonian circuit if each vertex has degree greater than or equal to $\frac{n}{2}$.

Proof. The sum of the degrees of any two vertices is at least $\frac{n}{2} + \frac{n}{2} = n$.

Theorem 3.28. *Let the number of edges of G be m. Then G has a Hamiltonian circuit if* $m \geq \frac{1}{2}(n^2 - 3n + 6)$.

(recall that n is the number of vertices)

Proof. Suppose that u and v are any two vertices of G that are not adjacent. We write deg (u) for the degree of u.

Let H be the graph produced by eliminating u and v from G along with any edges that have u or v as end points.

The H has $n - 2$ vertices and $m - \deg(u) - \deg(v)$ edges (one fewer edge would have been removed if u and v had been adjacent).

The maximum number of edges that H could possibly have is ${}_{n-2}C_2$.

This happens when there is an edge connecting every distinct pair of vertices.

Thus the number of edges of H is at most

$$ {}_{n-2}C_2 = \frac{(n-2)(n-3)}{2} \quad \text{or} \quad \frac{1}{2}(n^2 - 5n + 6) $$

We then have $m - \deg(u) - \deg(v) \leq \frac{1}{2}(n^2 - 5n + 6)$.

Therefore, $\deg(u) + \deg(v) \geq m - \frac{1}{2}(n^2 - 5n + 6)$

By the hypothesis of the theorem,

$$ \deg(u) + \deg(v) \geq \frac{1}{2}(n^2 - 3n + 6) - \frac{1}{2}(n^2 - 5n + 6) = n. $$

Problem 3.114. *Determine whether a Hamiltonian path or circuit exists in the graph of Figure.*

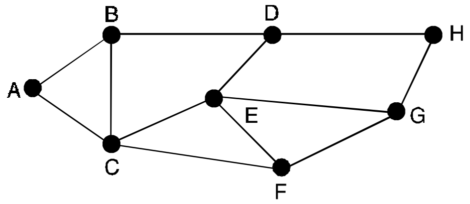


Fig. 3.146

Solution. Let us take node A to start with. Next, move to either B or C, say B. Extend the path upto B. Next move to D and not to C as a cycle of length 3 could be formed here. Extend the path upto D and drop node B and edges (B, A), (B, C) and (B, D). Then move to H. Drop D and edges from it. Then move to G, then to F (or E) then to E (or F), then to C and finally to A dropping the nodes and edges from them on the way. At the end, only one node A is left with degree zero and π is ABDHGFECA. This is a Hamiltonian cycle.

Problem 3.115. *Determine whether a Hamiltonian path or circuit exists in the graph of Figure.*

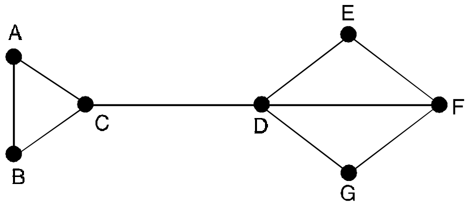


Fig. 3.147

Solution. Let us start with the node A. We can select any one but node C and D. Initialize the path π : A. Next move to the node B we cannot move to C from A. Because any move to D from B and to D from B need node C. Extend the path upto B. Then move to node C, extend the path upto C and drop node B together with edges (B, A) and (B, C). We have now the path π : ABC. Now move to D, extend the path upto D and drop node C together with arcs (C, A) and (C, D). Then move to either node G or E but not to F. Extend the path and do the rest. Finally, proceeding in this way, we get π : ABCDEFG. And two nodes A and G, with degree zero, are left. Thus, this graph has a Hamiltonian path π but no Hamiltonian circuit.

3.15 TREE

3.15.1. Acyclic graph

A graph is acyclic if it has no cycles.

3.15.2. Tree

A tree is a connected acyclic graph.

3.15.3. Forest

Any graph without cycles is a forest, thus the components of a forest are trees.

The tree with 2 points, 3 points and 4-points are shown below :

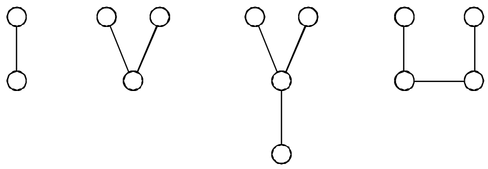


Fig. 3.148.

Note :

(1) Every edge of a tree is a bridge.

i.e., every block of G is acyclic.

Conversely, every edge of a connected graph G is a bridge, then G is a tree.

(2) Every vertex of G (tree) which is not an end vertex is neccessarily a cut-vertex.

(3) Every nontrivial tree G has at least two end vertices.

3.16 SPANNING TREE

A spanning tree is a spanning subgraph, that is a tree.

3.16.1. Branch of tree

An edge in a spanning tree T is called a branch of T.

3.16.2. Chord

An edge of G that is not in a given spanning tree is called a chord.

Note :

(1) The branches and chords are defined only with respect to a given spanning tree.

(2) An edge that is a branch of one spanning tree T_1 (in a graph G) may be chord, with respect to another spanning tree T_2.

3.17 ROOTED TREE

A rooted tree T with the vertex set V is the tree that can be defined recursively as follows :

T has a specially designated vertex $v_1 \in$ V, called the **root** of T. The subgraph of T_1 consisting of the vertices V – $\{v\}$ is partitionable into subgraphs.

T_1, T_2, ..., T_r each of which is itself a rooted tree. Each one of these r-rooted tree is called a **subtree of v_1.**

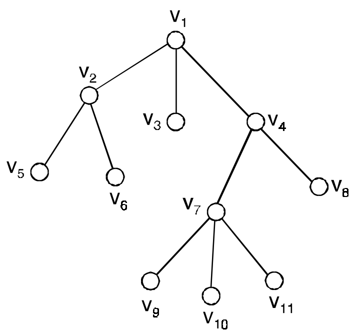


Fig. 3.149. A rooted tree.

3.17.1. Cotree

The cotree T* of a spanning tree T in a connected graph G is the spanning subgraph of G containing exactly those edges of G which are not in T. The edges of G which are not in T* are called its twigs.

For example :

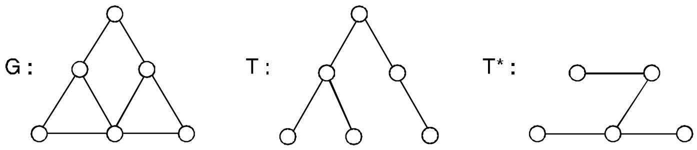


Fig. 3.150

3.18 BINARY TREES

A binary tree is a rooted tree where each vertex v has atmost two subtrees ; if both subtrees are present, one is called a left subtree of v and the other right-subtree of v. If only one subtree is present, it can be designated either as the left subtree or right subtree of v.

In other words, a binary tree is a 2-ary tree in which each child is designated as a left child or right child.

In a binary tree *e* very vertex has two children or no children.

Properties : (Binary trees) :

(1) The number of vertices n in a complete binary tree is always odd. This is because there is exactly one vertex of even degree, and remaining $n - 1$ vertices are of odd degree. Since from theorem (*i.e.,* the number of vertices of odd degree is even), $n - 1$ is even. Hence n is odd.

(2) Let P be the number of end vertices in a binary tree T. Then $n - p - 1$ is the number of vertices of degree 3. The number of edges in T is

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1 \qquad \text{or} \qquad p = \frac{n+1}{2} \qquad ...(1)$$

(3) A non end vertex in a binary tree is called an **internal vertex.** It follows from equation (1) that the number of internal vertices in a binary is one less than the number of end vertices.

(4) In a binary tree, a vertex v_i is said to be at **level l_i** if v_i is at a distance l_i from the root. Thus the root is at level O.

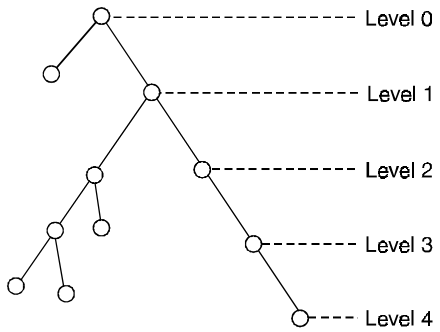


Fig. 3.151. 13-vertices, 4-level binary tree.

The maximum numbers of vertices possible in a k-level binary tree is $2^0 + 2^1 + 2^2 + ... + 2^k \geq n$,

The maximum level, l_{max} of any vertex in a binary tree is called the **height** of the tree.

On the other hand, to construct a binary tree for a given n such that the farthest vertex is as far as possible from the root, we must have exactly two vertices at each level, except at the O level.

Hence max $l_{max} = \dfrac{n-1}{2}$.

For example,

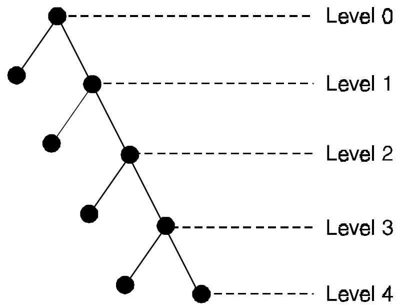


Fig. 3.152

Max $l_{max} = \frac{9-1}{2} = 4$

The minimum possible height of n-vertex binary tree is min $l_{max} = [\log_2(n + 1) - 1]$

In analysis of algorithm, we are generally interested in computing the sum of the levels of all end vertices. This quantity, known as the **path length** (or external path length) of a tree.

3.18.1. Path length of a binary tree

It can be defined as the sum of the path lengths from the root to all end vertices.

For example,

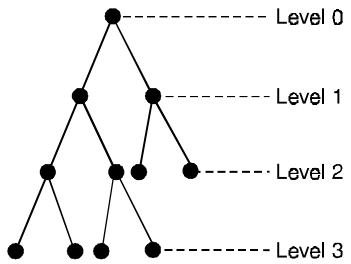


Fig. 3.153

Here the sum is 2 + 2 + 3 + 3 + 3 + 3 = 16 is the path length of a given above binary tree.

The path length of the binary tree is often directly related to the executive time of an algorithm.

3.18.2. Binary tree representation of general trees

There is a straight forward technique for converting a general tree to a binary tree form. The algorithm has two easy steps :

Step 1 :

Insert edges connecting siblings and delete all of a parents edges to its children except to its left most off spring.

Step 2 :

Rotate the resulting diagram 45° to distinguish between left and right subtrees.

For example,

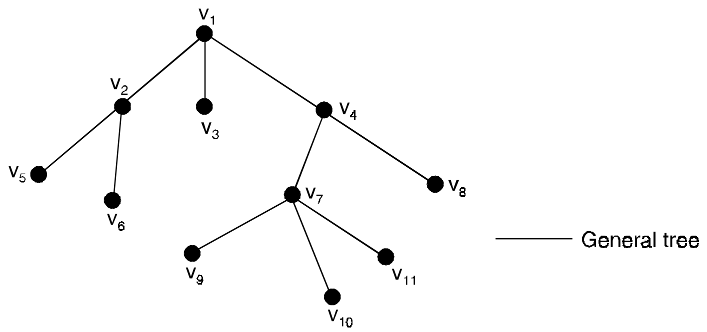


Fig. 3.154

Here v_2, v_3 and v_4 are siblings to the parent v_1, now apply the steps given above we have a binary tree as shown here.

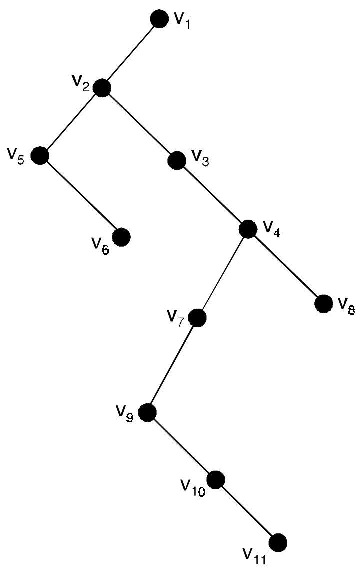


Fig. 3.155

Theorem 3.29. *A(p, q) graph is a tree if and only if it is acyclic and* $p = q + 1$ *or* $q = p - 1$.

Proof. If G is a tree, then it is acyclic.

By definition to verify the equality $p = q + 1$.

We employ induction on p.

For $p = 1$, the result is trivial.

Assume, then that the equality $p = q + 1$ holds for all (p, q) trees with $p \geq 1$ vertices.

Let G_1 be a tree with $p + 1$ vertices.

Let v be an end-vertex of G_1.

The graph $G_2 = G_1 - v$ is a tree of order p, and so $p = |E(G_2)| + 1$.

Since G_1 has one more vertex and one more edge than that of G_2.

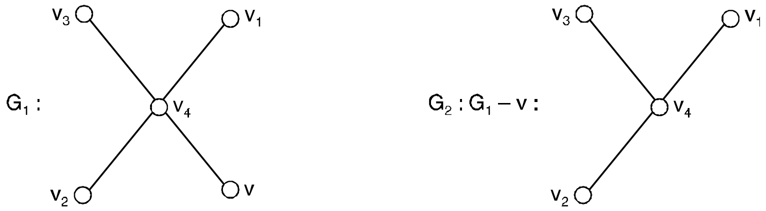


Fig. 3.156

$$|V(G_1)| = p + 1 = (|E(G_2)| + 1) + 1$$
$$= |E(G_1)| + 1$$

$\therefore \quad |V(G_1)| = |E(G_1)| + 1.$

Conversely : Let G be an acyclic (p, q) graph with $p = q + 1$.

To show G is a tree, we need only verify that G is connected. Denote by $G_1, G_2, ..., G_k$, the components of G, where $k \geq 1$.

Furthermore, let G_i be a (p_i, q_i) graph.

Since each G_i is a tree, $p_i = q_i + 1$.

Hence $$p - 1 = q = \sum_{i=1}^{k} q_i$$

$$= \sum_{i=1}^{k} (p_i - 1) = p - k$$

$\Rightarrow \quad p - 1 = p - k \quad \Rightarrow \quad k = 1$ and G is connected.

Hence, (p, q) graph is a tree.

Hence the proof.

Corollary : A forest G of vertices p has $p - k$ edges where k is the number of components.

Theorem 3.30. *A (p, q) graph G is a tree if and only if G is connected and p = q + 1.*

Proof. Let G be a (p, q) tree.

By definition of G, it is connected and by theorem : *i.e.,* A(p, q) graph is a tree if and only if it is acyclic and $p = q + 1$), $p = q + 1$.

Conversely : We assume G is connected (p, q) graph with $p = q + 1$.

It is sufficient to show that G is acyclic.

If G contains a cycle C and e is an edge of C, then $G - e$ is a connected graph with p vertices having $p - 2$ edges.

This is impossible by the definition (*i.e.,* A(p, q) graph has $q < p - 1$ then G is disconnected).

This contradicts our assumption.

Hence G is connected.

Theorem 3.31. *A complete **n-ary** tree with m internal nodes contains n × m + 1 nodes.*

Proof. Since there are m internal nodes, and each internal node has n descendents, there are $n \times m$ nodes in three other than root node.

Since there is one and only one root node in a tree, the total number of nodes in the tree will $n \times m + 1$.

Problem 3.116. *A tree has five vertices of degree 2, three vertices of degree 3 and four vertices of degree 4. How many vertices of degree 1 does it have ?*

Solution. Let x be the number of nodes of degree one.

Thus, total number of vertices

$$= 5 + 3 + 4 + x = 12 + x.$$

The total degree of the tree $= 5 \times 2 + 3 \times 3 + 4 \times 4 + x = 35 + x$

Therefore number of edges in the tree is half of the total degree of the tree.

If G = (V, E) be the tree, then, we have

$$| V | = 12 + x \text{ and } | E | = \frac{35 + x}{2}$$

In any tree, $| E | = | V | - 1$.

Therefore, we have $\dfrac{35 + x}{2} = 12 + x - 1$

$\Rightarrow \quad 35 + x = 24 + 2x - 2$

$\Rightarrow \quad x = 13$

Thus, there are 13 nodes of degree one in the tree.

Problem 3.117. *A tree has 2n vertices of degree 1, 3n vertices of degree 2 and n vertices of degree 3. Determine the number of vertices and edges in the tree.*

Solution. It is given that total number of vertices in the tree is $2n + 3n + n = 6n$.

The total degree of the tree is $2n \times 1 + 3n \times 2 + n \times 3 = 11n$.

The number of edges in the tree will be half of $11n$.

If G = (V, E) be the tree then, we have

$$| V | = 6n \quad \text{and} \quad | E | = \frac{11n}{2}$$

In any tree, $| E | = | V | - 1$.

Therefore, we have

$$\frac{11n}{2} = 6n - 1$$

$\Rightarrow \quad 11n = 12n - 2$

$\Rightarrow \quad n = 2$

Thus, there are $6 \times 2 = 12$ nodes and 11 edges in the tree.

Theorem 3.32. *There are at the most* $\boldsymbol{n^h}$ ***leaves in an n-ary tree*** *of height h.*

Proof. Let us prove this theorem by mathematical induction on the height of the tree.

As basis step take $h = 0$, *i.e.,* tree consists of root node only.

Since $n^\circ = 1$, the basis step is true.

Now let us assume that the above statement is true for $h = k$.

i.e., an n-ary tree of height k has at the most n^k leaves.

If we add n nodes to each of the leaf node of n-ary tree of height k, the total number of leaf nodes will be at the most $n^h \times n = n^{h+1}$.

Hence inductive step is also true.

This proves that above statement is true for all $h \geq 0$.

Theorem 3.33. *In a complete n-ary tree with m internal nodes, the number of leaf node l is given by the formula*

$$l = \frac{(n-1)(x-1)}{n}.$$

where, x is the total number of nodes in the tree.

Proof. It is given that the tree has m internal nodes and it is complete n–ary, so total number of nodes

$$x = n \times m + 1.$$

Thus, we have $\quad m = \dfrac{(x-1)}{n}$

It is also given that l is the number of leaf nodes in the tree.

Thus, we have $\quad x = m + l + 1$

Substituting the value of m in this equation, we get

$$x = \left(\frac{x-1}{n}\right) + l + 1$$

or
$$l = \frac{(n-1)(x-1)}{n}$$

Theorem 3.34. *If T = (V, E) be a rooted tree with v_0 as its root then*

(*i*) *T is a acyclic*

(*ii*) *v_0 is the only root in T*

(*iii*) *Each node other than root in T has in degree 1 and v_0 has indegree zero.*

Proof. We prove the theorem by the method of contradiction.

(*i*) Let there is a cycle π in T that begins and end at a node v.

Since the in degree of root is zero, $v \neq v_0$.

Also by the definition of tree, there must be a path from v_0 to v, let it be p.

Then πp is also a path, distinct from p, from v_0 to v.

This contradicts the definition of a tree that there is unique path from root to every other node.

Hence T cannot have a cycle in it.

i.e., a tree is always acyclic.

(*ii*) Let v_1 is another root in T.

By the definition of a tree, every node is reachable from root.

This v_0 is reachable from v_1 and v_1 is reachable from v_0 and the paths are π_1 and π_2 respectively.

Then $\pi_1\pi_2$ combination of these two paths is a cycle from v_0 and v_0.

Since a tree is always acyclic, v_0 and v_1 cannot be different.

Thus, v_0 is a unique root.

(*iii*) Let w be any non-root node in T.

Thus, $\exists$ a path $\pi : v_0, v_1, ..., v_kw$ from v_0 to w in T.

Now let us suppose that indegree of w is two.

Then $\exists$ two nodes w_1 and w_2 in T such that edges (w_1, v_0) and (w_2, v_0) are in E.

Let π_1 and π_2 be paths from v_0 to w_1 and w_2 respectively.

Then $\pi_1 : v_0v_1 ...... v_kw_1w$ and $\pi_2 : v_0v_1 ...... v_kw_2w$ are two possible paths from v_0 to w.

This is in contradiction with the fact that there is unique path from root to every other nodes in a tree.

Thus indegree of w cannot be greater than 1.

Next, let indegree of $v_0 > 0$. Then $\exists$ a node v in T such that $(v, v_0) \in$ E.

Let π be a path from v_0 to v, thus $\pi(v, v_0)$ is a path from v_0 to v_0 that is a cycle.

This is again a contradiction with the fact that any tree is acyclic.

Thus indegree of root node v_0 cannot be greater than zero.

Problem 3.118. *Let T = (V, E) be a rooted tree. Obviously E is a relation on set V. Show that*

(*i*) *E is irreflexive*

(*ii*) *E is asymmetric*

(*iii*) *If $(a, b) \in E$ and $(b, c) \in E$ then $(a, c) \notin E, \forall a, b, c \in V$.*

Solution. Since a tree is acyclic, there is no cycle of any length in a tree.

This implies that there is no loop in T.

Thus, $(v, v) \notin$ E $\forall a \in$ V.

Thus E is an irreflexive relation on V.

Let $(x, y) \in$ E. If $(y, x) \in$ E, then there will be cycle at node x as well as on node y.

Since no cycle is permissible in a tree, either pair (x, y) or (y, x) can be in E but never both.

This implies that presence of (x, y) excludes the presence of (y, x) in E and *vice versa*.

Thus E is a asymmetric relation on V.

Let $(a, c) \in$ E.

Thus presence of pairs (b, c) and (a, c) in E implies that c has indegree > 1.

Hence $(a, c) \notin$ E.

Problem 3.119. *Prove that a tree T is always separable.*

Solution. Let w be any internal node in T and node v is the parent of w.

By the definition of a tree, in degree of w is one.

If w is dropped from the tree T, the incoming edge from v to w is also removed.

Therefore all children of w will be unreachable from root and tree T will become disconnected.

See the forest of the Figure (3.157), which has been obtained after removal of node F from the tree of Figure (3.158).

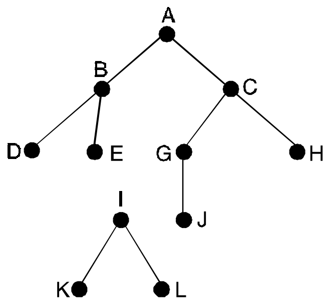


Fig. 3.157

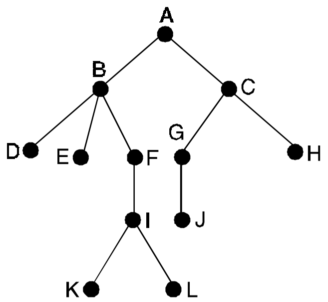


Fig. 3.158

Problem 3.120. *Let A = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and let*

T = $\{(v_2, v_3), (v_2, v_1), (v_4, v_5), (v_4, v_6), (v_5, v_8), (v_6, v_7), (v_4, v_2), (v_7, v_9), (v_7, v_{10})\}$.

Show that T is a rooted tree and identify the root.

Solution. Since no paths begin at vertices v_1, v_3, v_8, v_9 and v_{10}, these vertices cannot be roots of a tree.

There are no paths from vertices v_6, v_7, v_2 and v_5 to vertex v_4, so we must eliminate these vertices as possible roots.

Thus, if T is a rooted tree, its root must be vertex v_4.

It is easy to show that there is a path from v_4 to every other vertex.

For example, the path v_4, v_6, v_7, v_9 leads from v_4 and v_9, since (v_4, v_6), (v_6, v_7) and (v_7, v_9) are all in T.

We draw the digraph of T, beginning with vertex v_4, and with edges shown downward.

The result is shown in Fig. (3.159). A quick inspection of this digraph shows that paths from vertex v_4 to every other vertex are unique, and there are no paths from v_4 and v_4.

Thus T is a tree with root v_4.

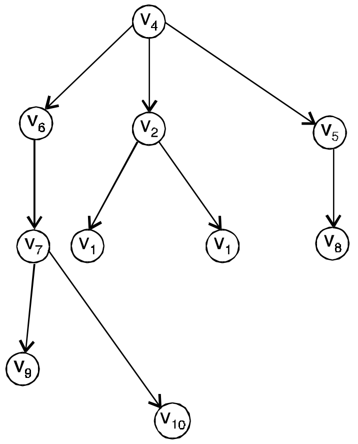


Fig. 3.159

Theorem 3.35. *There is one and only one path between every pair of vertices in a tree T.*

Proof. Since T is a connected graph, there must exist atleast one path between every pair of vertices in T.

Let there are two distinct paths between two vertices u and v of T.

But union of these two paths will contain a cycle and then T cannot be a tree.

Theorem 3.36. *If in a graph G there is one and only one path between every pair of vertices, G is a tree.*

Proof. Since there exists a path between every pair of vertices then G is connected.

A cycle in a graph (with two or more vertices) implies that there is atleast one pair of vertices u, v such that there are two distinct paths between u and v.

Since G has one and only one path between every pair of vertices, G can have no cycle.

Therefore, G is a tree.

Theorem 3.37. *A tree T with n vertices has n – 1 edges.*

Proof. The theorem is proved by induction on n, the number of vertices of T.

Basis of Inductive : When $n = 1$ then T has only one vertex. Since it has no cycles, T can not have any edge.

i.e., it has $e = 0 = n - 1$

Induction step : Suppose the theorem is true for $n = k \geq 2$ where k is some positive integer.

We use this to show that the result is true for $n = k + 1$.

Let T be a tree with $k + 1$ vertices and let uv be edge of T. Let uv be an edge of T. Then if we remove the edge uv from T we obtain the graph T – uv. Then the graph is disconnected since T – uv contains no (u, v) path.

If there were a path, say u, v_1, v_2 ...... v from u to v then when we added back the edge uv there would be a cycle $u, v_1, v_2, ... v, u$ in T.

Thus, T – uv is disconnected. The removal of an edge from a graph can disconnected the graph into at most two components. So T – uv has two components, say, T_1 and T_2.

Since there were no cycles in T to begin with, both components are connected and are without cycles.

Thus, T_1 and T_2 are trees and each has fewer than n vertices.

This means that we can apply the induction hypothesis to T_1 and T_2 to give

$$e(T_1) = v(T_1) - 1$$
$$e(T_2) = v(T_2) - 1$$

But the construction of T_1 and T_2 by removal of a single edge from T gives that

$$e(T) = e(T_1) + e(T_2) + 1$$

and that $v(T) = v(T_1) + v(T_2)$

it follows that

$$e(T) = v(T_1) - 1 + v(T_2) - 1 + 1$$
$$= v(T) - 1$$
$$= k + 1 - 1 = k.$$

Thus T has k edges, as required.

Hence by principle of mathematical induction the theorem is proved.

Theorem 3.38. *For any positive integer n, if G is a connected graph with n vertices and n – 1 edges, then G is a tree.*

Proof. Let n be a positive integer and suppose G is a particular but arbitrarily chosen graph that is connected and has n vertices and $n - 1$ edges.

We know that a tree is a connected graph without cycles. (We have proved in previous theorem that a tree has $n - 1$ edges).

We have to prove the converse that if G has no cycles and $n - 1$ edges, then G is connected.

We decompose G into k components, c_1, c_2, ... c_k.

Each component is connected and it has no cycles since G has no cycles.

Hence, each C_k is a tree.

Now $e_1 = n_1 - 1$ and $\sum_{i=1}^{k} e_i = \sum_{i=1}^{k} (n_i - 1) = n - k$

$\Rightarrow \quad e = n - k$

Then it follows that $k = 1$ or G has only one component.

Hence G is a tree.

Problem 3.121. *Consider the rooted tree in Figure (3.160).*

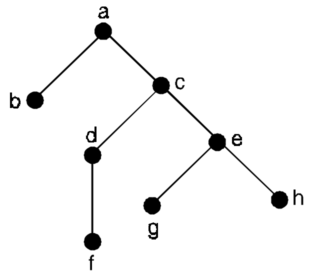


Fig. 3.160

(*a*) *What is the root of T ?*

(*b*) *Find the leaves and the internal vertices of T.*

(*c*) *What are the levels of c and e.*

(*d*) *Find the children of c and e.*

(*e*) *Find the descendants of the vertices a and c.*

Solution. (*a*) Vertex *a* is distinguished as the only vertex located at the top of the tree.

Therefore *a* is the root.

(*b*) The leaves are those vertices that have no children. These *b*, *f*, *g* and *h*. The internal vertices are *c*, *d* and *e*.

(*c*) The levels of *c* and *e* are 1 and 2 respectively.

(*d*) The children of *c* are *d* and *e* and of *e* are *g* and *h*.

(*e*) The descendants of *a* are *b*, *c*, *d*, *e*, *f*, *g*, *h*.

The descendants of *c* are *d*, *e*, *f*, *g*, *h*.

Theorem 3.39. *A full m-ary tree with i internal vertex has n = mi + 1 vertices.*

Proof. Since the tree is a full *m*-ary, each internal vertex has *m* children and the number of internal vertex is *i*, the total number of vertex except the root is *mi*.

Therefore, the tree has $n = mi + 1$ vertices.

Since 1 is the number of leaves, we have $n = l + i$ using the two equalities $n = mi + 1$ and $n = 1 + i$, the following results can easily be deduced.

A full *m*-ary tree with

(*i*) *n* vertices has $i = \dfrac{(n-1)}{m}$ internal vertices and $l = \dfrac{[(m-1)(n+1)]}{m}$ leaves.

(*ii*) *i* internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves.

(*iii*) *l* leaves has $n = \dfrac{(ml-1)}{(m-1)}$ vertices and $i = \dfrac{(l-1)}{(m-1)}$ internal vertices.

Theorem 3.40. *There are at most m^h leaves in an m-ary tree of height h.*

Proof. We prove the theorem by mathematical induction.

Basis of Induction :

For $h = 1$, the tree consists of a root with no more than *m* children, each of which is a leaf.

Hence there are no more than $m^1 = m$ leaves in an *m*-ary of height 1.

Induction hypothesis :

We assume that the result is true for all *m*-ary trees of heights less than *h*.

Induction step :

Let T be an *m*-ary tree of height *h*. The leaves of T are the leaves of subtrees of T obtained by deleting the edges from the roots to each of the vertices of level 1.

Each of these subtrees has at most m^{h-1} leaves. Since there are at most *m* such subtrees, each with a maximum of m^{h-1} leaves, there are at most $m \cdot m^{h-1} = m^h$.

Problem 3.122. *Find all spanning trees of the graph G shown in Figure 3.161.*

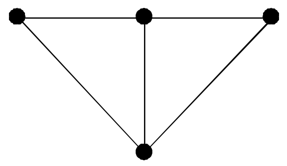


Fig. 3.161

Solution. The graph G has four vertices and hence each spanning tree must have 4 – 1 = 3 edges.
Thus each tree can be obtained by deleting two of the five edges of G.
This can be done in 10 ways, except that two of the ways lead to disconnected graphs.
Thus there are eight spanning trees as shown in Figure (3.162).

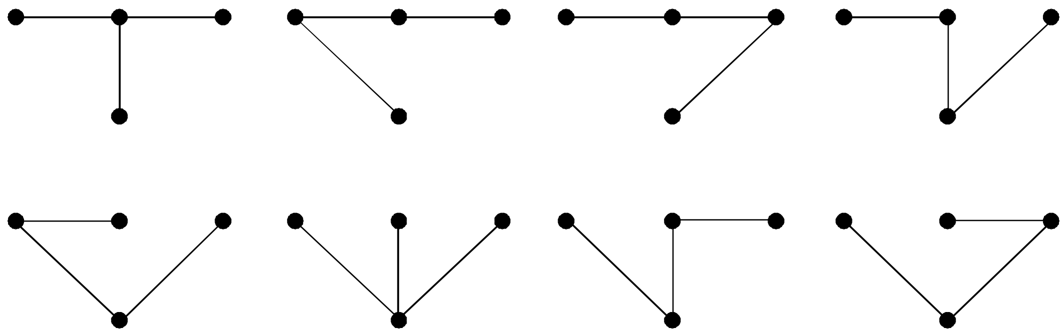


Fig. 3.162

Problem 3.123. *Find all spanning trees for the graph G shown in Figure 3.163, by removing the edges in simple circuits.*

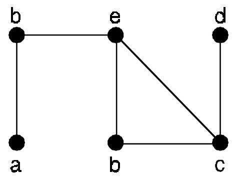


Fig. 3.163

Solution. The graph G has one cycle *cbec* and removal of any edge of the cycle gives a tree.
There are three trees which contain all the vertices of G and hence spanning trees.

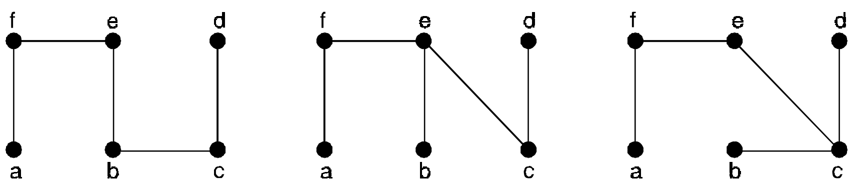


Fig. 3.164

Theorem 3.41. *A simple graph G has a spanning tree if and only if G is connected.*

Proof. First, suppose that a simple graph G has a spanning tree T. T contains every vertex of G. Let a and b be vertices of G. Since a and b are also vertices of T and T is a tree, there is a path P between a and b.

Since T is subgraph, P also serves as path between a and b in G.

Hence G is connected.

Conversely, suppose that G is connected.

If G is not a tree, it must contain a simple circuit. Remove an edge from one of these simple circuits. The resulting subgraph has one fewer edge but still contains all the vertices of G and is connected.

If this subgraph is not a tree, it has a simple circuit, so as before, remove an edge that is in a simple circuit.

Repeat this process until no simple circuit remain.

This is possible because there are only a finite number of edges in the graph, the process terminates when no simple circuits remain.

Thus we eventually produce an acyclic subgraph T which is a tree.

The tree is a spanning tree since it contains every vertex of G.

Theorem 3.42. *There is one and only path between every pair of vertices in a tree.*

(OR)

A graph G is a tree if and only if every two distinct vertices of G are joined by a unique path of G.

Proof. Since T is a connected graph, there must exist atleast one path between pair of vertices in T.

Now suppose that between two vertices a and b of T there are two distinct paths.

The union of these two paths will contain a cycle, and T cannot be a tree.

Conversely, suppose in a graph G there is one and only one path between every pair of vertices, then G is a tree.

If there exists a path between every pair of vertices, then G is connected.

A cycle in a graph implies that there is atleast one pair of vertices a and b such that there are two distinct paths between a and b.

Since G has one and only one path between every pair of vertices, G can have no cycle.

Therefore, G is a tree.

Theorem 3.43. *Every non trivial tree contains atleast two end vertices.*

Proof. Suppose that T is a tree with p-vertices and q-edges and let d_1, d_2, ... d_p denotes the degrees of its vertices, ordered so that $d_1 \le d_2 \le .... \le d_p$.

Since T is connected and non trivial, $d_i \ge 1$ for each $i(1 \le i \le p)$.

If T does not contain two end vertices, then $d_i \ge 1$ and $d_i \ge 2$ for $2 \le i \le p$,

$$\text{So} \quad \sum_{i=1}^{p} d_i \ge 1 + 2\,(p-1) = 2p - 1 \qquad ...(1)$$

However, from the results *i.e.,* $\sum_{i=1}^{p} \deg v_i = 2q$ and a tree with p-vertices has $p - 1$ edges.

$$\sum_{i=1}^{p} d_i = 2q = 2(p-1) = 2p-2$$ which contradicts in equality (1).

Hence T contains atleast two end vertices.

Theorem 3.44. *If G is a tree and if any two non adjacent vertices of G are joined by an edge e, then G + e has exactly one cycle.*

Proof. Suppose G is a tree. Then there is exactly one path joining any two vertices of G.

If we add an edge of G, that edge together with unique path joining u and v forms a cycle.

Theorem 3.45. *A graph G is connected if and only if it contains a spanning tree.*

Proof. It is immediate that, if a graph contains a spanning tree, then it must be connected.

Conversely, if a connected graph does not contain any cycle then it is a tree.

For a connected graph containing one or more cycles, we can remove an edge from one of the cycles and still have a connected subgraph. Such removal of edges from cycles can be repeated until we have a spanning tree.

Theorem 3.46. *If u and v are distinct vertices of a tree T contains exactly one u – v path.*

Proof. Suppose, to the contrary that T contains two $u - v$ paths say P and Q are different $u - v$, paths there must be a vertex x (*i.e.*, $x = u$) belonging to both P and Q such that the vertex immediately following x on Q. See Figure 3.165.

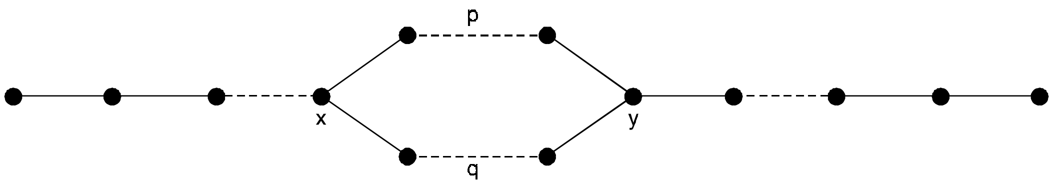


Fig. 3.165

Let y be the first vertex of P following x that also belongs to Q (y could be v).

Then this produces to $x - y$ paths that have only x and y in common.

These two paths produces a cycle in T, which contradicts the fact that T is a tree.

Therefore, T has only one $u - v$ path.

Problem 3.124. *Construct two non-isomorphic trees having exactly 4 pendant vertices on 6 vertices.*

Solution.

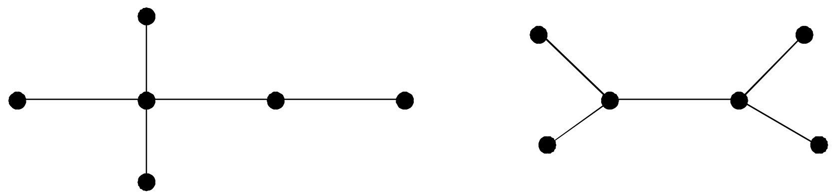


Fig. 3.166

Problem 3.125. *Construct three distinct trees with exactly*

(*i*) *one central vertex* (*ii*) *two central vertices.*

Solution. (*i*) The following trees contain only one central vertex.

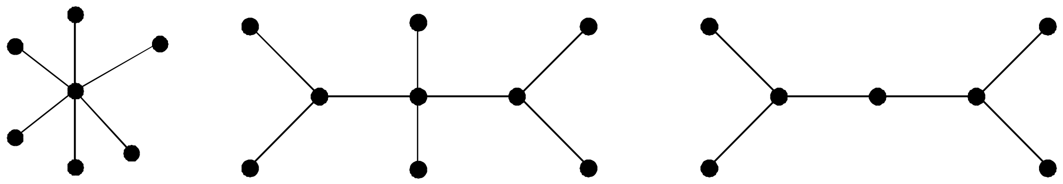


Fig. 3.167.

(*ii*) The following trees contain exactly two central vertices.

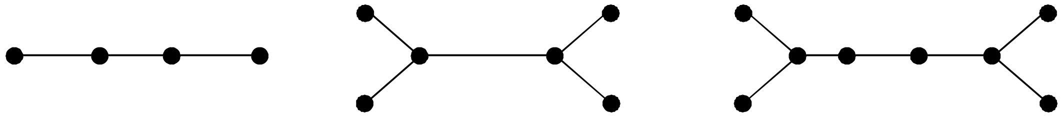


Fig. 3.168

Problem 3.126. *Count the number of vertices of degree three in a binary tree on n vertices having k number of pendant vertices.*

Solution. Since the binary tree contains k number of pendant vertices and one vertex of degree two, we have total number of remaining vertices which are of degree three is $n - k - 1$.

Problem 3.127. *Let T be a tree with 50 edges. The removal of certain edge from T yields two disjoint trees T_1 and T_2. Given that the number of vertices in T_1 equals the number of edges in T_2, determine the number of vertices and the number of edges in T_1 and T_2.*

Solution. We have removal of an edge from a graph will not remove any vertex from the graph.

Thus $|V(T_1)| + |V(T_2)| = |V(T)|$

Since T_1 and T_2 are trees and number of vertices of T_1 is equal to the number of edges in T_2, we get

$$\begin{aligned} |V(T)| &= |V(T_1)| + |V(T_2)| \\ &= (|V(T_2)| - 1) + |V(T_2)| \\ &= 2|V(T_2)| - 1 \end{aligned}$$

but $|V(T)| = |E(T) + 1| = 50 + 1 = 51$

Hence $2|V(T_2)| - 1 = 51$

$\Rightarrow \quad |V(T_2)| = 26$ and $|V(T_1)| = 25$

Therefore, there are 26 vertices and hence 25 edges in T_2 and there are 25 vertices hence 24 edges in T_1.

Thus $50 - (25 + 24) = 1$ edge is removed from the tree T.

Problem 3.128. *What is the maximum number of end vertices a tree on n vertices may have ?*

Solution. The graph $K_{1,\,n}$ contains maximum number of end vertices.

Thus a tree on n vertices may contain a maximum of $n - 1$ end vertices.

Problem 3.129. *Prove that a pendant edge in a connected graph G is contained in every spanning tree of G.*

Solution. By a pendant edge, we mean an edge whose one end vertex is a pendant vertex.

Let e be a pendant edge of a connected graph G and let v be the corresponding pendant vertex.

Then e is the only edge that is incident on v.

Suppose there is a spanning tree of T for which e is not a branch.

Then, T cannot contain the vertex v.

This is not possible, because T must contain every vertex of G.

Hence there is no spanning tree of G for which e is not a branch.

Problem 3.130. *Show that a Hamiltonian path is a spanning tree.*

Solution. Recall that a Hamiltonian path P in a connected graph G, if there is a path which contains every vertex of G and that if G has n vertices then P has $n - 1$ edges.

Thus, P is a connected subgraph of G with n vertices and $n - 1$ edges.

Therefore, P is a tree. Since P contains all vertices of G, it is a spanning tree of G.

Problem 3.131. *Prove that the number of branches of a spanning tree T of a connected graph G is equal to the rank of G and the number of the corresponding chords is equal to the nullity of G.*

Solution. Let n be the number of vertices and m be the number of edges in a connected graph G. Then

Rank of G = $\rho(G) = n - 1$

= no. of branches of a spanning tree T of G.

Nullity of G = $\mu(G) = m - (n - 1)$

= no. of chords relative to T.

Problem 3.132. *Prove that every circuit in a graph G must have atleast one edge in common with a chord set.*

Solution. Recall that a chord set is the complement of a spanning tree.

If there is a circuit that has no common edge with this set, the circuit must be containined in a spanning tree.

This is impossible, because a tree does not contain a circuit.

Problem 3.133. *Let G be a graph with k components, where each component is a tree. If n is the number of vertices and m is the number of edges in G, prove that n = m + k.*

Solution. Let $H_1, H_2, ... H_k$ be the components of G.

Since each of these is a tree, if n_i is the number of vertices in H_i and m_i is the number of edges in H_i

We have $m_i = n_i - 1, \quad i = 1, 2, ... k$

this gives $m_1 + m_2 + ... + m_k = (n_1 - 1) + (n_2 - 1) + ... (n_k - 1)$

$= n_1 + n_2 + ... + n_k - k$

But $m_1 + m_2 + ... + m_k = m$ and

$n_1 + n_2 + ... + n_k = n$

Therefore $m = n - k$

$\Rightarrow \quad n = m + k.$

Problem 3.134. *Show that, in a tree ; if the degree of every non-pendant vertex is 3, the number of vertices in the tree is even.*

Solution. Let n be the number of vertices in a tree T.

Let k be the number of pendant vertices.

Then, if each non-pendant vertex is of degree 3, the sum of the degrees of vertices is $k + 3(n - k)$.

This must be equal to $2(n - 1)$

Thus, $k + 3(n - k) = 2(n - 1)$

$\Rightarrow \quad n = 2(k - 1)$

Therefore, n is even.

Problem 3.135. *Suppose that a tree T has N_1 vertices of degree 1, N_2 vertices of degree 2, N_3 vertices of degree 3, ... N_k vertices of degree k. Prove that*

$$N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (K - 2)\, N_k.$$

Solution. Note that a tree T,

The total number of vertices = $N_1 + N_2 + ... + N_k$

Sum of the degrees of vertices = $N_1 + 2N_2 + 3N_3 + ... kN_k$

Therefore, the total number of edges in T is

$$N_1 + N_2 + ... + N_k - 1, \text{ and}$$

the handshaking property, gives

$$N_1 + 2N_2 + 3N_3 + 4N_4 + 5N_5 + ... kN_k$$
$$= 2(N_1 + N_2 + ... + N_k - 1)$$

Rearranging terms, which gives

$$N_3 + 2N_4 + 3N_5 + ... + (k - 2)\, N_k = N_1 - 2$$

$\Rightarrow \quad N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (k - 2)\, N_k.$

Problem 3.136. *Show that if a tree has exactly two pendant vertices, the degree of every other vertex is two.*

Solution. Let n be the number of vertices in a tree T.

Suppose, it has exactly two pendant vertices, and let $d_1, d_2, ... d_{n-2}$ be the degrees of the other vertices.

Then, since T has exactly $n - 1$ edges.

We have $\quad 1 + 1 + d_1 + d_2 + ... + d_{n-2} = 2(n - 1)$

$\Rightarrow \quad d_1 + d_2 + ... + d_{n-2} = 2n - 4 = 2(n - 2)$

The left hand side of the above condition has $n - 2$ terms d's, and none of these is one or zero.

Therefore, this condition holds only if each of the $d_i s$ is equal to two.

Problem 3.137. *Show that the complete graph K_n is not a tree, when $n > 2$.*

Solution. If v_1, v_2, v_3 are any three vertices of K_n, $n > 2$ then the closed walk $v_1v_2v_3v_1$ is a circuit in K_n.

Since K_n has a circuit, it cannot be a tree.

Problem 3.138. *Suppose that a tree T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4. Find the number of pendant vertices in T.*

Solution. Let N be the number of pendant vertices in T.

It is given that T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4.

Therefore, the total number of vertices

$$= N + 2 + 4 + 3$$
$$= N + 9.$$

Sum of the degrees of vertices $= N + (2 \times 2) + (4 \times 3) + (3 \times 4)$

$$= N + 28.$$

Since T has N + 9 vertices, it has N + 9 – 1 = N + 8 edges.

Therefore, by handshaking property, we have

$$N + 28 = 2(N + 8)$$

$\Rightarrow$ $N = 12$

Thus, the given tree has 12 pendant vertices.

Problem 3.139. *Show that the complete bipartite graph $K_{r,\,s}$ is not a tree if $r \geq 2$.*

Solution. Let v_1 and v_2 be any two vertices in the first partition and v_1', v_2' be any two vertices in the second partition of $K_{r,\,s}$ $s \geq r > 1$.

Then the closed walk $v_1v_1'v_2v_2'v_1$ is a circuit in $K_{r,\,s}$.

Since $K_{r,\,s}$ has a circuit, it cannot be a tree.

Problem 3.140. *Prove that, in a tree with two or more vertices, there are atleast two leaves (pendant vertices).*

Solution. Consider a tree T with n vertices, $n \geq 2$. Then, it has $n - 1$ edges.

Therefore, the sum of the degrees of the n vertices must be equal to $2(n - 1)$.

Thus, if $d_1, d_2, ... d_n$ are the degrees of vertices.

We have $d_1 + d_2 + ... + d_n = 2(n - 1) = 2n - 2$.

If each of $d_1, d_2, ... d_n$ is ≥ 2, then their sum must be at least $2n$.

Since this is not true, atleast one of the d's is less than 2.

Thus, there is a d which is equal to 1.

Without loss of generality, let us take this to be d_1. Then

$$d_2 + d_3 + ... + d_n = (2n - 2) - 1 = 2n - 3.$$

This is possible only if atleast one of $d_2, d_3 ...... d_n$ is equal to 1.

So, there is atleast one more d which is equal to 1.

Thus, there are atleast two vertices with degree 1.

Problem 3.141. *Prove that a graph with n vertices, n – 1 edges, and no circuits is connected.*

Solution. Consider a graph G which has n vertices, $n - 1$ edges and no circuits.

Suppose G is not connected.

Let the components of G be H_i, $i = 1, 2, ... k$.

If H_i has n_i vertices, we have

$$n_1 + n_2 + ... + n_k = n.$$

Since G has no circuits, $H_i s$ is also do not have circuits.

Further, they are all connected graphs.

Therefore, they are trees.

Consequently, each H_i must have $n_i - 1$ edges.

Therefore, the total number of edges in these $H_i s$ is $(n_1 - 1) + (n_2 - 1) + ... (n_k - 1) = n - k$.

This must be equal to the total number of edges in G, that is $n - k = n - 1$.

This is not possible, since $k > 1$.

Therefore, G must be connected.

Problem 3.142. *Construct three distinct binary trees on 11 vertices.*

Solution. (*i*) (*ii*) (*iii*)

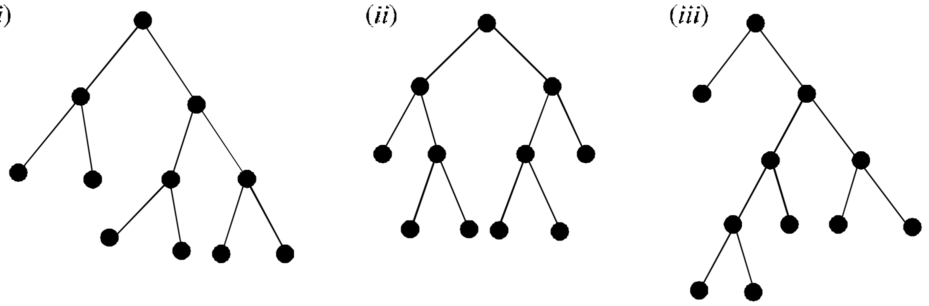


Fig. 3.169

Problem 3.143. *What is the minimum possible height of a binary tree on 2n – 1 (n ≥ 1) vertices ?*

Solution. Let k be the minimum height of a binary tree on $2n - 1$ vertices.

For minimum height we have to keep maximum number of vertices in the previous level before placing any vertex in the next level.

Thus, k should satisfy the inequality

$$2n - 1 \leq 2^0 + 2^1 + 2^2 + ... + 2^k$$

$$= \frac{1(1 - 2^{k+1})}{1 - 2}$$

Since right hand side is a G.P. series with first term is 1 and common ratio having $k + 1$ terms.

i.e., $2n - 1 \leq 2^{k+1} - 1 \quad \Rightarrow \quad 2n \leq 2^{k+1}$

$$\Rightarrow \quad n \leq 2^k.$$

Now taking natural log on both sides we get

$$\log_2 n \leq k \quad \Rightarrow \quad k \geq \log_2 n.$$

Since k is an integer, this implies that the minimum value of $k = [\log_2 n]$.

Problem 3.144. *What is the maximum possible number of vertices in any k-level tree ?*

Solution. The level of a root is zero and it is the only one vertex at level zero.

There are two vertices that are adjacent to the root, at which are at levels one.

From these vertices we can find maximum four vertices at level 2 so on ...... to get a minimum heighten tree we have to keep the vertex at higher level only after filling all the vertices in its lower level.

Trees maximum number of vertices possible for such a k-level tree is therefore

$$n \leq 2^0 + 2^1 + 2^2 + \ldots\ldots 2^k = \frac{1(1-2^{k+1})}{1-2} = 2^{k+1} - 1.$$

Problem 3.145. *What is the maximum possible level (height) of a binary tree on 2n + 1 (n ≥ 0) vertices.*

Solution. Let k be the height of a binary tree on $2n + 1$ vertices.

To get a vertex in maximum level we must keep exactly two (minimum) vertices in each level except the root vertex.

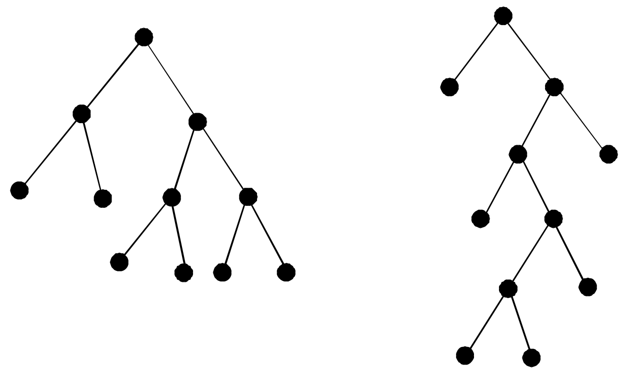
That is out of $2n + 1$ vertices one is a root and the remaining $2n$ vertices can keep in exactly n levels.

Thus the maximum height of a tree is n.

Hence maximum possible value of k is n.

Problem 3.146. *Sketch two different binary trees on 11 vertices with one having maximum height and the other with minimum height.*

Solution. Required binary trees on 11 vertices are



(*i*) **With minimum height 3** (*ii*) **With maximum height 5.**

Fig. 3.170

Problem 3.147. *Show that the number of vertices in a binary tree is always odd.*

Solution. Consider a binary tree on n vertices. Since it contains exactly one vertex of degree two and other vertices are of degree one or three, it follows that there are $n - 1$ odd degree vertices in the graph.

But if the number of odd degree vertices of a graph is even, it follows that $n - 1$ is even and hence n is odd.

Problem 3.148. *In any binary tree T on n vertices, show that the number of pendant vertices (edges) is equal to* $\frac{(n+1)}{2}$.

Solution. Let the number of pendant edges in a binary tree on n vertices be k.

Then we have there are $n - k - 1$ vertices of degree three, one vertex of degree two, k vertices of degree one and $n - 1$ edges.

Therefore, sum of degrees of vertices = 2 × number of edges.

$$(n - k - 1) \times 3 + 2 + k \times 1 = 2(n - 1)$$

$$\Rightarrow \quad 3n - 3k - 3 + 2 + k = 2n - 2$$

$$\Rightarrow \quad 2k = 3n - 2n + 1 = n + 1$$

$$\Rightarrow \quad k = \frac{(n+1)}{2}.$$

Problem 3.149. *Draw a tree with 6 vertices, exactly 3 of which have degree 1.*

Solution. A tree with 6 vertices which contains 3 pendant vertices is given in Figure (3.171).

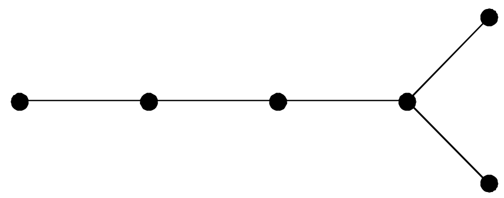


Fig. 3.171

Problem 3.150. *Which trees are complete bipartite graphs ?*

Solution. Suppose T is a tree which is a complete bipartite graph.

Let T = $K_{m, n}$ then the number of vertices in T is $(m + n)$.

Hence the tree T contains $(m + n - 1)$ number of edges.

But the graph $K_{m, n}$ has (m, n) number of edges.

Therefore $\quad m + n - 1 = mn$

$$\Rightarrow \quad mn - m - n + 1 = 0$$

$$\Rightarrow \quad m(n - 1) - 1(n - 1) = 0$$

$$\Rightarrow \quad (m - 1)(n - 1) = 0$$

$$\Rightarrow \quad m = 1 \text{ or } n = 1$$

This means T is either $K_{1, n}$ or $K_{m, 1}$ that is T is a star.

Problem 3.151. *Draw all non-isomorphic trees with 6 vertices.*

Solution. All non isomorphic trees with 6 vertices are shown below :

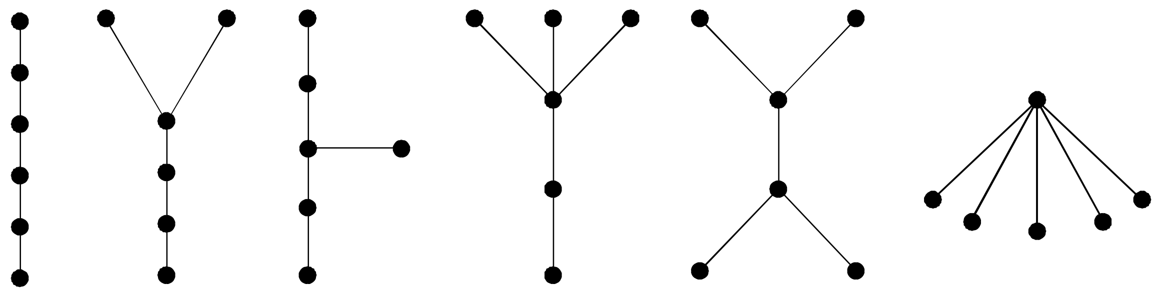


Fig. 3.172

Problem 3.152. *Is it possible to draw a tree with five vertices having degrees 1, 1, 2, 2, 4.*

Solution. Since the tree has 5 vertices hence it has 4 edges.

Now given the vertices of tree are having degrees

1, 1, 2, 2, 4.

i.e., the sum of the degrees of the tree = 10

By handshaking lemma, $2q = \sum_{i=1}^{5} d(v_i)$

Where q is the number of edges in the graph

$$2q = 10 \quad \Rightarrow \quad q = 5$$

Which is contradiction to the statement that the tree has 4 edges with 5 vertices.

Hence the tree with given degrees of vertices does not exist.

3.19. COUNTING TREES

The subject of graph enumeration is concerned with the problem of finding out how many non-isomorphic graphs possess a given property. The subject was initiated in the 1850's by Arthur Cayley, who later applied it to the problem of enumerating alkanes $C_n H_{2n+2}$ with a given number of carbon atoms. This problem is that of counting the number of trees in which the degree of each vertex is either 4 or 1. Many standard problems of graph enumeration have been solved.

For example, it is possible to calculate the number of graphs, connected graphs, trees and Eulerian graphs with a given number of vertices and edges, corresponding general results for planar graphs and Hamiltonian graphs have, however, not yet been obtained. Most of the known results can be obtained by using a fundamental enumeration theorem due to Polya, a good account of which may be found in Harary and Palmer.

Unfortunately, in almost every case it is impossible to express these results by means of simple formulas.

Consider Fig. (3.173), which shows three ways of labelling a tree with four vertices. Since the second labelled tree is the reverse of the first one, these two labelled trees are the same. On the other hand, neither is isomorphic to the third labelled tree, as you can see by comparing the degrees of vertex 3.

Thus, the reverse of any labelling does not result in a new one, and so the number of ways of labelling this tree is $\frac{(4!)}{2} = 12$.

Similarly, the number of ways of labelling the tree in Fig. (3.28) is 4, since the central vertex can be labelled in four different ways, and each one determines the labelling.

Thus, the total number of non-isomorphic labelled trees on four vertices is 12 + 4 = 16.

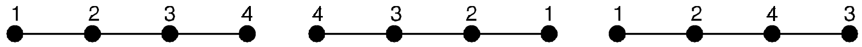


Fig. 3.173

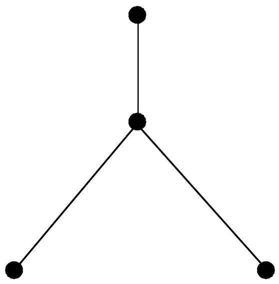


Fig. 3.174

Theorem 3.47. *Let T be a graph with n vertices. Then the following statements are equivalent :*

(*i*) *T is a tree*

(*ii*) *T contains no cycles, and has n – 1 edges*

(*iii*) *T is connected and has n – 1 edges*

(*iv*) *T is connected and each edge is a bridge*

(*v*) *Any two vertices of T are connected by exactly one path*

(*vi*) *T contains no cycles, but the addition of any new edge creates exactly one cycle.*

Proof. If $n = 1$, all six results are trivial, we therefore assume that $n \geq 2$.

(*i*) ⇒ (*iii*)

Since T contains no cycles, the removal of any edge must disconnect T into two graphs, each of which is a tree.

It follows by induction that the number of edges in each of these two trees is one fewer than the number of vertices. We deduce that the total number of edges of T is $n - 1$.

(*ii*) ⇒ (*iii*)

If T is disconnected, then each component of T is a connected graph with no cycles and hence, by the previous part, the number of vertices in each component exceeds the number of edges by 1.

It follows that the total number of vertices of T exceeds the total number of edges by atleast 2, contradicting the fact that T has $n - 1$ edges.

$(iii) \Rightarrow (iv)$

The removal of any edge results in a graph with n vertices and $n - 2$ edges, which must be disconnected.

$(iv) \Rightarrow (v)$

Since T is connected, each pair of vertices is connected by atleast one path.

If a given pair of vertices is connected by two paths, then they enclose a cycle, contradicting the fact that each edge is a bridge.

$(v) \Rightarrow (vi)$

If T contained a cycle, then any two vertices in the cycle would be connected by atleast two paths, contradicting statement (v).

If an edge e is added to T, then, since the vertices incident with e are already connected in T, a cycle is created.

The fact that only one cycle is obtained.

$(vi) \Rightarrow (i)$

Suppose that T is disconnected.

If we add to T any edge joining a vertex of one component to a vertex in another, then no cycle is created.

Corollary :

If G is a forest with n vertices and k components, then G has $n - k$ edges.

Theorem 3.48. *If T is any spanning forest of a graph G, then*

(i) *each cutset of G has an edge in common with T*

(ii) *each cycle of G has an edge in common with the complement of T.*

Proof. (i) Let C* be a cutset of G, the removal of which splits a component of G into two subgraphs H and K.

Since T is a spanning forest, T must contain an edge joining a vertex of H to a vertex of K, and this edge is the required edges.

(ii) Let C be a cycle of G having no edge is common with the complement of T.

Then C must be contained in T, which is a contradiction.

3.19.1. Cayley theorem (3.49)

There are n^{n-2} distinct labelled trees on n vertices.

Remark. The following proofs are due to Prüfer and Clarke.

Proof. First proof :

We establish a one-one correspondence between the set of labelled trees of order n and set of sequences $(a_1, a_2, ... a_{n-2})$, where each a_i is an integer satisfying $1 \le a_i \le n$.

Since there are precisely n^{n-2} such sequence, the result follows immediately.

We assume that $n \ge 3$, since the result is trivial if $n = 1$ or 2.

In order to establish the required correspondence, we first let T be a labelled tree of order n, and show how the sequence can be determined.

If b_1 is the smallest label assigned to an end-vertex, we let a_1 be the label of the vertex adjacent to the vertex b_1.

We then remove the vertex b_1 and its incident edge, leaving a labelled tree of order $n - 1$.

We next let b_2 be the smallest label assigned to an end-vertex of our new tree, and let a_2 be the label of the vertex adjacent to the vertex b_2.

We then remove the vertex b_2 and its incident edge, as before.

We proceed in this way until there are only two vertices left, and the required sequence is (a_1, a_2, ... a_{n-2}).

For example, if T is the labelled tree in Figure (3.175),

then $b_1 = 2$, $a_1 = 6$, $b_2 = 3$, $a_2 = 5$, $b_3 = 4$, $a_3 = 6$

$b_4 = 6$, $a_4 = 5$, $b_5 = 5$, $a_5 = 1$

The required sequence is therefore (6, 5, 6, 5, 1)

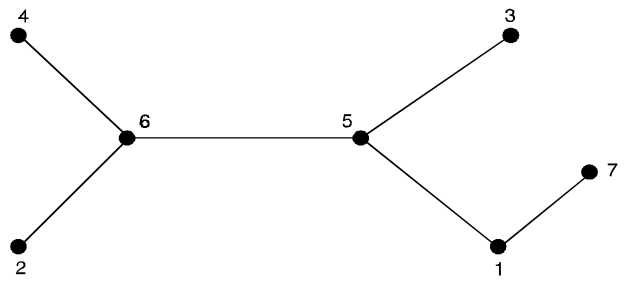


Fig. 3.175

To obtain the reverse correspondence, we take a sequence (a_1, ... a_{n-2}).

Let b_1 be the smallest number that does not appear in it, and join the vertices a_1 and b_1.

We then remove a_1 from the sequence, remove the number b_1 from consideration, and proceed as before.

In this way we build up the tree, edge by edge,

For example, if we start with the sequence (6, 5, 6, 5, 1), then $b_1 = 2$, $b_2 = 3$, $b_3 = 4$, $b_4 = 6$, $b_5 = 5$, and the corresponding edges are 62, 53, 64, 56, 15.

We conclude by joining the last two vertices not yet crossed out-in this case, 1 and 7.

It is simple to check that if we start with any labelled tree, find the corresponding sequence, and then find the labelled tree corresponding to that sequence, then we obtain the tree we started from.

We have therefore established the required correspondence and the result follows.

Second Proof :

Let T(n, k) be the number of labelled trees on n vertices in which a given vertex v has degree k.

We shall derive an expression for T(n, k), and the result follows on summing from $k = 1$ to $k = n - 1$.

Let A be any labelled tree in which deg (v) = $k - 1$.

The removal from A of any edge wz that is not incident with v leaves two subtrees, one containing v and either w or z (w, say), and the other containing z.

If we now join the vertices v and z, we obtain a labelled tree B in which deg (v) = K see Fig. (3.30).

We call a pair (A, B) of labelled trees of linkage if B can be obtained from A by the above construction.

Our aim is to count the possible linkages (A, B).

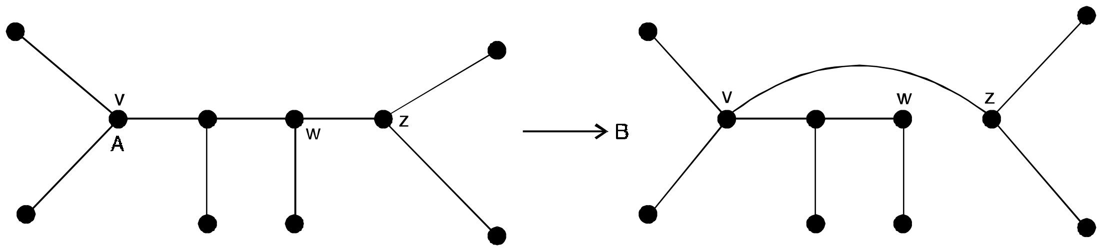


Fig. 3.176

Since A may be chosen in T(n, k – 1) ways, and since B is uniquely defined by the edge wz which may be chosen in (n – 1) – (k – 1) = (n – k) ways, the total number of linkages (A, B) is (n – k) T(n, k – 1).

On the other hand, let B be a labelled tree in which deg (v) = k, and let T_1, ... T_k be the subtrees obtained from B by removing the vertex v and each edge incident with v.

Then we obtain a labelled tree A with deg (v) = k – 1 by removing from B just one of these edges (vw_i, say, where w_i lies in T_i), and joining w_i to any vertex u of any other subtree T (see Fig. 3.177).

Note that the corresponding pair (A, B) of labelled trees is a linkage, and that all linkages may be obtained in this way.

Since B can be chosen in T(n, k) ways, and the number of ways of joining w_i to vertices in any other T_j is (n – 1) – n_i, where n_i is the number of vertices of T_i, the total number of linkages (A, B) is

$T(n, k) \{(n - 1 - n_1) + ... + (n - 1 - n_k)\} = (n - 1)(k - 1) T(n, k)$, since $n_1 + ... + n_k = n - 1$

We have thus shown that

$$(n - k) T(n, k - 1) = (n - 1)(k - 1) T(n, k).$$

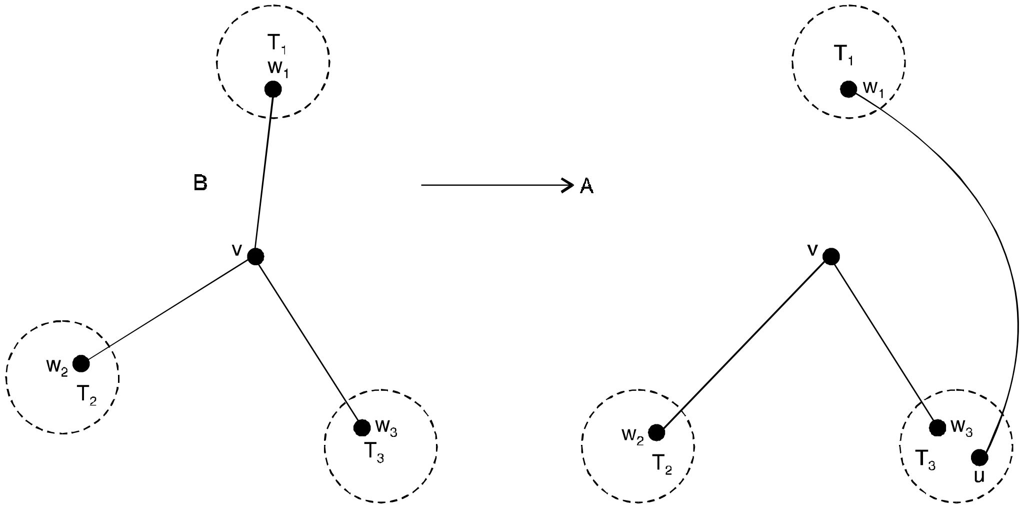


Fig. 3.177

On iterating this result, and using the obvious fact that T(n, n – 1) = 1, we deduce immediately that

$$T(n, k) = \binom{n-2}{n-1} (n-1)_{n-k-1}$$

On summing over all possible values of k, we deduce that the number T(n) of labelled trees on n vertices is given by

$$T(n) = \sum_{k=1}^{n-1} T(n, k) = \sum_{k=1}^{n-1} \binom{n-2}{k-1}^{n-k-1}$$

$$= \{(n-1)+1\}^{n-2} = n^{n-2}.$$

Corollary :

The number of spanning trees of K_n is n^{n-2}.

Proof. To each labelled tree on n vertices there corresponds a unique spanning tree of K_n.

Conversely, each spanning tree of K_n gives rise to a unique labelled tree on n vertices.

Theorem 3.50. *Prove that the maximum number of vertices in a binary tree of depth d is* $2^d - 1$, *where* $d \geq 1$.

Proof. We shall prove the theorem by induction.

Basis of induction :

The only vertex at depth $d = 1$ is the root vertex.

Thus the maximum number of vertices on depth

$d = 1$ is $2^1 - 1 = 1$.

Induction hypothesis :

We assume that the theorem is true for depth k,

$d > k \geq 1$

Therefore, the maximum number of vertices on depth k is $2^k - 1$.

Induction step :

By induction hypothesis, the maximum number of vertices on depth $k - 1$ is $2^{k-1} - 1$.

Since, we know that each vertex in a binary tree has maximum degree 2, therefore, the maximum number of vertices on depth $d = k$ is twice the maximum number of vertices on depth $k - 1$.

So, at depth k, the maximum number of vertices is $2.2^{k-1} - 1 = 2^k - 1$.

Hence proved.

Problem 3.153. *What are the left and right children of b shown in Fig. 3.178 ? What are the left and right subtrees of a ?*

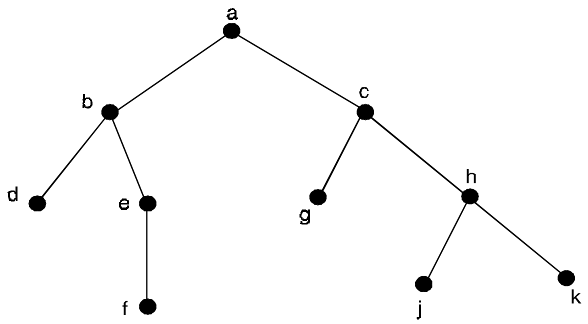


Fig. 3.178

Solution. The left child of *b* is *d* and the right child is *e*. The left subtree of the vertex *a* consists of the vertices *b*, *d*, *e* and *f* and the right subtree of *a* consists of the vertices *c*, *g*, *h*, *j* and *k* whose figures are shown in Fig. 3.179. (*a*) and (*b*) respectively.

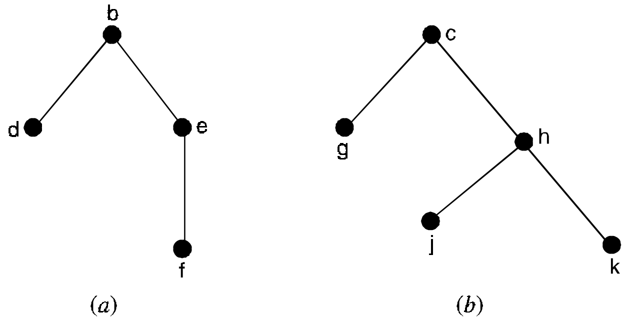


Fig. 3.179

Theorem 3.51. *Prove that the maximum number of vertices on level n of a binary tree is 2^n, where $n \geq 0$.*

Proof. We prove the theorem by mathematical induction.

Basis of induction :

When $n = 0$, the only vertex is the root.

Thus the maximum number of vertices on level $n = 0$ is $2^0 = 1$.

Induction hypothesis :

We assume that the theorem is true for level K, where $n \geq k \geq 0$.

So the maximum number of vertices on level k is 2^k.

Induction step :

By induction hypothesis, maximum number of vertices on level $k - 1$ is 2^{k-1}.

Since each vertex in binary tree has maximum degree 2, then the maximum number of vertices on level k is twice the maximum number of level $k - 1$.

Hence, the maximum number of vertices at level k is $= 2.2^{k-1} = 2^k$.

Hence, the theorem is proved.

3.19.2. Reachability

A node v in a simple graph G is said to be reachable from the vertex u of G if there exists a from u to v the set of vertices which a path from u to v, the set of vertices which are reachable from a given vertex v is called the reachable set of v and is denoted by R(v).

For any subset U of the vertex set V, the reachable set of U is the set of all vertices which are reachable from any vertex set of S and this set is denoted by R(S).

For example, in the graph given below :

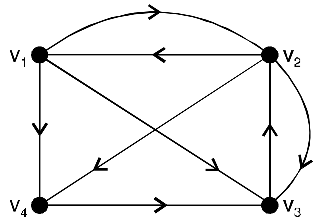


Fig. 3.180

$R(v_1) = \{v_2, v_3, v_4\}$, $R(v_2) = \{v_1, v_3, v_4\}$ and $R(\{v_1, v_2\}) = \{v_3, v_4\}$.

3.19.3. Distance and diameter

In a connected graph G, the distance between the vertices u and v, denoted by $d(u, v)$ is the length of the shortest path.

In Fig. 3.181(a), $d(a, f) = 2$ and in Fig. 3.181(b), $d(a, e) = 3$.

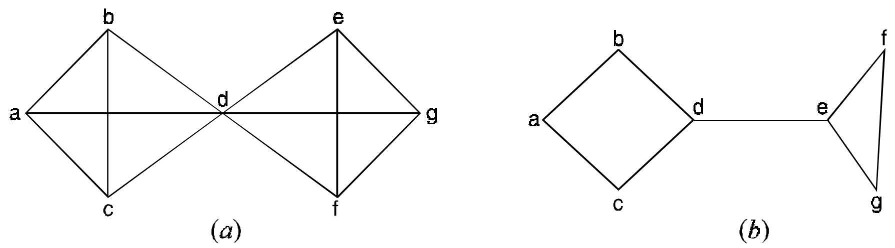


Fig. 3.181

The distance function as defined above has the following properties.

If u, v and w are any three vertices of a connected graph then

(i) $d(u, v) \geq 0$ and $d(u, v) = 0$ if $u = v$.

(ii) $d(u, v) = d(v, u)$ and

(iii) $d(u, v) \geq d(u, w) + d(w, v)$

This shows that distance in a graph is metric.

The diameter of G, written as diam (G) is the maximum distance between any two vertices in G.

In Fig. 3.181(a), diam (G) = 2 and in Fig. 3.181(b), diam (G) = 4.

3.20. MINIMAL SPANNING TREES

3.20.1. Weighted graph

A weighted graph is a graph G in which each edge e has been assigned a non-negative number $w(e)$, called the weight (or length) of e. Figure (3.182) shows a weighted graph. The weight (or length) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. Many optimisation problems amount to finding, in a suitable weighted graph, a certain type of subgraph with minimum (or maximum) weight.

3.20.2. Minimal spanning tree

Let G be weighted graph. A minimal spanning tree of G is a spanning tree of G with minimum weight. The weighted graph G of Figure (3.182) shows six cities and the costs of laying railway links between certain pairs of cities. We want to set up railway links between the cities at minimum costs. The solution can be represented by a subgraph. This subgraph must be spanning tree since it covers all the vertices (so that each city is in the road system), it must be connected (so that any city can be reached from any other), it must have unique simple path between each pair of vertices.

Thus what is needed is a spanning tree the sum of whose weights is minimum, *i.e.,* a minimal spanning tree.

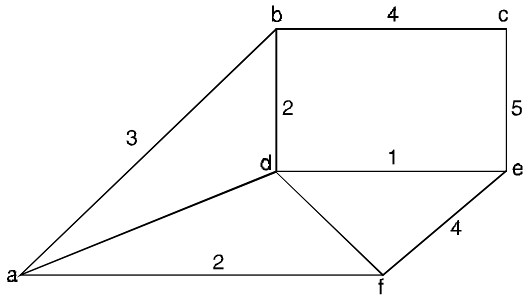


Fig. 3.182

3.20.3. Algorithm for minimal spanning tree

There are several methods available for actually finding a minimal spanning tree in a given graph. Two algorithms due to Kruskal and Prim of finding a minimal spanning tree for a connected weighted graph where no weight is negative are presented below. These algorithms are example of greedy algorithms. A greedy algorithm is a procedure that makes an optimal choice at each of its steps without regard to previous choices.

3.20.4. Kruskal's algorithm

Kruskal's algorithm for finding a minimal spanning tree :

Input : A connected graph G with non-negative values assigned to each edge.

Output : A minimal spanning tree for G

Let G = (V, E) be graph and S = (V_s, E_s) be the spanning tree to be found from G. Let $|V| = n$ and E = {e_1, e_2, ... e_m}. The stepwise algorithm is given below :

Method :

Step 1 : Select any edge of minimal value that is not a loop. This is the first edge of T. If there is more than one edge of minimal value, arbitrarily choose one of these edges.

i.e., select an edge e_1 from E such that e_1 has least weight. Replace $E = E - \{e_1\}$ and $E_s = \{e_1\}$

Step 2 : Select any remaining edge of G having minimal value that does not form a circuit with the edges already included in T.

i.e., select an edge e_i from E such that e_i has least weight and that it does not form a cycle with members of E_s. Set $E = E - \{e_i\}$ and $E_s = E_s \cup \{e_i\}$.

Step 3 : Continue step 2 until T contains $n - 1$ edges, where n is the number of vertices of G.

i.e., Repeat step 2 until $|E_s| = |V| - 1$.

Suppose that a problem calls for finding an optimal solution (either maximum or minimum).

Suppose, further, than an algorithm is designed to make the optimal choice from the available data at each stage of the process. Any algorithm based on such an approach is called a greedy algorithm.

A greedy algorithm is usually the first heuristic algorithm one may try to implement and it does lead to optimal solutions sometimes, but not always. Kruskal's algorithm is an example of a greedy algorithm that does, in fact, lead to an optimal solution.

Theorem 3.52. *Let $G = (V, E)$ be a loop-free weighted connected undirected graph. Any spanning tree for G that is obtained by Kruskal's algorithm is optimal.*

Proof. Let $|V| = n$, and let T be a spanning tree for G obtained by Kruskal's algorithm.

The edges in T are labeled $e_1, e_2, ... e_{n-1}$, according to the order in which they are generated by the algorithm.

For each optimal tree T′ of G, define $d(T') = k$ if k is the smallest positive integer such that T and T′ both contain $e_1, e_2, .., e_{k-1}$, but $e_k \notin T'$.

Let T_1 be an optimal tree for which $d(T_1) = r$ is maximal.

If $r = n$, then $T = T_1$ and the result follows.

Otherwise, $r \le n - 1$ and adding edge e_r (of T) to T_1 produces the cycle C, where there exists an edge e_r' if C that is in T_1 but not in T.

Start with tree T_1. Adding e_r to T_1 and deleting e_r', we obtain a connected graph with n vertices and $n - 1$ edges.

This graph is a tree, T_2. The weights of T_1 and T_2 satisfy $wt(T_2) = wt(T_1) + wt(e_r) - wt(e_r')$.

Following the selection of $e_1, e_2, ..., e_{r-1}$ in Kruskal's algorithm, the edge e_r is chosen so that $wt(e_r)$ is minimal and no cycle results when e_r is added to the subgraph H of G determined by $e_1, e_2, ..., e_{r-1}$.

Since e_r' produces no cycle when added to the subgraph H, by the minimality of $wt(e_r)$ it follows that $wt(e_r') \ge wt(e_r)$.

Hence $wt(e_r) - wt(e_r') \le 0$, so $wt(T_2) \le wt(T_1)$. But with T_1 optimal, we must have $wt(T_2) = wt(T_1)$, so T_2 is optimal.

The tree T_2 is optimal and has the edges $e_1, e_2, ..., e_{r-1}, e_r$ in common with T, so $d(T_2) \ge r + 1 > r = d(T_1)$, contradicting the choice of T_1.

Consequently, $T_1 = T$ and the tree T produced by Kruskal's algorithm is optimal.

Theorem 3.53. *Let G be a connected graph where the edges of G are labelled by non-negative numbers. Let T be an economy tree of G obtained from Kruskal's Algorithm. Then T is a minimal spanning tree.*

Proof. As before, for each edge e of G, let C(e) denote the value assigned to the edge by the labelling.

If G has n vertices, an economy tree T must have $n - 1$ edges.

Let the edges $e_1, e_2, ..., e_{n-1}$ be chosen as in Kruskal's Algorithm. Then $C(T) = \sum_{i=1}^{n-1} C(e_i)$.

Let T_0 be a minimal spanning tree of G.

We show that $C(T_0) = C(T)$, and thus conclude that T is also minimal spanning tree.

If T and T_0 are not the same let e_i be the first edge of T not in T_0.

Add the edge e_i to T_0 to obtain the graph G_0.

Suppose $e_i = \{a, b\}$, then a path P from a to b exists in T_0 and so P together with e_i produces a circuit C in G_0.

Since T contains no citcuits, there must be an edge e_0 in C that is not in T.

The graph $T_1 = G_0 - e_0$ is also a spanning tree of G since T_1 has $n - 1$ edges.

Moreover, $C(T_1) = C(T_0) + C(e_i) - C(e_0)$.

However, we know that $C(T_0) \leq C(T_1)$ since T_0 was a minimal spanning tree of G.

Thus, $C(T_1) - C(T_0) = C(e_i) - C(e_0) \geq 0$.

Implies that $C(e_i) \geq C(e_0)$.

However, since T was constructed by Kruskal's algorithm e_i is an edge of smallest value that can be added to the edges $e_1, e_2, ..., e_{i-1}$ without producing a circuit. Also, if e_0 is added to the edges $e_1, e_2, ..., e_{i-1}$, no circuit is produced because the graph thus formed is a subgraph of the tree T_0.

Therefore, $C(e_i) = C(e_0)$, so that $C(T_1) = C(T_0)$.

We have constructed from T_0 a new minimal spanning tree T_1 such that the number of edges common to T_1 and T exceeds the number of edges common to T_0 and T by one edge, namely e_i.

Repeat this procedure, to construct another minimal spanning tree T_2 with one more edge in common with T than was in common between T_1 and T.

By continuing this procedure, we finally arrive at a minimal spanning tree with all edges in common with T, and thus we conclude that T is itself a minimal spanning tree.

Problem 3.154. *Using Kruskal's algorithm, find the minimum spanning tree for the weighted graph of the Fig. (3.183).*

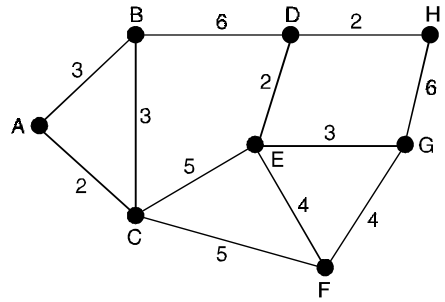


Fig. 3.183

Solution. Let S = (V_s, E_s) be the spanning tree to be found from G.

Initialize, there are eight nodes so the spanning tree will have seven arcs.

The iterations of algorithm applied on the graph are given below and it runs at the most seven times.

The number indicates iteration number.

1. Since arcs AC, ED, and DH have minimum weight 2. Since they do not form a cycle, we select all of them and E_s = {(A, C), (E, D), (D, H)} and E = E – ({A, C), (E, D), (D, H)}.
2. Next arcs with minimum weights 3 are AB, BC, and EG. We can select only one of the AB and BC. Also we can select EG.

   Therefore, E_s = ({A, C), (E, D), (D, H), (A, B), (E, G)} and E = E – {(A, B), (E, G)}
3. Next arcs with minimum weights 4 are EF and FG. We can select only one of them.

   Therefore, E_s = {(A, C), (E, D), (D, H), (A, B), (E, G), (F, G)} and E = E – {(F, G)}.
4. Next arcs with minimum weights 5 are CE and CF. We can select only one of them.

   Therefore, E_s = ({A, C), (E, D), (D, H), (A, B), (E, G), (F, G), (C, E)} and E = E – {(C, E)}.

   Since number of edges in E_s is seven process terminates here. The spanning tree so obtained is shown in the Fig. (3.184).

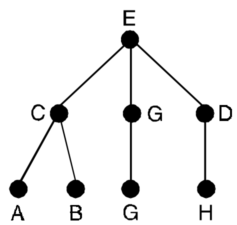


Fig. 3.184

Problem 3.155. *Show how Kruskal's algorithm find a minimal spanning tree for the graph of Fig. (3.185).*

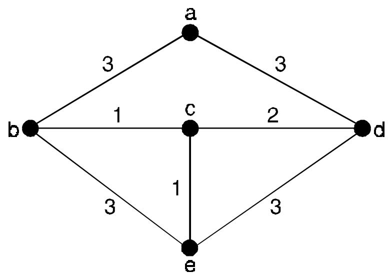


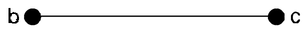
Fig. 3.185

Solution. We collect the edges with their weights into a table

Edge	*Weight*
(b, c)	1
(c, e)	1
(c, d)	2
(a, b)	3
(e, d)	3
(a, d)	4
(b, e)	4

The steps of finding a minimal spanning tree are shown below.

1. Choose the edge (b, c) as it has a minimal weight



2. Add the next edge (c, e)

Fig. 3.186

3. Add the edge (c, d)

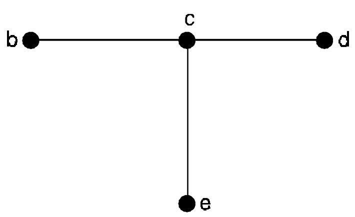


Fig. 3.187

4. Add the edge (*b*, *a*)

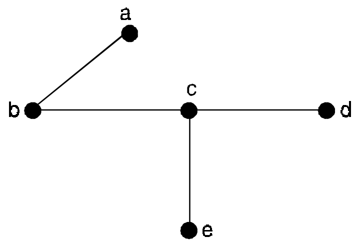


Fig. 3.188

Since vertices are 5 and we have choosen 4 edges, we stop the algorithm and the minimal spanning tree is produced.

Problem 3.156. *Show how Kruskal's algorithm find a minimal spanning tree of the graph of Fig. (3.189).*

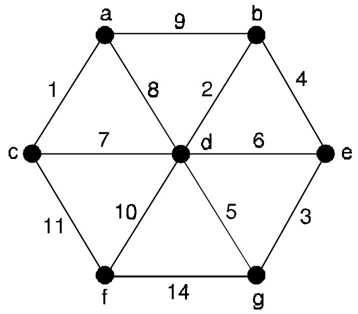


Fig. 3.189

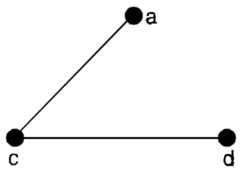
Solution. We collect the edges with their weights into a table.

Edge	*Weight*
(*a*, *c*)	1
(*b*, *d*)	2
(*e*, *g*)	3
(*b*, *e*)	4
(*d*, *g*)	5
(*d*, *e*)	6
(*d*, *c*)	7
(*a*, *d*)	8
(*a*, *b*)	9
(*d*, *f*)	10
(*c*, *f*)	11
(*f*, *g*)	14

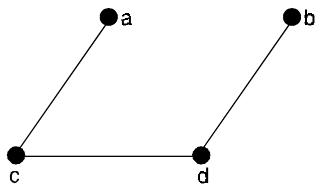
The steps of finding a minimal spanning tree are shown below :

1. Choose the edge (a, c) as it has minimal weight

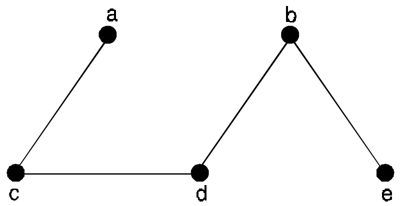
2. Add the next edge (c, d)



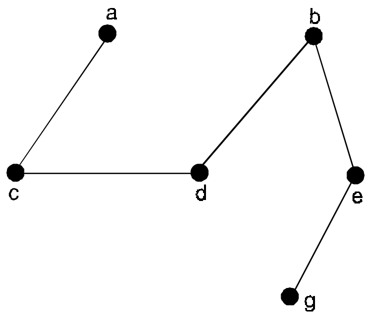
3. Add the edge (d, b)



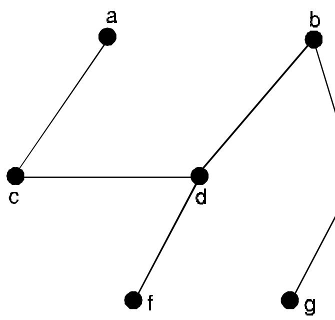
4. Add the edge (b, e)



5. Add the edge (e, g)



6. Add the edge (d, f)



Since vertices are 7 and we have chosen 6 edges, we stop the algorithm and the minimal spanning tree is produced.

Problem 3.157. *Use Kruskal's algorithm to find a minimum spanning tree in the weighted graph shown in Fig. (3.190).*

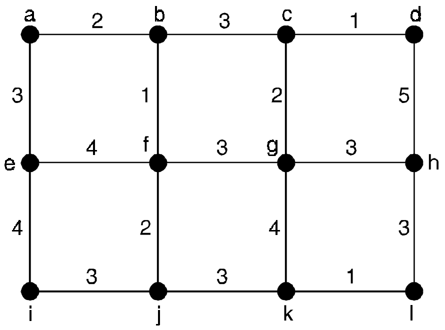


Fig. 3.190

Solution. A minimum spanning tree and the choices of edges at each stage of Kruskal's algorithm are shown in Fig. (3.191)

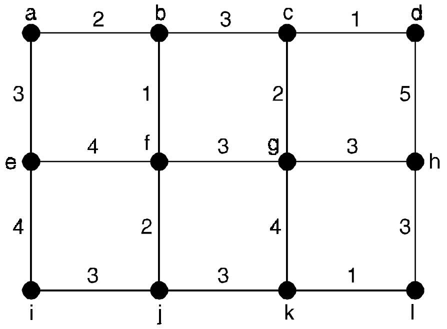


Fig. 3.191

Choice	*Edge*	*Weight*
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
		Total : 24

Problem 3.158. *Determine a railway network of minimal cost for the cities in Fig. (3.192).*

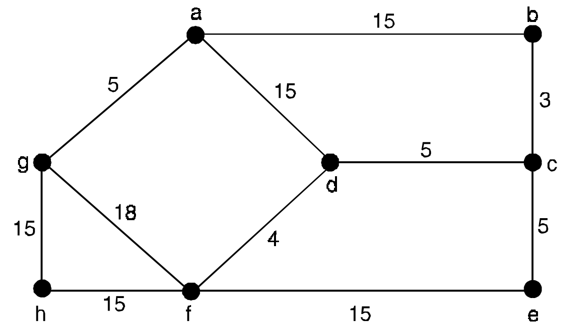


Fig. 3.192.

Solution. We collect lengths of edges into a table :

Edge	*Cost*
$\{b, c\}$	3
$\{d, f\}$	4
$\{a, g\}$	5
$\{c, d\}$	5
$\{c, e\}$	5
$\{a, b\}$	15
$\{a, d\}$	15
$\{f, h\}$	15
$\{g, h\}$	15
$\{e, f\}$	15
$\{f, g\}$	18

1. Choose the edges $\{b, c\}$, $\{d, f\}$, $\{a, g\}$, $\{c, d\}$, $\{c, e\}$
2. Then we have options : we may choose only one of $\{a, b\}$ and $\{a, d\}$ for the selection of both creates a circuit. Suppose that we choose $\{a, b\}$.
3. Likewise we may choose only one of $\{g, h\}$ and $\{f, h\}$. Suppose we choose $\{f, h\}$.
4. We then have a spanning tree as illustrated in Fig. (3.193).

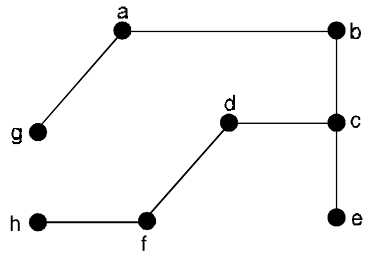


Fig. 3.193

The minimal cost for construction of this tree is

$$3 + 4 + 5 + 5 + 5 + 15 + 15 = 52.$$

Problem 3.159. *Apply Kruskal's algorithm to the graph shown in Fig. (3.194).*

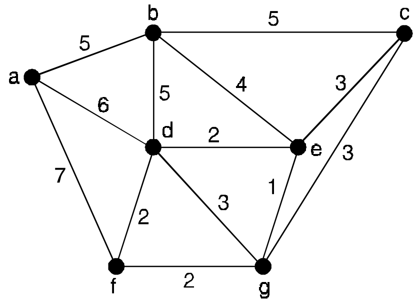


Fig. 3.194

Solution. Initialization : ($i = 1$) since there is a unique edge-namely, $\{e, g\}$, of smallest weight 1, start with T = $\{\{e, g\}\}$. (T starts as a tree with one edge, and after each iteration it grows into a larger tree or forest. After the last iteration the subgraph T is an optimal spanning tree for the given graph G).

First iteration

Among the remaining edges in G, three have the next smallest weight 2. Select $\{d, f\}$, which satisfies the conditions in step (2).

Now T is the forest $\{\{e, g\}, \{d, f\}\}$, and i is increased to 2. With $i = 2 < 6$, return to step (2).

Second iteration

Two remaining edges have weight 2. Select $\{d, e\}$.

Now T is the tree $\{\{e, g\}, \{d, f\}, \{d, e\}\}$, and i increases to 3. But because $3 < 6$, the algorithm directs us back to step (2).

Third iteration

Among the edges of G that are not in T, edge {*f*, *g*} has minimal weight 2.

However, if this edge is added to T, the result contains a cycle, which destroys the tree structure being sought.

Consequently, the edges {*c*, *e*}, {*c*, *g*} and {*d*, *g*} are considered.

Edge {*d*, *g*} brings about a cycle, but either {*c*, *e*} or {*c*, *g*} satisfies the conditions in step (2).

Select {*c*, *e*}. T grows to {{*e*, *g*}, {*d*, *f*}, {*d*, *e*}, {*c*, *e*}} and *i* is increased to 4.

Returning to step (2), we find that the fourth and fifth iterations provide the following.

Fourth iteration

T = {{*e*, *g*}, {*d*, *f*}, {*d*, *e*}, {*c*, *e*}, {*b*, *e*}}, *i* increases to 5.

Fifth iteration

T = {{*e*, *g*}, {*d*, *f*}, {*d*, *e*}, {*c*, *e*}, {*b*, *e*}, {*a*, *b*}}.

The counter *i* now becomes 6 = (number of vertices in G) – 1.

So T is an optimal tree for graph G and has weight

$$1 + 2 + 2 + 3 + 4 + 5 = 17$$

Fig. (3.195) shows this spanning tree of minimal weight.

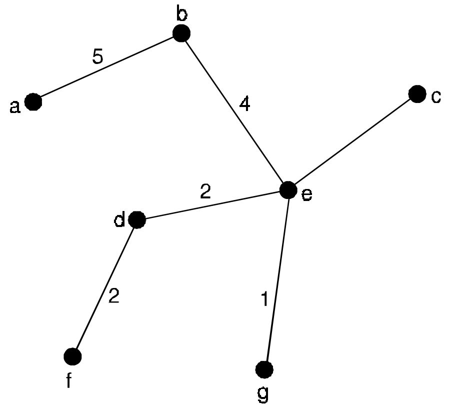


Fig. 3.195

Problem 3.160. *Using the Kruskal's algorithm, find a minimal spanning tree of the weighted graph shown below :*

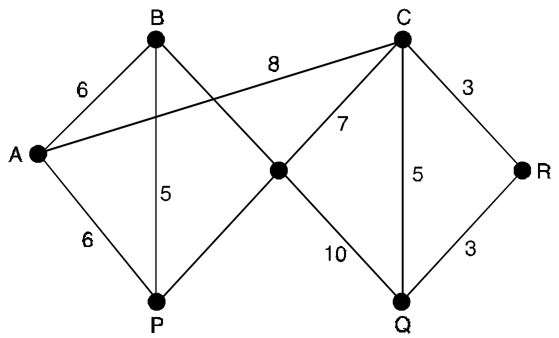


Fig. 3.196

Solution. We observe that the given graph has 6 vertices, hence a spanning tree will have 5 edges. Let us put the edges of the graph in the non-decreasing order of their weights and successively select 5 edges in such a way that no circuit is created.

Edges	CR	QR	BP	CQ	AB	AP	CP	AC	BQ
Weight	3	3	5	5	6	6	7	8	10
Select	Yes	Yes	Yes	No	Yes	No	Yes		

Observe that CQ is not selected because CR and QR have already been selected and the selection of CQ would have created a circuit. Further, AP is not selected because it would have created a circuit along with BP and AB which have already been selected we have stopped the process when exactly 5 edges are selected.

Thus, a minimal spanning tree of the given graph contains the edges CR, QR, BP, AB, CP. This tree as shown in Fig. (3.197). The weight of this tree is 24 units.

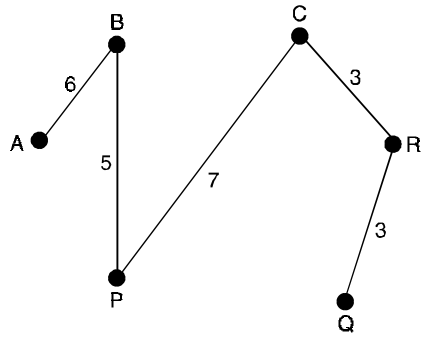


Fig. 3.197

Problem 3.161. *Eight cities A, B, C, D, E, F, G, H are required to be connected by a new railway network. The possible tracks and the cost of involved to lay them (in crores of rupees) are summarized in the following table :*

Track between	*Cost*	*Track between*	*Cost*
A and B	*155*	*D and F*	*100*
A and D	*145*	*E and F*	*150*
A and G	*120*		
B and C	*145*	*F and G*	*140*
C and D	*150*	*F and H*	*150*
C and E	*95*	*G and H*	*160*

Determine a railway network of minimal cost that connects all these cities.

Solution. Let us first prepare a graph whose the vertices represent the cities, edges represent the possible tracks and weights represent the cost. The graph is as shown in Fig. (3.198).

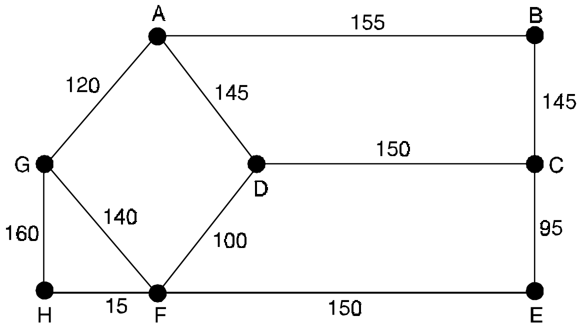


Fig. 3.198

A minimal spanning tree of this graph represents the required network. Since there are eight vertices, seven edges should be there in a minimal spanning tree.

Let us put the edges of the graph in the non-decreasing order of their weights and select seven edges one by one in such a way that no circuit is created.

Edges	CE	DF	AG	FG	AD	BC	CD	EF	FH	AB	GH
Weight	95	100	120	140	145	145	150	150	150	155	160
Select	Yes	Yes	Yes	Yes	No	Yes	Yes	No	Yes.		

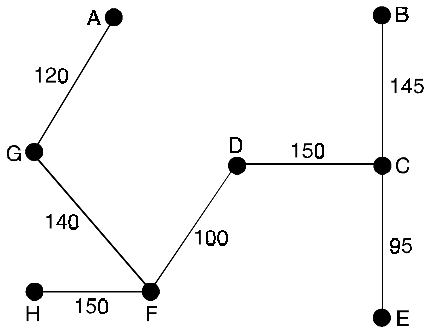


Fig. 3.199

Thus, a minimal spanning tree of the given graph consists of the branches CE, DF, AG, FG, BC, CD, FH.

This tree represents the required railway network. The network is shown in Fig. (3.199). The cost involved is 95 + 100 + 120 + 140 + 145 + 150 + 150 = 900 (in crores of rupees).

Problem 3.162. *Using the Kruskal's algorithm, find a minimal spanning tree for the weighted graph shown below :*

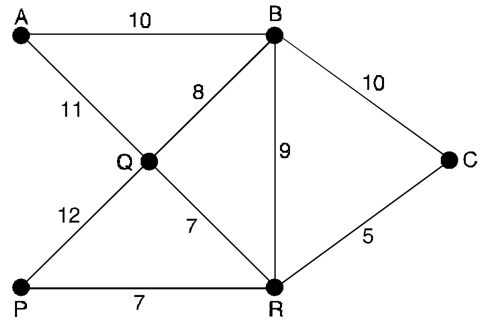


Fig. 3.200

Solution. The given graph has 6 vertices and therefore a spanning tree will have 5 edges.

Let us put the edges of the graph in the non-decreasing order of their weights and go on selecting 5 edges one by one in such a way that no circuit is created.

Edges	CR	PR	QR	BQ	BR	AB	BC	AR	PQ
Weight	5	7	7	8	9	10	10	11	12
Select	Yes	Yes	Yes	Yes	No	Yes			

Thus, a minimal spanning tree of the given graph contains the edges CR, PR, QR, BQ, AB. The tree is shown in Fig. (3.201). The weight of the tree is 37 units.

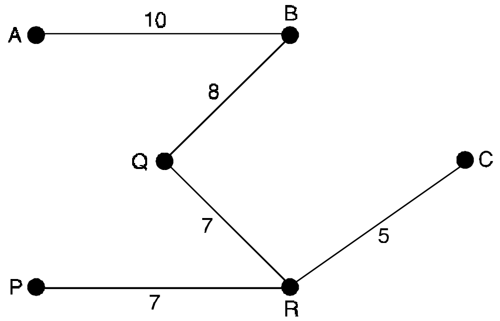


Fig. 3.201

Problem Set 3.1

1. How many vertices do the following graphs have if they contains
   (*i*) 16 edges and all vertices of degree 2
   (*ii*) 21 edges, 3 vertices of degree 4 and others each of degree 3.

2. If G = (V, E) be an undirected graph with e edges then show that $\sum_{V \in V} \deg_G(V) = 2e$.

3. Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

4. How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree 2.

5. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

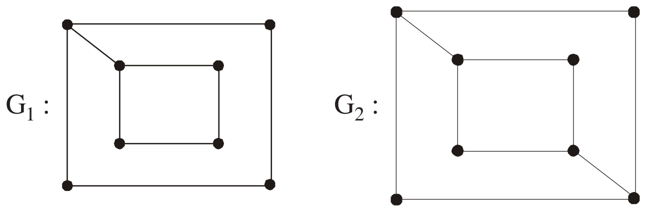
6. Does a 3-regular graph on 14 vertices exist ? Whan can you say on 17 vertices ?

7. Draw all six graphs with five vertices and five edges.

8. Find all possible non-isomorphic induced subgraphs of the following graph.

9. If a graph G of n vertices is isomorphic to its complement $\overline{G}$, show that n or $(n - 1)$ must be multiple of 4.

10. Determine whether the following graphs are isomorphic or not



11. Prove that any party with six people, there are three mutual acquaintances or three mutual nonacquaintances.

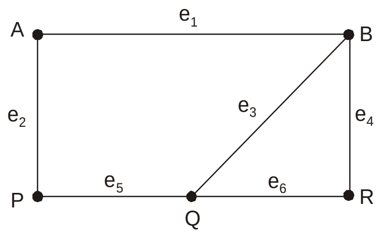
12. Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

13. Prove that A simple graph with n vertices and k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

14. Show that a simple (p, q)-graph is connected then $p \leq q + 1$.

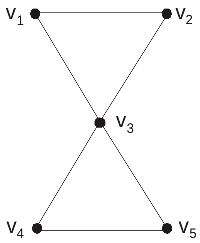
15. Let G be a disconnected graph with n vertices where n is even. If G has two components each of which is complete, prove that G has a minimum of $\frac{n(n-2)}{4}$ edges.

16. Consider the graph shown in figure, find all paths from vertex A to vertex R. Also, indicate their lengths.



17. Prove that A connected graph G has an Eulerian trial if and only if it has atmost odd vertices.

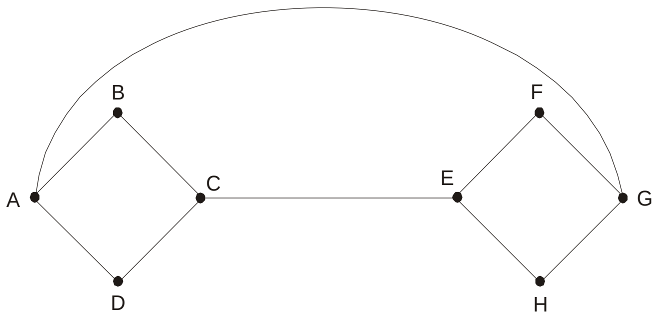
18. Let G be a graph of figure. Verify that G has an Eulerian circuit.



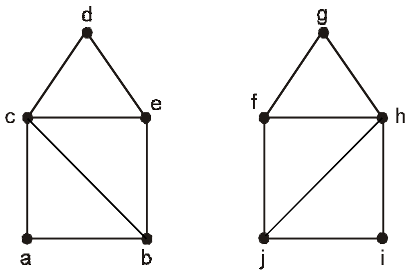
19. If G is a graph in which the degree of each vertex is atleast 2 then prove that G contains a cycle.

20. If a graph G has more than two vertices of odd degree then prove that, there can be no Euler path in G.

21. Use Fleury's algorithm to construct an Euler circuit for the graph in Figure below.



22. Using Fleury's algorithm, find Euler circuit in the graph of figure below.

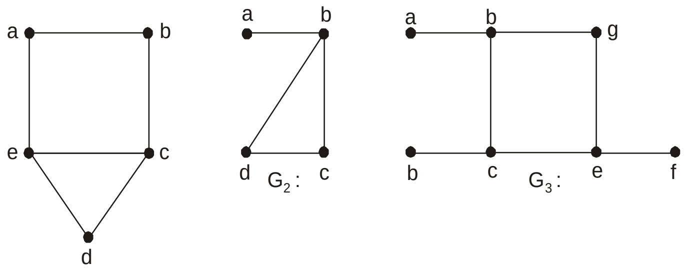


23. If G is a connected graph of order three or more which is not hamiltonian then prove that the legnth k of a longest path of G satisfies $k \geq 2\delta(G)$.

24. If G is a group with $p \geq 3$ vertices such that for all non adjacent vertices u and v, deg u + deg v $\geq$ P then prove that G is hamiltonian.

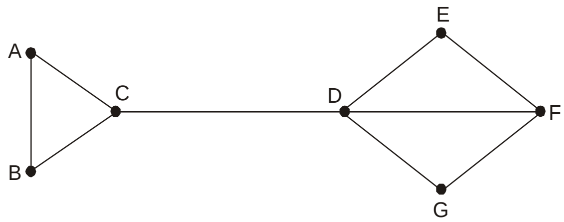
25. Let G be a simple graph with n vertices and m edges where m is atleast 3. If $m \geq \frac{1}{2}(n-1)(n-2) + 2$. Prove that G is Hamiltonian.

26. Which of the simple graphs in Figure have a Hamilton circuit or if not, a Hamilton path.



27. Let the number of edges of G be m, then prove that G has a Hamiltonian circuit if $m \geq \frac{1}{2}(n^2 - 3n + 6)$.

28. Determine whether a Hamiltonian path or circuit exists in the graph of figure.



29. A tree has five vertices of degree 2, three vertices of degree 3 and four vertices of degree 4. How many vertices of degree one does it have ?

30. Prove that in a complete n-ary tree with m internal nodes, the number of leaf node l is given by the formula

$$l = \frac{(n-1)(x-1)}{n}$$

where, x is the total number of nodes in the tree.

31. Prove that a tree T is always separable.

32. Prove that, A tree T with n vertices has $n - 1$ edges.

33. Prove that, there are atmost m^h leaves in an m-ary tree of height h.

34. Construct two non-isomorphic trees having exactly 4 pendant vertices on 6 vertices.

35. Let T be a tree with 50 edges. The removal of certain edge from T yields two disjoint trees T_1 and T_2. Given that the number of vertices in T_1 equals the number of edges in T_2. Determine the number of vertices and the number of edges in T_1 and T_2.

36. Show that a Hamiltonian path is a spanning tree.

37. Show that the complete graph k_n is not a tree, when $n > 2$.

38. Suppose that a tree T has N_1 vertices of degree 1, N_2 vertices of degree 2, N_3 vertices of degree 3, ... N_k vertices of degree k. Prove that $N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (K - 2) N_k$.

39. What is the maximum possible number of vertices in any k-level tree ?

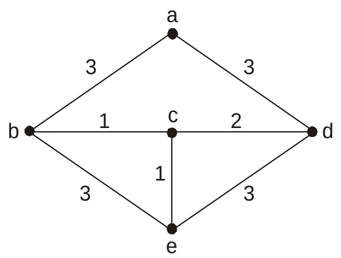
40. Which trees are complete bipartite graphs ?

41. In any binary tree T on n vertices, show that the number of pendant vertices (edges) is equal to $\dfrac{(n+1)}{2}$.

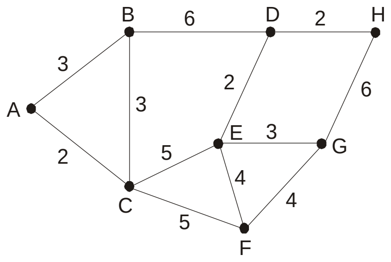
42. Prove that, there are n^{n-2} distinct labelled trees on n-vertices.

43. Write a short note on ''Kruskals'' algorithm for finding a minimal spanning tree.

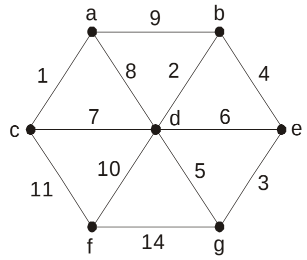
44. Show how Kruskal's algorithm find a minimal spanning tree for the graph of figure below.



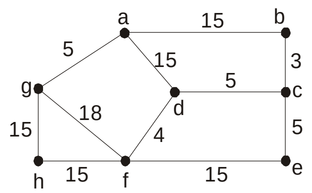
45. Using Kruskal's algorithm, find the minimum spanning tree for the weighted graph of the figure below.



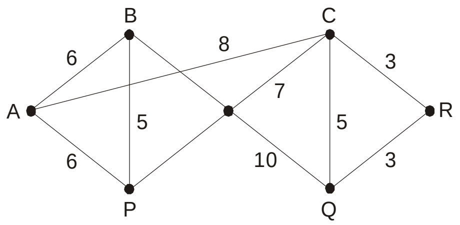
46. Show how Kruskal's algorithm find a minimal spanning tree of the graph of figure below.



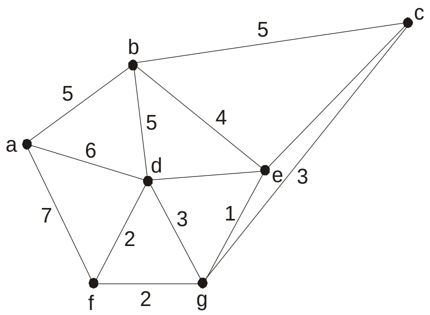
47. Determine a railway network of minimal cost for the cities in figure below.



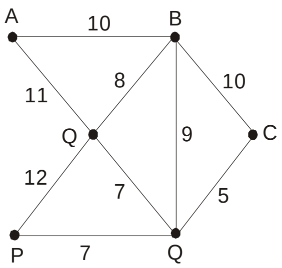
48. Using the Kruskal's algorithm, find a minimal spanning tree of the weighted graph shown below.



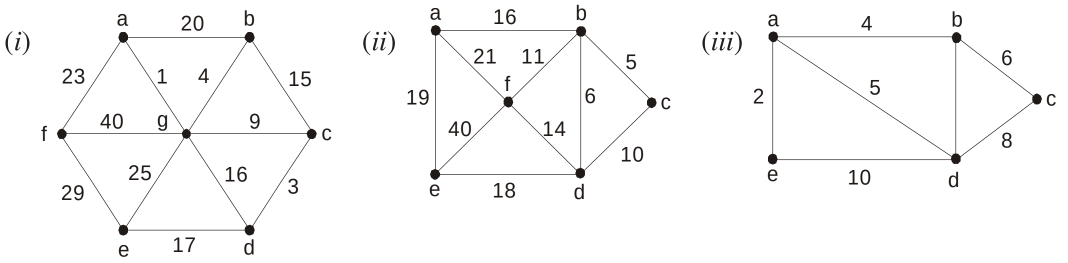
49. Apply Kruskal's algorithm to the graph shown in figure below.



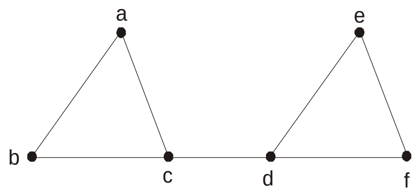
50. Using the Kruskal's algorithm find a minimal spanning tree for the weighted graph show below.



51. Use Kruskal's algorithm to find a minimum spanning tree for the given weighted graphs.



52. Draw all spanning trees of the following graph shown below.



53. Show that a tree with two vertices of degree 3 must have atleast four vertices of degree 1.

54. Prove that a tree with $n \geq 2$ vertices is a bipartite graph.

55. Let T be a tree with n vertices v_1, v_2, ... v_h. Prove that the number of leaves in T is

$$\sum_{\deg v_i \geq 3} [\deg v_i - 2].$$

56. Suppose a graph G has two connected components T_1, T_2 each of which is a tree. Suppose we add a new edge to G by joining a vertex of T_1 to a vertex in T_2. Proev that the new graph is a tree.

57. Let e be an edge in a tree T. Prove that the graph consisting of all the vertices of T but with the single edge e deleted is not connected.

58. Suppose some edge of a connected graph G belongs to every spanning tree of G. What can you conclude and why ?

59. Prove that a connected graph with n vertices is a tree if and only if the sum of the degrees of the vertices is $2(n - 1)$.

60. Prove that any two edges of a connected graph are part of some spanning tree.

61. Prove that every edge in a connected graph is part of some spanning tree.

Problem Set 3.2

1. Define (*i*) Graph (*ii*) Directed and un-directed graphs (*iii*) multigraph (*iv*) simple graph, given en example.

2. Define (*i*) Degree of vertex (*ii*) Indegree and out degree (*iii*) Isolated and pendent vertices.

3. Define (*i*) Complete graph (*ii*) Regular graph (*iii*) Platonic graph (*iv*) Subgraph (*v*) Spanning subgroup (*vi*) Induced subgraph.

4. Define (*i*) Graphs isomorphism (*ii*) Union (*iii*) Intersection (*iv*) Sum of two graphs (*v*) Ring sum (*vi*) Product of graphs.

5. Define (*i*) Composition (*ii*) complement (*iii*) Fusion.

6. Define (*i*) Connected and Disconnected graphs (*ii*) Path graphs and cycle graphs.

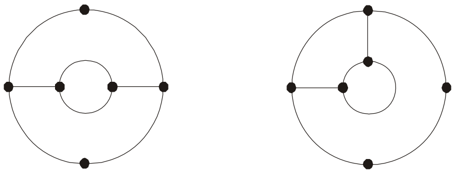
7. Define (*i*) Walks, paths and circuits (*ii*) Length (*iii*) Euler path (*iv*) Euler circuit (*v*) Hamiltonian graphs.

8. Define (*i*) Tree (*ii*) Spanning tree (*iii*) Rooted tree (*iv*) Binary trees

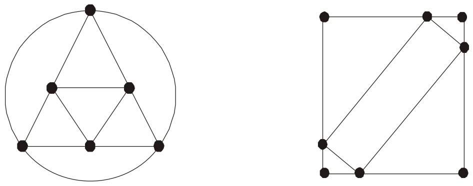
9. Define (*i*) Reachability (*ii*) Distance and diameter.

10. Define (*i*) Weighted graph (*ii*) Minimal spanning tree.

11. Show that the following graphs are not isomorphic.

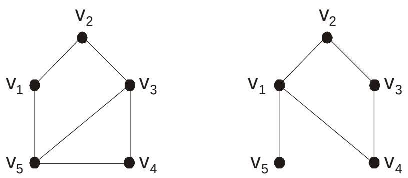


12. Show that the following graphs are not isomorphic



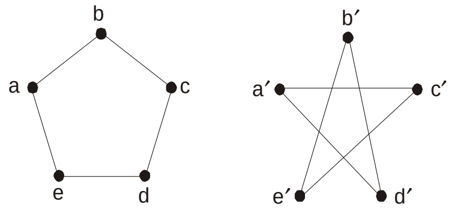
13. Suppose a graph has vertices of degree 0, 2, 2, 3 and 9. How many edegs does the graph have ?

14. Show that graphs are not isomorphic.

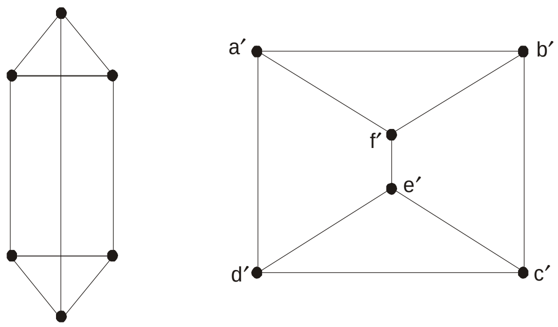


15. Show that the given pairs of graphs are isomorphic.

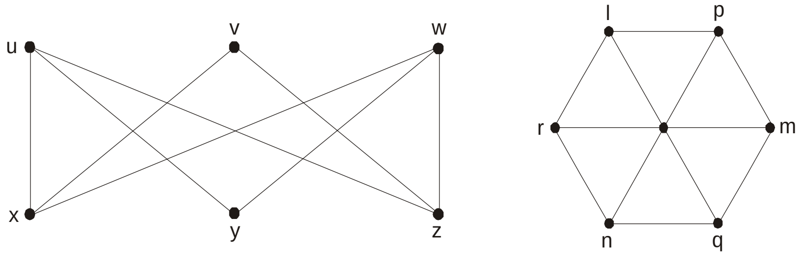
(*i*)



(*ii*)



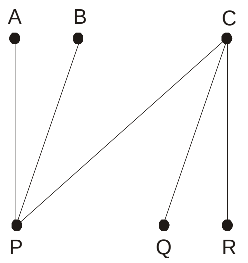
16. Write down the vertex set and edge set of each graph in figure below.



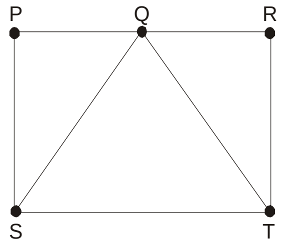
17. Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices. How many of these have exactly m edges ?

18. For the graphs shown below, indicate the number of vertices, the number of edges and the degrees of vertices.

(*i*)



(*ii*)



19. Prove that if a connected graph G is decomposed into two subgraph H_1 and H_2 there must be atleast one vertex common to H_1 and H_2.

20. Let G be a simple graph. Show that If G is not connected then its complement $\overline{G}$ is connected.

21. Prove that a connected graph of order n contains exactly one circuit if and only if its size is also n.

22. If G is a simple graph with n vertices and k components prove that G has atleast $n - k$ number of edges.

23. Let G be a simple with 15 vertices and 4 components. Prove that atleast one component of G has atleast 4 vertices.

24. Show that if G is a connected graph in which every vertex has degree either 1 or 0 then G is either a path or a cycle.

25. Prove that any two simple connected graphs with n vertices all of degree two are isomorphic.

26. Suppose G_1 and G_2 are isomorphic. Prove that if G_1 is connected then G_2 is also connected.

27. In a graph G, let P_1 and P_2 be two different paths between two given vertices. Prove that G has a circuit in it.

28. Prove that if u is an odd vertex in a graph G then there must be a path in G from u to another odd vertex v in G.

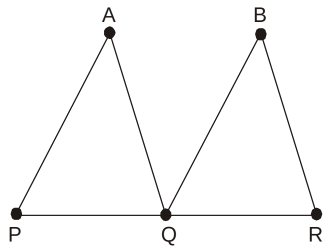
29. Show that in a graph with n vertices, the length of a path cannot exceed $n - 1$ and the length of a circuit cannot exceed n.

30. Prove that the edge set of every closed walk can be partitioned into pairwise edge-disjoint circuit.

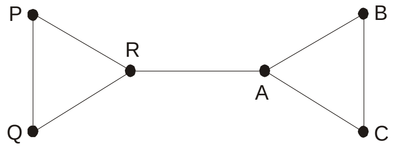
31. Prove that the petersen graph is neither Eulerian nor semi-Eulerian.

32. Prove that a connected graph is semi-Eulerian if and only if it has exactly zero or two vertices of odd dgree.

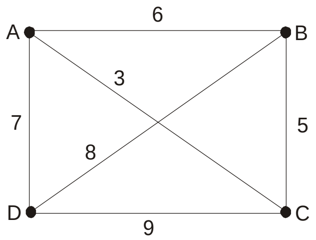
33. Show that the following graph is Eulerian.



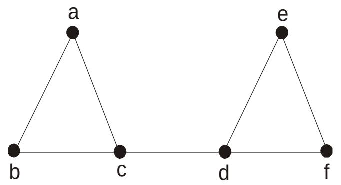
34. Show that the following graph is not Eulerian.



35. Solve the travelling salesman problem for the weighted graph shown below.



36. Draw all the spanning trees of the following graph shown below.



37. Draw three distinct rooted trees that have 4 vertices.

38. If G is a graph and e is an edge which is not part of a circuit then e must belong to every spanning tree of G. Why ?

39. How many graphs have n vertices labelled $v_1, v_2, \ldots\ldots v_n$ and $n - 1$ edges ? Compare this number with the number of trees with vertices $v_1, \ldots\ldots v_n$ for $2 \le n \le 6$.

40. How will you differentiate between a general tree and a binary tree ?

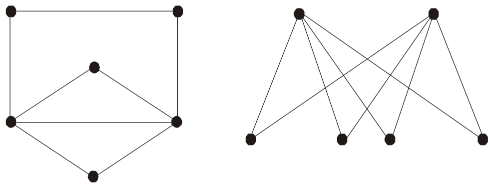
41. How many binary trees are possible with three vertices ?

42. Determine the number of spanning trees of the complete bipartite graph $K_{2,\,n}$.

Answers 3.1

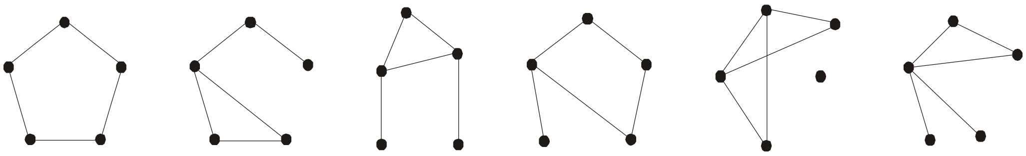
1. (*i*) 16 (*ii*) 13

3. $e = 8$;



4. P = 6

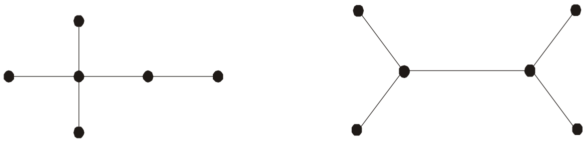
7.



16. Ae_1Be_4R, $Ae_1Be_3Qe_6R$, $Ae_2Pe_5Qe_6R$, $Ae_2Pe_5Qe_3Be_4R$
Lengths : two, three and four.

29. 13

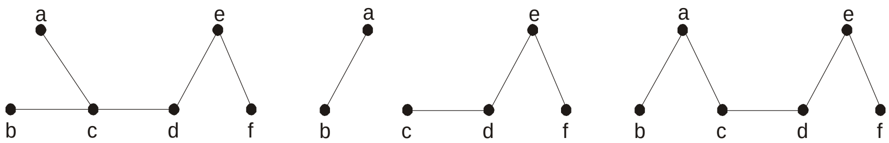
34.



35. 26 vertices and 25 edges in T_2 ; 25 vertices and 24 edges in T_1 and 1 edge is removed from the tree T.

39. $2^{K+1} - 1$.

52.



Answers 3.2

13. 8

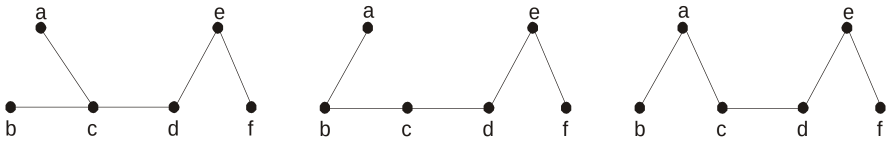
18. (*i*) There are 6 vertices adn 5 edges, vertices A, B, Q, R are pendant vertices and vertices C and P have degree 3.

(*ii*) There are 5 vertices and 7 edges ; vertices P and Q have degree 2, S and T have degree 3 and Q has degree 4.

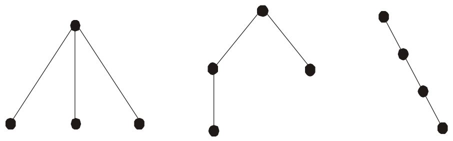
35. Circuit of least weight : ADBCA

least total weight 23.

36.



37.

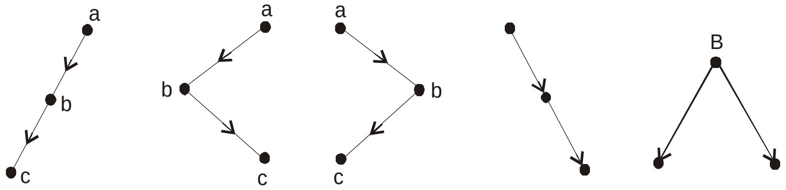


39. There are $\binom{n}{2}$ possible edges from which we choose $n - 1$. The number of graphs is therefore

$\binom{\binom{n}{2}}{n-1}$. The number of trees on n labelled vertices is n^{n-2}. For $n \le 6$ the table shows the numbers of trees V'*s* graphs

n	No. of trees	No. of graphs
2	1	1
3	3	3
4	16	20
5	125	210
6	1296	3003

41. Five



Unit-4

Cutsets, Networkflows and Planar Graphs

4.1 CUT VERTEX, CUT SET AND BRIDGE

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components. A cut vertex of a connected graph G is a vertex whose removal increases the number of components. Clearly if v is a cut vertex of a connected graph G, G – v is disconnected.

A cut vertex is also called a **cut point.**

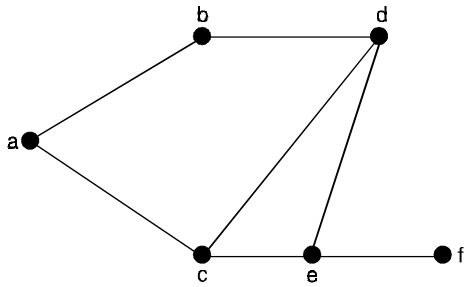
Analogously, an edge whose removal produces a graph with more connected components then the original graph is called a **cut edge** or **bridge.**

The set of all minimum number of edges of G whose removal disconnects a graph G is called a **cut set** of G. Thus a cut set S of a satisfy the following :

(*i*) S is a subset of the edge set E of G.

(*ii*) Removal of edges from a connected graph G disconnects G.

(*iii*) No proper subset of G satisfy the condition.



Fig, 4.1

In the graph in Figure below, each of the sets $\{\{b, d\}, \{c, d\}, \{c, e\}\}$ and $\{\{e, f\}\}$ is a cut set. The edge $\{e, f\}$ is the only bridge. The singleton set consisting of a bridge is always a cut of set of G.

4.2 CONNECTED OR WEAKLY CONNECTED

A directed graph is called connected at weakly connected if it is connected as an undirected graph in which each directed edge is converted to an undirected graph.

4.3 UNILATERALLY CONNECTED

A simple directed graph is said to be unilaterally connected if for any pair of vertices of the graph atleast one of the vertices of the pair is reachable from other vertex.

4.4 STRONGLY CONNECTED

A directed graph is called strongly connected if for any pair of vertices of the graph both the vertices of the pair are reachable from one another.

For the diagraphs is Fig. (4.2) the digraph in (*a*) is strongly connected, in a (*b*) it is weakly connected, while in (*c*) it is unilaterally connected but not strongly connected.

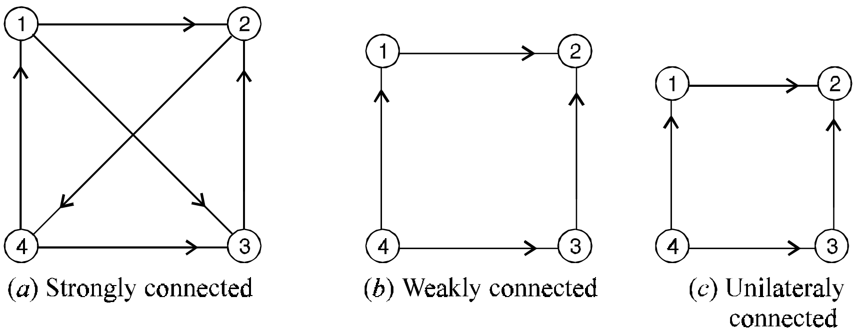


Fig. 4.2. Connectivity in digraphs.

Note that a unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilateraly connected. A strongly connected digraph is both unilaterally and weakly connected.

4.5 CONNECTIVITY

To study the measure of connectedness of a graph G we consider the minimum number of vertices and edges to be removed from the graph in order to disconnect it.

4.6 EDGE CONNECTIVITY

Let G be a connected graph. The edge connectivity of G is the minimum number of edges whose removal results in a disconnected or trivial graph. The edge connectivity of a connected graph G is denoted by λ(G) or E(G).

(*i*) If G is a disconnected graph, then λ(G) or E(G) = 0.

(*ii*) Edge connectivity of a connected graph G with a bridge is 1.

4.7 VERTEX CONNECTIVITY

Let G be a connected graph. The vertex connectivity of G is the minimum number of vertices whose removal results in a disconnected or a trivial graph. The vertex connectivity of a connected graph is denoted by *k*(G) or V(G)

(*i*) If G is a disconnected graph, then λ(G) or E(G) = 0.

(*ii*) Edge connectivity of a connected graph G with a bridge is 1.

(*iii*) The complete graph k_n cannot be disconnected by removing any number of vertices, but the removal of $n - 1$ vertices results in a trivial graph. Hence $k(k_n) = n - 1$.

(*iv*) The vertex connectivity of a graph of order atleast there is one if and only if it has a cut vertex.

(*v*) Vertex connectivity of a path is one and that of cycle C_n ($n \geq 4$) is two.

Problem 4.1. *Find the (i) vertex sets of components*

(ii) cut-vertices and (iii) cut-edges of the graph given below.

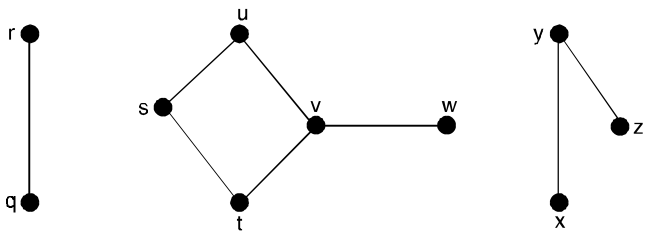


Fig. 4.3.

Solution. The graph has three components. The vertex set of the components are {*q*, *r*}, {*s*, *t*, *u*, *v*, *w*} and {*x*, *y*, *z*}. The cut vertices of the graph are *t* and *y*.

Its cut-edges are *qr*, *st*, *xy* and *yz*.

Problem 4.2. *Is the directed graph given below strongly connected ?*

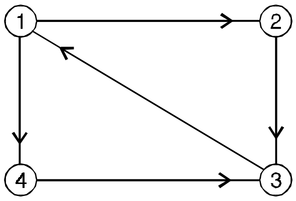


Fig. 4.4.

Solution. The possible pairs of vertices and the forward and the backward paths between them are shown below for the given graph.

Pairs of Vertices	*Forward path*	*Backward path*
(1, 2)	1–2	2–3–1
(1, 3)	1–2–3	3–1
(1, 4)	1–4	4–3–1
(2, 3)	2–3	3–1–2
(2, 4)	2–3–1–4	4–3–1–2
(3, 4)	4–3	4–3

Therefore, we see that between every pair of distinct vertices of the given graph there exists a forward as well as backward path, and hence it is strongly connected.

Theorem 4.1. *The edge connectivity of a graph G cannot exceed the minimum degree of a vertex in G, i.e.,* $\lambda(G) \leq \delta(G)$.

Theorem 4.7. *Let v be a point a connected graph G. The following statements are equivalent*

(1) v is a cutpoint of G

(2) There exist points u and v distinct from v such that v is on every u–w path.

(3) There exists a partition of the set of points V–{v} into subsets U and W such that for any points $u \in U$ *and* $w \in W$*, the point v is on every u–w path.*

Proof. (1) implies (3)

Since v is a cutpoint of G, G–v is disconnected and has atleast two components. Form a partition of V–{v} by letting U consist of the points of one of these components and W the points of the others.

The any two points $u \in$ U and $w \in$ W lie in different components of G–v.

Therefore every u–w path in G contains v.

(3) implies (2)

This is immediate since (2) is a special case of (3).

(2) implies (1)

If v is on every path in G joining u and w, then there cannot be a path joining these points in G–v.

Thus G–v is disconnected, so v is a cutpoint of G.

Theorem 4.2. *Every non trivial connected graph has atleast two points which are not cutpoints.*

Proof. Let u and v be points at maximum distance in G, and assume v is a cut point.

Then there is a point w in a different component of G–v than u.

Hence v is in every path joining u and w, so $d(u, w) > d(u, v)$ which is impossible.

Therefore v and similarly u are not cut points of G.

Theorem 4.3. *Let x be a line of a connected graph G. The following statements are equivalent :*

(1) x is a bridge of G

(2) x is not on any cycle of G

(3) There exist points u and v of G such that the line x is on every path joining u and v.

(4) These exists a partition of V into subsets U and W such that for any points $u \in U$ *and* $w \in W$*, the line x is on every path joining u and w.*

Theorem 4.4. *A graph H is the block graph of some graph if and only if every block of H is complete*.

Proof. Let H = B(G), and assume there is a block H_i of H which is not complete.

Then there are two points in H_i which are non adjacent and lie on a shortest common cycle Z of length atleast 4.

But the union of the blocks of G corresponding to the points of H_i which lie on Z is then connected and has no cut point, so it is itself contained in a block, contradicting the maximality property of a block of a graph.

On the otherhand, let H be a given graph in which every block is complete.

From B(H), and then form a new graph G by adding to each point H_i of B(H) a number of end lines equal to the number of points of the block H_i which are not cut points of H. Then it is easy to see that B(G) is isomorphic to H.

Theorem 4.5. *Let G be a connected graph with atleast three points. The following statements are equivalent :*

(1) *G is a block*

(2) *Every two points of G lie on a common cycle*

(3) *Every point and line of G lie on a common cycle.*

(4) *Every two lines of G lie on a common cycle*

(5) *Given two points and one line of G, there is a path joining the points which contains the line.*

(6) *For every three distinct points of G, there is a path joining any two of them which contains the third.*

(7) *For every three distinct points of G, there is a path joining any two of them which does not contain the third.*

Proof. (1) implies (2)

Let u and v be distinct points of G and let U be the set of points different from u which lie on a cycle containing u.

Since G has atleast three points and no cutpoints, it has no bridges.

Therefore, every point adjacent to u is in U, so U is not empty.

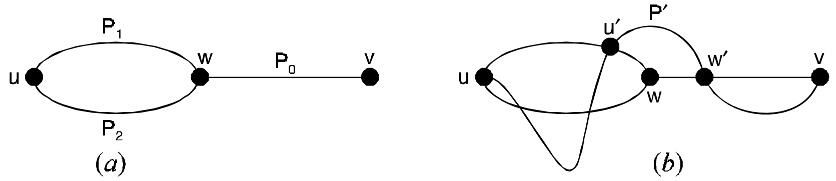


Fig. 4.5. Paths in blocks.

Suppose v is not in U. Let w be a point in U for which the distance $d(w, v)$ is minimum.

Let P_0 be a shortest w–v path, and let P_1 and P_2 be the two u–w paths of a cycle containing u and w (see Fig. 4.5(*a*)).

Since w is not a cutpoint, there is a u–v path P′ not containing w (see Fig. 4.5(*b*)).

Let w' be the point nearest u in P′ which is also in P_0 and let u be the last point of the u–w subpath of P′ in either P_1 or P_2. Without loss of generality, we assume u' is in P_1.

Let Q_1 be the u–w' path consisting of the u–u' subpath of P_1 and the u'–w' subpath of P′.

Let Q_2 be the u–w' path consisting of P_2 followed by the w–w' subpath of P_0. Then Q_1 and Q_2 are disjoint u–w' paths. Together they form a cycle, so w' is in U. Since w' is on a shortest w–v path, $d(w', v) < d(w, v)$. This contradicts our choice of w, proving that u and v do lie on a cycle.

(2) implies (3)

Let u be a point and vw a line of G.

Let z be a cycle containing u and v. A cycle z' containing u and vw can be formed as follows.

If w is on z then z' consists of vw together with the v–w path of z containing u.

If w is not on z there is a w–u path P not containing v, since otherwise v would be a cutpoint.

Let u' be the first point of P in z. Then z' consists of vw followed by the w–u' subpath of P and the u'–v path in z containing u.

(3) implies (4)

This proof is analogous to the preceding one, and the details are omitted.

(4) implies (5)

Any two points of G are incident with one line each, which lie on a cycle by (4).

Hence any two points of G lie on a cycle, and we have (2) so also (3).

Let u and v be distinct points and x a line of G.

By statement (3), there are cycles z_1 containing u and x, and z_2 containing v and x.

If v is on z_1 or u is on z_2, there is clearly a path joining u and v containing x.

Thus we need only consider the case where v is not on z_1 and u is not on z_2.

Begin with u and proceed along z_1 until reaching the first point w of z_2, then take the path on z_2 joining w and v which contains x.

This walk constitutes a path joining u and v that contains x.

(5) implies (6)

Let u, v and w be distinct points of G and let x be any line incident with w. By (5), there is a path joining u and v which contains x and hence must contain w.

(6) implies (7)

Let u, v and w be distinct points of G. By statement (6) there is a u–w path P containing v. The u–v subpath of P does not contain w.

(7) implies (1)

By statement (7), for any two points u and v, no point lies on every u–v path.

Hence, G must be a block.

Problem 4.3. *Find the V(G), E(G) and deg (G) for the graph of the Figure (4.6).*

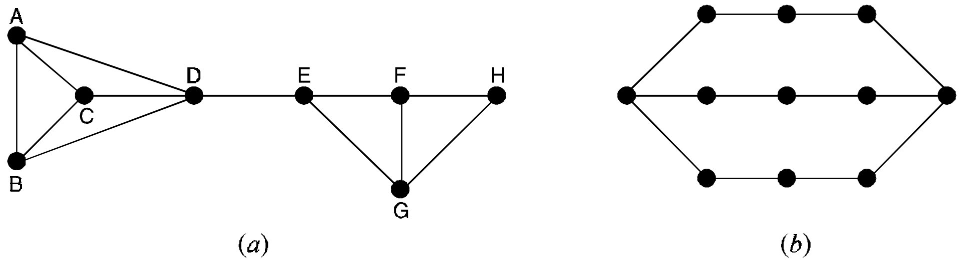


Fig. 4.6

Solution. (*a*) The degree of the graph G, deg (G) = 5.

If we remove node D from the graph then graph becomes two components graph.

Thus, V(G) = 1.

By the removal of arcs (D, H) and (D, E) the graph G turns into two components graph.

Hence E(G) = 2.

(*b*) Here deg(G) = 3

V(G) = 2 and E(G) = 2.

Problem 4.4. *Find the E(G) and V(G) of the graph shown in Figure (4.7).*

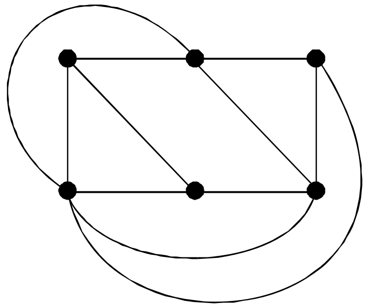


Fig. 4.7.

Solution. To calculate number of arc disjoint paths between any pair of nodes, maximum flow between that pair of node is calculated.

The procedure is shown in the given table. It is assumed that :

(*i*) an arc can carry only one unit of flow and

(*ii*) a node has infinite capacity.

S.No.	*Node Pair*	*Maximum Flow*	*Remark*
1.	(1, 2)	3	Three arcs from node 1 can carry at the most 3 units of flow and node 2 can receive all of them.
2.	(1, 3)	3	same as above
3.	(1, 4)	3	same as above
4.	(1, 5)	3	same as above
5.	(1, 6)	3	same as above
6.	(2, 3)	3	Though node 2 can send 4 units of flow, node 3 can accept only 3 units.
7.	(2, 4)	4	Node 2 can sent 4 units and node 4 can accept all of them.
8.	(2, 5)	3	same as in sl. no. 6
9.	(2, 6)	4	same as in sl. no. 7
10.	(3, 4)	3	same as in sl. no. 1
11.	(3, 5)	3	same as in sl. no. 1
12.	(3, 6)	3	same as in sl. no. 1
13.	(4, 5)	3	same as in sl. no. 6
14.	(4, 6)	4	same as in sl. no. 7
15.	(5, 6)	3	same as in sl. no. 1

The minimum value of maximum flow between any pair of node 3. This is the count of minimum number of arc disjoint path between any pair of nodes in G.

Hence E(G) = 3.

Similarly, we can compute V(G) of the graph.

The following assumptions are made to compute node disjoint path between one node to another :

(*i*) Arc has infinite capacity so it can carry any amount of flow.

(*ii*) Any intermediate node in the path can accept I units of flow along any one incoming arc and can pass only one unit at a time along any one outgoing arc. If an intermediate node *b* has 5 incoming arcs from a node *a* then *b* can accept only one unit of flow from *a*.

Similarly if *b* has 4 outgoing arcs, it can pass only one unit of flow along any one out of four arcs.

(*iii*) If nodes are adjacent then it can sustain loss of all other nodes, so maximum flow is assumed to be $n - 1$, where n is $|V|$.

The calculation is shown in the given table. From the table, it is clear that V(G) = 3.

S.No.	*Node Pair*	*Maximum Flow*	*Remark*
1.	(1, 2)	$n - 1$	Both nodes 1 and 2 are adjacent.
2.	(1, 3)	3	Node disjoint paths are (1, 2, 3), (1, 5, 4, 3) and (1, 6, 3)
3.	(1, 4)	3	Node disjoint paths are (1, 2, 4), (1, 5, 4) and (1, 6, 4)
4.	(1, 5)	$n - 1$	Both nodes 1 and 5 are adjacent
5.	(1, 6)	$n - 1$	Both nodes 1 and 6 are adjacent
6.	(2, 3)	$n - 1$	Both nodes 2 and 3 are adjacent
7.	(2, 4)	$n - 1$	Both nodes 2 and 4 are adjacent
8.	(2, 5)	3	Noded disjoint paths are : (2, 1, 5), (2, 4, 5) and (2, 6, 5)
9.	(2, 6)	$n - 1$	Both nodes 2 and 6 are adjacent
10.	(3, 4)	$n - 1$	Both nodes 3 and 4 are adjacent
11.	(3, 5)	3	Noded disjoint paths are : (3, 4, 5), (3, 2, 1, 5) and (3, 6, 5)
12.	(3, 6)	$n - 1$	Both nodes 1 and 2 are adjacent
13.	(4, 5)	$n - 1$	Both nodes 1 and 2 are adjacent
14.	(4, 6)	$n - 1$	Both nodes 1 and 2 are adjacent
15.	(5, 6)	$n - 1$	Both nodes 1 and 2 are adjacent

Theorem 4.6. *In any graph G, V(G) ≤ E(G) ≤ deg (G).*

Proof. Let deg(G) = n.

Then there exists a node V in G such that degree of V is n.

If we drop all arcs for which V is an incidence (a node is called an incidence of an arc if the node is either a start or an end point of the arc), the graph becomes disconnected.

Thus, E(G) cannot exceed n otherwise there exists a node which is incidence of $m > n$ number of arcs. That is in contradiction with the assumption that

$$\text{deg(G)} = n$$

Thus, $$\text{E(G)} \le \text{deg(G)} \quad ...(1)$$

Next, Let E(G) = r.

Then there exist a pair of nodes such that there are r disjoint paths between them.

These r paths may cross through S ≤ r number of nodes.

If we remove these s nodes from the graph, the r arcs get deleted from the graph making the graph a disconnected.

That means V(G) cannot exceed r.

Thus V(G) ≤ E(G) ...(2)

Combining results (1) and (2), we have

$$V(G) \leq E(G) \leq \deg(G).$$

Theorem 4.7. *Let v is a cut point of a connected graph G = (V, E). The remaining set of vertex V–{v} can be partitioned into two non empty disjoint subsets U and W such that for any node u ∈ U and w ∈ W, the node v lies on every u–w path.*

Proof. When cut point v is removed from G it becomes disconnected.

Let U be a set of vertices of the largest connected subgraph of G and W = V – {v} – U.

Let v is not on every u–w path.

This implies that a path from u to w exists even after removal of v from G.

That means U is not the set of vertices of largest connected subgraph of G after removal of v.

This is a contrary to the assumption that U is the largest component.

Hence v lies on every u–w path.

4.8 TRANSPORT NETWORKS

Let N = (V, E) be a loop-free connected directed graph. Then N is called a **network,** or **transport network,** if the following conditions are satisfied :

(*i*) There exists a unique vertex $a \in$ V with $id(a)$, the in degree of a, equal to O. This vertex a is called the **source.**

(*ii*) There is a unique vertex $z \in$ V, called the **sink,** where $od(z)$, the out degree of z, equals O.

(*iii*) The graph N is weighted, so there is a function from E to the set of non negative integers that assigns to each edge $e = (v, w) \in$ E a capacity, denoted by $c(e) = c(v, w)$.

If N = (V, E) is a transport network, a function f from E to the non negative integers is called a **flow** for N if

(*i*) $f(e) \leq c(e)$ for each edge $e \in$ E, and

(*ii*) for each $v \in$ V, other than the source a or the sink z, $\sum_{w \in V} f(w, v) = \sum_{w \in V} f(v, w)$

If there is no edge (v, w), then $f(v, w) = 0$.

Let f be a flow for a transport network N = (V, E)

(*i*) An edge e of the network is called **saturated** if $f(e) = c(e)$. When $f(e) < c(e)$, the edge is called **unsaturated.**

(*ii*) If a is the source of N, then $\text{val}(f) = \sum_{v \in V} f(a, v)$ is called the value of the **flow.**

If N = (V, E) is a transport network and C is a cut-set for the undirected graph associated with N, then C is called a **cut,** or an ***a–z* cut,** if the removal of the edges in C from the network results in the separation of a and z.

For example, the graph in Fig. (4.8) is a transport network. Here vertex a is the source, the sink is at vertex z, and capacities are shown beside each edge. Since $c(a, b) + c(a, g) = 5 + 7 = 12$, the amount of the commodity being transported from a to z cannot exceed 12. With $c(d, z) + c(h, z) = 5 + 6 = 11$, the amount is further restricted to be no greater than 11.

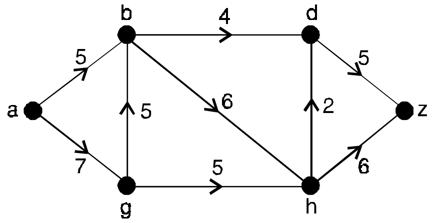


Fig. 4.8.

For the network in Fig. (4.9), the label x, y on each edge e is determined so that $x = c(e)$ and y is the value assigned for a possible flow f. The label on each edge e satisfies $f(e) \le c(e)$.

In part (a) of the Fig. (4.9), the flow into vertex g is 5, but the flow out from that vertex is $2 + 2 = 4$.

Hence the function f is not a flow in this case.

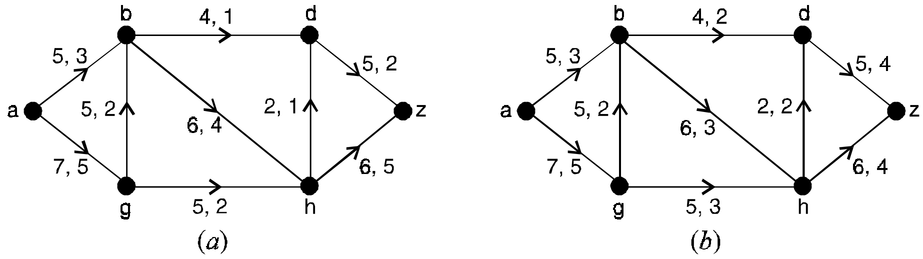


Fig. 4.9.

For the network in Fig. 4.9(*b*), only the edge (h, d) is saturated. All other edges are unsaturated.

The value of the flow in this network is

$$\text{val}(f) = \sum_{v \in V} f(a, v) = f(a, b) + f(a, g) = 3 + 5 = 8.$$

We observe that in the network of Fig. (4.68) (*b*)

$$\sum_{v \in V} f(a, v) = 3 + 5 = 8 = 4 + 4 = f(d, z) + f(h, z) = \sum_{v \in V} f(v, z).$$

Consequently, the total flow leaving the source a equals the total flow into the sink z.

Fig. (4.10) indicates a cut for the given network (dotted curves). The cut C, consists of the undirected edges $\{a, g\}$, $\{b, d\}$, $\{b, g\}$ and $\{b, h\}$. This cut partitions the vertices of the network into the two sets $P = \{a, b\}$ and its complement $\overline{P} = \{d, g, h, z\}$, so C_1 is denoted as $(P, \overline{P})$.

The capacity of a cut, denoted $C(P, \overline{P})$, is defined by $C(P, \overline{P}) = \sum_{\substack{v \in P \\ w \in \overline{P}}} C(v, w)$, the sum of the capacities of all edges (v, w), where $v \in P$ and $w \in \overline{P}$.

In this example, $C(P, \overline{P}) = c(a, g) + c(b, d) + c(b, h) = 7 + 4 + 6 = 17$.

The cut c_2 induces the vertex partition $Q = \{a, b, g\}$.

$\overline{Q} = \{d, h, z\}$ and has capacity $c(Q, \overline{Q}) = c(b, d) + c(b, h) + c(g, h) = 4 + 6 + 5 = 15$.

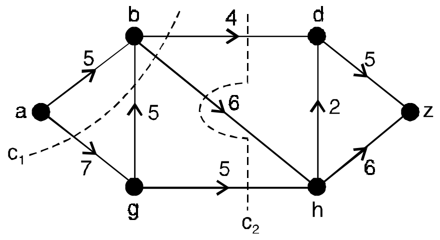


Fig. 4.10.

Theorem 4.8. *Let f be a flow in a network N = (V, E). If C = (P, $\overline{P}$) is any cut in N, then val(f) cannot exceed c(P, $\overline{P}$).*

Proof. Let vertex a be the source in N and vertex z the sink. Since $id(a) = 0$, it follows that for all $w \in V, f(w, a) = 0$.

Consequently, $\text{val}(f) = \sum_{v \in V} f(a, v) = \sum_{v \in V} f(a, v) - \sum_{w \in V} f(w, a)$

By the definition of a flow, for all $x \in P$, $x \neq a$,

$$\sum_{v \in V} f(x, v) - \sum_{w \in V} f(w, x) = 0.$$

Adding the results in the above equations yields

$$\text{val}(f) = \left[\sum_{v \in V} f(a, v) - \sum_{w \in V} f(w, a)\right] + \sum_{\substack{w \in P \\ x \neq a}} \left[\sum_{v \in V} f(x, v) - \sum_{w \in V} f(w, x)\right]$$

$$= \sum_{\substack{x \in P \\ v \in V}} f(x, v) - \sum_{\substack{x \in P \\ w \in V}} f(w, x)$$

$$= \left[\sum_{\substack{x\in P\\ v\in P}} f(x,v) + \sum_{\substack{x\in P\\ v\in \overline{P}}} f(x,v)\right] - \left[\sum_{\substack{x\in P\\ w\in P}} f(w,x) + \sum_{\substack{x\in P\\ w\in \overline{P}}} f(w,x)\right]$$

Since $\sum_{\substack{x\in P\\ v\in P}} f(x,v)$ and $\sum_{\substack{x\in P\\ w\in P}} f(w,x)$ are summed over the same set of all ordered pairs in $P \times P$, these summations are equal.

Consequently, $\text{val}(f) = \sum_{\substack{x\in P\\ v\in \overline{P}}} f(x,v) - \sum_{\substack{x\in P\\ w\in \overline{P}}} f(w,x)$

For all $x, w \in v, f(w, x) \geq 0$, so

$$\sum_{\substack{x\in P\\ w\in \overline{P}}} f(w,x) \geq 0 \text{ and } \text{val}(f) \leq \sum_{\substack{x\in P\\ v\in \overline{P}}} f(x,v) \leq \sum_{\substack{x\in P\\ v\in \overline{P}}} c(x,v) = c(P, \overline{P}).$$

Corollary :

If f is a flow in a transport network N = (V, E), then the value of the flow from the source a is equal to the value of the flow into the sink z.

Proof. Let $P = \{a\}$, $\overline{P} = V - \{a\}$, and $Q = V - \{z\}$.

From the above observation,

$$\sum_{\substack{x\in P\\ v\in \overline{P}}} f(x,v) - \sum_{\substack{x\in P\\ w\in \overline{P}}} f(w,x) = \text{val}(f) = \sum_{\substack{y\in Q\\ v\in \overline{Q}}} f(y,v) - \sum_{\substack{y\in Q\\ w\in \overline{Q}}} f(w,y)$$

With $P = \{a\}$ and $id(a) = 0$, we find that

$$\sum_{\substack{x\in P\\ w\in \overline{P}}} f(w,x) = \sum_{w\in \overline{P}} f(w,a) = 0.$$

Similarly, for $\overline{Q} = \{z\}$ and $od(z) = 0$, it follows that

$$\sum_{\substack{y\in Q\\ w\in \overline{Q}}} f(w,y) = \sum_{y\in Q} f(z,y) = 0$$

Consequently, $\sum_{\substack{x\in P\\ v\in \overline{P}}} f(x,v) = \sum_{v\in \overline{P}} f(a,v) = \text{val}(f)$

$$= \sum_{\substack{y\in Q\\ v\in \overline{Q}}} f(y,v) = \sum_{y\in Q} f(y,z).$$

Theorem 4.9. *The value of any flow in a given transport network is less than or equal to the capacity of any cut in the network.*

Proof. Let ϕ be a flow and $(P, \bar{P})$ be a cut in a transport network. For the source a,

$$\sum_{\text{all } i} \phi(a, i) - \sum_{\text{all } j} \phi(j, a) = \sum_{\text{all } i} \phi(a, i) = \phi v \qquad ...(1)$$

Since $\phi(j, a) = 0$ for any j. For a vertex P other than a in P,

$$\sum_{\text{all } i} \phi(P, i) - \sum_{\text{all } j} \phi(j, P) = 0 \qquad ...(2)$$

Combining (1) and (2), we have

$$\phi_v = \sum_{p \in P} \left[\sum_{\text{all } i} \phi(P, i) - \sum_{\text{all } j} \phi(j, P) \right]$$

$$= \sum_{p \in P\,;\,\text{all } i} \phi(P, i) - \sum_{p \in P\,;\,\text{all } j} \phi(j, P)$$

$$= \sum_{p \in P\,;\,i \in P} \phi(P, i) + \sum_{p \in P\,;\,i \in \bar{P}} \phi(P, i) - \left[\sum_{p \in P\,;\,j \in P} \phi(j, P) + \sum_{p \in P\,;\,j \in \bar{P}} \phi(j, P) \right] \qquad ...(3)$$

Note that $\displaystyle\sum_{p \in P\,;\,i \in P} \phi(P, i) = \sum_{p \in P\,;\,j \in P} \phi(j, P)$

because both sums run through all the vertices in P. Thus, (3) becomes

$$\phi_v = \sum_{p \in P\,;\,i \in \bar{P}} \phi(P, i) - \sum_{p \in P\,;\,j \in \bar{P}} \phi(j, P) \qquad ...(4)$$

But, sine $\displaystyle\sum_{p \in P\,;\,j \in \bar{P}} \phi(j, P)$ is always a non-negative quantity.

We have

$$\phi_v \leqq \sum_{p \in P\,;\,i \in \bar{P}} \phi(P, i) \leqq \sum_{p \in P\,;\,i \in \bar{P}} w(P, i) = w(P, \bar{P}).$$

Theorem 4.10. *In any directed network, the value of an (s, t)–flow never exceeds the capacity of any (s, t)–cut.*

Proof. Let $F = \{f_{uv}\}$ be any (s, t)–flow and $\{S, T\}$ any (s, t)–cut.

Conservation of flow tells us that $\displaystyle\sum_{v} f_{uv} - \sum_{v} f_{vu} = 0$

for any $u \in s, u \neq s$. (the possibility $u = t$ is excluded because $t \notin S$)

Hence, $\text{val}(F) = \displaystyle\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

$$= \sum_{u \in S} \left(\sum_{v \in V} f_{uv} - \sum_{v \in V} f_{vu} \right)$$

(since the term in parentheses is 0 except for $u = s$)

$$= \sum_{u \in S,\, v \in V} f_{uv} - \sum_{u \in S,\, v \in V} f_{vu}\ .$$

Since {S, T} is a partition, this last sum can be written

$$\sum_{u \in S,\, v \in S} f_{uv} + \sum_{u \in S,\, v \in T} f_{uv} - \sum_{u \in S,\, v \in S} f_{vu} - \sum_{u \in S,\, v \in T} f_{vu}$$

$$= \sum_{u \in S,\, v \in S} f_{uv} - \sum_{u \in S,\, v \in S} f_{vu} + \sum_{u \in S,\, v \in T} (f_{uv} - f_{vu})$$

The first two terms in the line are the same, so we obtain

$$\text{val(F)} = \sum_{u \in S,\, v \in T} (f_{uv} - f_{vu})\ .$$

But $f_{uv} \le C_{uv}$ and $f_{vu} \ge 0$, so $f_{uv} - f_{vu} \le C_{uv}$ for all u and v.

Therefore, $\text{val(F)} \le \sum_{u \in S,\, v \in T} C_{uv} = \text{cap(S, T)}$ as desired.

Corollary 1.

If F is any (s, t)–flow and (S, T) is any (s, t)–cut, then val(F), $\sum_{u \in S,\, v \in T} (f_{uv} - f_{vu})$.

With refernece to the network in Fig. (4.11) and the cut S = {s, a, c}, T = {b, d, t}, the sum specified in the corollary is

$$\sum_{u \in S,\, v \in T} (f_{uv} - f_{vu}) = f_{sb} + f_{ad} - f_{bc} + f_{ct} = 2 + 0 - 1 + 11 = 12,$$

which is the value of the flow in this network.

Corollary 2.

Suppose there exists some (s, t)–flow F and some (s, t)–cut {S, T} such that the value of F equals the capacity of {S, T}. Then val(F) is the maximum value of any flow and cap(S, T) is the minimum capacity of any cut.

Proof. Let F_1 be any flow. To see that $\text{val}(F_1) \le \text{val(F)}$, note that the theorem says that $\text{val}(F_1) \le$ cap(S, T) and, by hypothesis, cap(S, T) = val(F).

So val(F) is maximum.

In any directed network, there is always a flow and a cut such that the value of the flow is the capacity of the cut, such a flow has maximum value.

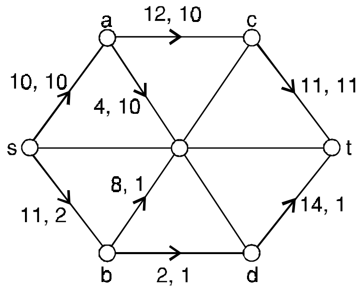


Fig. 4.11. **(*s*, *t*)-flow.**

4.9 MAX–FLOW MIN–CUT THEOREM

In any network, the value of any maximum flow is equal to the capacity of any minimum cut.

First proof :

Suppose first that the capacity of each arc is an integer. Then the network can be regarded as a digraph D whose capacities represent the number of arcs connecting the various vertices (as in Figs. (4.12) and (4.13)).

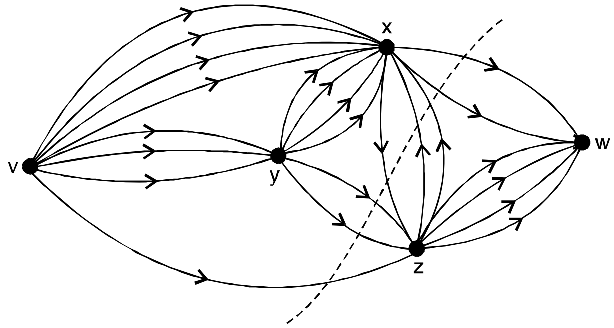


Fig. 4.12.

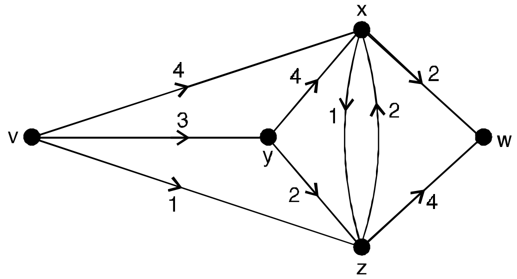


Fig. 4.13

The value of a maximum flow is the total number of arc-disjoint paths from v to w in D, and the capacity of a minimum cut is the minimum number of arcs in a vw-disconnecting set of D.

The extension of this result to networks in which the capacities are rational numbers is effected by multiplying these capacities by a suitable integer d to make them integers.

We then have the case described above, and the result follows on dividing by d.

Finally, if some capacities are irrational, then we approximate them as closely as we please by rationals and use the above result.

By choosing these rationals carefully, we can ensure that the value of any maximum flow and the capacity of any minimum cut are altered by an amount that is as small as we wish.

Note that, in practical examples, irrational capacities rarely occur, since the capacities are usually given in decimal form.

Second Proof

Since the value of any maximum flow cannot exceed the capacity of any minimum cut, it is sufficient to prove the existence of a cut whose capacity is equal to the value of a given maximum flow.

Let ϕ be a maximum flow. We define two sets V and W of vertices of the network as follows.

If G is the underlying graph of the network, then a vertex z is contained in V, if and only if there exists in G a path $v = v_0 \to v_1 \to v_2 \to \ldots\ldots \to v_{m-1} \to v_m = z$, such that each edge v_iv_{i+1} corresponds either to an unsaturated arc v_iv_{i+1}, or to an arc $v_{i+1}v_i$ that carries a non-zero flow. The set W consists of all those vertices that do not lie in V.

For example, in Fig. (4.14), the set V consists of the vertices v, x and y, and the set W consists of the vertices z and w.

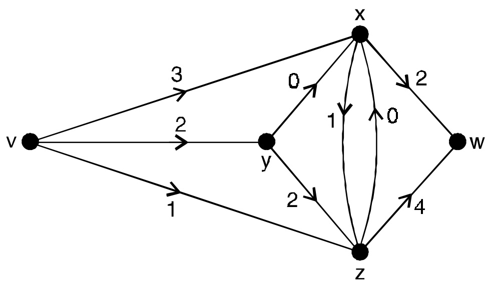


Fig. 4.14.

Clearly, v is contained in V. We now show that W contains the vertex w.

If this is not so, then w lies in V, and hence there exists in G a path $v \to v_1 \to v_2 \to \ldots\ldots \to v_{m-1} \to w$ of the above type.

We now choose a positive number ε that does not exceed the amount needed to saturate any unsaturated arc v_iv_{i+1}, and does not exceed the flow in any arc $v_{i+1}v_i$ that carries a non-zero flow.

It is now easy to see that, if we increase by ε the flow in all arcs of the first type and decrease by ε the flow in all arcs of the second type, then we increase the value of the flow by ε, contradicting our assumption that ϕ is a maximum flow.

It follows that w lies in W.

To complete the argument, we let E be the set of all arcs of the form *xz,* where *x* is in V and *z* is in W.

Clearly E is a cut. Moreover, each arc *xz* of E is saturated and each arc *zx* carries a zero flow, since otherwise *z* would also be an element of V. If follows that the capacity of E must equal the value of ϕ, and that E is the required minimum cut.

Remark. When applying this theorem, it is often simplest to find a flow and a cut such that the value of the flow equals the capacity of the cut. It follows from the theorem that the flow must be a maximum flow and that the cut must be a minimum cut. If all the capacities are integers, then the value of a maximum flow is also an integer, this turns out to be useful in certain applications of network flows.

Problem 4.5. *Find a maximum flow in the directed network shown in Fig. (4.15) and prove that it is a maximum.*

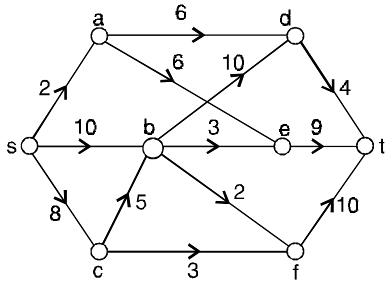


Fig. 4.15. A directed network.

Solution. We start by sending a flow of 2 units through the path *sadt*, a flow of 3 units through *sbet*, and a flow of 3 units through *scft*, obtaining the flow shown on the left in Fig. (4.16).

Continue by sending flows of 2 units through *sbdt* and 2 units through *sbft*, obtaining the flow shown on the right in Fig. (4.16)

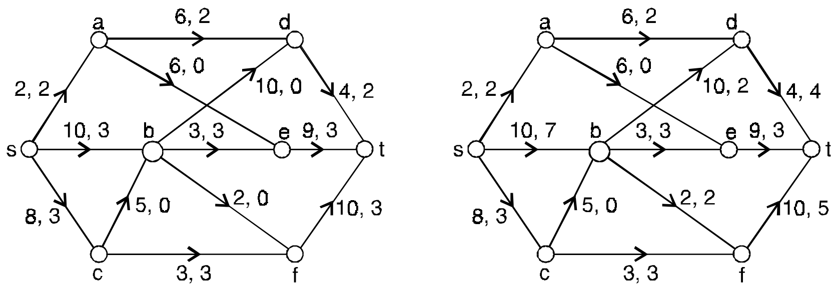


Fig. 4.16.

At this point, there are no further flow-augmenting chains from *s* to *t* involving only forward arcs.

However, we can use the backward arc *da* to obtain a flow-augmenting chain *scbdaet*.

Since the slack of this chain is 2, we add a flow of 2 to *sc*, *cb, bd, ae*, and *et*, and subtract 2 from *ad*.

The result is shown in Fig. (4.17).

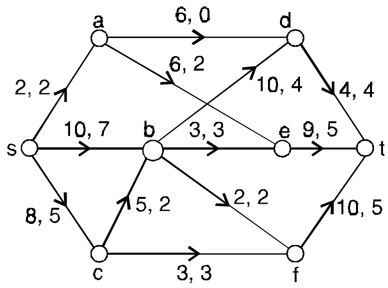


Fig. 4.17.

A search for further flow-augmenting chains takes us from *s* to *c* or *b* and on to *d*, where we are stuck.

This tells us that the current flow (of value 14) is maximum.

It also presents us with a cut verifying maximality, namely, S = {*s*, *b*, *c*, *d*} (those vertices reachable from *s* by flow-augmenting chains) and T = {*a*, *e*, *f*, *t*} (the complement of *s*).

The capacity of this cut is

$$C_{sa} + C_{be} + C_{bf} + C_{cf} + C_{dt} = 2 + 3 + 2 + 3 + 4 = 14.$$

Since this is the same as the value of the flow, we have verified that our flow is maximum.

Problem 4.6. *Why does the procedure just described of adding an amount q to the forward arcs of a chain and subtracting the same amount from the backward arcs preserve conservation of flow at each vertex ?*

Solution. The flow on the arcs incident with a vertex not on the chain are not changed, so conservation of flow continues to hold at such a vertex. What is the situation at a vertex on the chain. Remember that a chain in a directed network is just a trail whose edges can be followed in either direction, this, each vertex on a chain is incident with exactly two arcs.

Suppose a chain contains the arcs *uv, vw* (in that order) and that the flows on these arcs before changes are f_{uv} and f_{vw}.There are essentially two cases to consider.

Case 1.

Suppose the situation at vertex *v* in the network is $u \to v \to w$.

In this case, both *uv* and *vw* are forward arcs, so each has the flow increased by *q*.

The total flow into *v* increases by *q*, but so does the total flow out of *v*, so there is still conservation of flow at *v*. (the analysis is similar if the situation at *v* is $u \leftarrow v \leftarrow w$).

Case 2.

The situation at *v* is $u \to v \leftarrow w$.

Here the flow on the forward arc *uv* is increased by *q* and the flow in the backward arc *wv* is decreased by *q*. There is no change in the flow out of *v*.

Neither is there any change in the flow in *v* since the only terms in the sum $\sum_r f_{rv}$ which change occur with $r = u$ and $r = w$, and these become, respectively, $f_{uv} + q$ and $f_{vw} - q$. (the analysis is similar if the situation at *v* is $u \leftarrow v \to w$).

Problem 4.7. *Verify the law of conservation at vertices a, b and d.*

Solution. The law of conservation holds at a because

$$\sum_v f_{va} = f_{sa} = 10 \text{ and } \sum_v f_{av} = f_{ac} + f_{ad} = 10 + 0 = 10$$

It holds at b because $\sum_v f_{vb} = f_{sb} = 2$ and $\sum_v f_{bv} = f_{bc} + f_{bd} = 1 + 1 = 2$

It holds at d because $\sum_v f_{vd} = f_{ad} + f_{bd} = 0 + 1 = 1$ and $\sum_v f_{dv} = f_{dt} = 1$.

Problem 4.8. *What does it mean to say that {S, T} is a partition of V ?*

Solution. To say that sets S and T comprise a partition of V is to say that S and T are disjoint subsets of V whose union is V.

Problem 4.9. *With reference to the directed network of Fig. (4.18), find a flow whose value exceeds 12.*

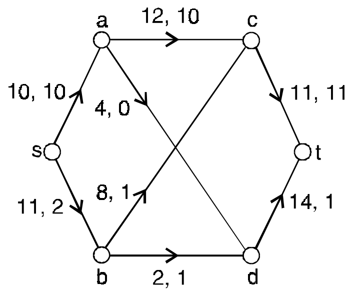


Fig. 4.18.

Solution. A flow with value 13 appears in Fig. (4.19) and one with value 17 is shown in Fig. (4.20).

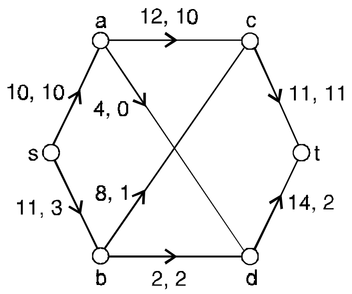


Fig. 4.19.

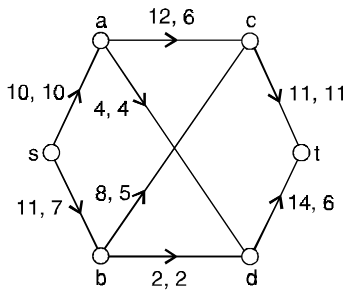


Fig. 4.20.

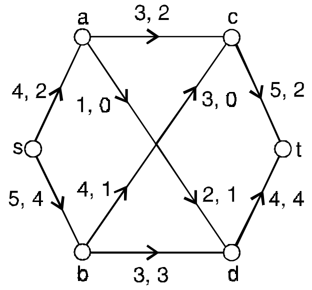
Problem 4.10. *(i) Verify the law of conservation of flow at a, e, and d.*

(ii) Find the value of the indicated flow.

(iii) Find the capacity of the (s, t)-cut defined by S = {s, a, b} and T = {c, d, e, t}

(iv) Can the flow be increased along the path sbedt ? If so, by how much ?

(v) Is the given flow maximum ? Explain.



Solution. (*i*) The law of conservation holds at *a* because

$$\sum_v f_{va} = f_{sa} = 2 \text{ and } \sum_v f_{av} = f_{ac} + f_{ae} = 2 + 0 = 2.$$

It holds at *e* because $\sum_v f_{ve} = f_{ae} + f_{be} = 0 + 1 = 1$

and $\sum_v f_{ev} = f_{ec} + f_{ed} = 0 + 1 = 1$

It holds at *d* because $\sum_v f_{vd} = f_{bd} + f_{ed} = 3 + 1 = 4$

and $\sum_v f_{dv} = f_{dt} = 4.$

(*ii*) the value of the flow is 6.

(*iii*) The capacity of the cut is $C_{ac} + C_{ae} + C_{be} + C_{bd} = 3 + 1 + 4 + 3 = 11$.

(*iv*) No. Arc *dt* is saturated.

(*v*) The flow is not maximum. For instance, it can be increased by adding 1 to the flow in the arcs along *sact*.

Problem 4.11. *Find the capacity of the (s, t)-cut defined by S = {s, a, b, d} and T = {c, e, f, t}.*

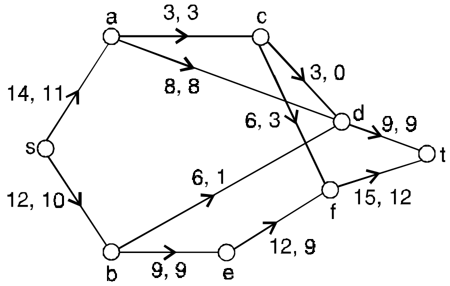


Fig. 4.21.

Solution. The capacity of the cut is $C_{ac} + C_{be} + C_{dt} = 3 + 9 + 9 = 21$.

Problem 4.12. *Answer the following questions for each of the networks shown in Fig. (4.80).*

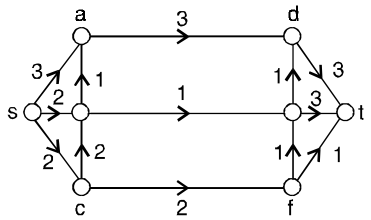


Fig. 4.22.

(i) Exhibit a unit flow

(ii) Exhibit a flow with a saturated arc.

(iii) Find a ''good'' and, if possible, a maximum flow in the network. State the value of your flow.

Solution. (*i*) Send one unit through the path *sbet*.

(*ii*) The flow in Fig. (4.22) has a saturated arc, *be*.

(*iii*) Here is a maximum flow, of value 6.

To see that the flow is maximum, consider the cut S = {*s*, *a*, *b*}, T = {*c*, *d*, *e*, *f*, *t*}.

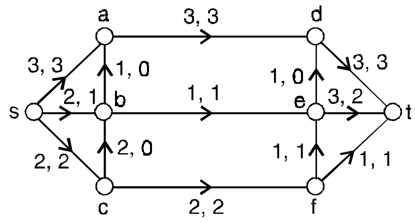
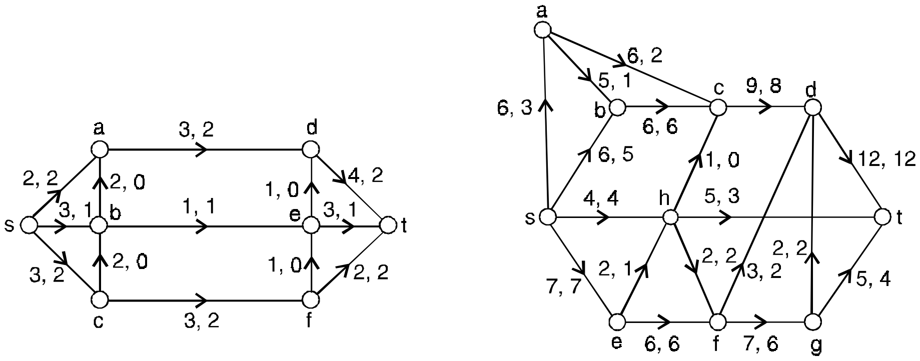


Fig. 4.23.

Problem 4.13. *Answer the following two questions for each of the directed networks shown.*

(i) Show that the given flow is not maximum by finding flow augmenting chain from s to t. What is the slack in your chain ?

(ii) Find a maximum flow, give its value, and prove that it is maximum by appealing to max-flow-mincut theorem.



(*a*) (*b*)

Fig. 4.24.

Solution. (*a*) (*i*) One flow-augmenting chain is *sbadt* in which the slack is 1.

(*ii*) Here is a maximum flow, of value 7. We can see this is maximum by examining the cut S = {*s*, *a*, *b*} T = {*c*, *d*, *e*, *f*, *t*}, of capacity

$$C_{sc} + C_{ad} + C_{be} = 3 + 3 + 1 = 7, \text{ the value of the flow.}$$

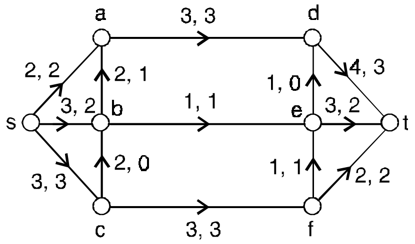


Fig. 4.25.

4.10 COMBINATORIAL AND GEOMETRIC GRAPHS (REPRESENTATION)

An abstract graph G can be defined as G = (V, E, Ψ) where the set V consists of the five objects named *a*, *b*, *c*, *d* and *e*, that is, V = {*a, b, c, d, e*} and the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6 and 7, that is,

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

and the relationship between the two sets is defined by the mapping Ψ, which consists of combinatorial representation of the graph.

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b) \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix} \longrightarrow \text{Combinatorial representation of a graph}$$

Here, the symbol 1 ⟶ (*a*, *c*) says that object 1 from set E is mapped onto the (unordered) pair (*a*, *c*) of objects from set V.

It can be represented by means of geometric figure as shown below. It is true that graph can be represented by means of such configuration.

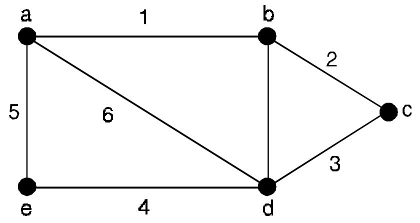


Fig. 2.2. Geometric representation of a graph.

4.11 PLANAR GRAPHS

A graph G is said to be **planar** if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. The points of intersection are called crossovers.

A graph that cannot be drawn on a plane without a crossover between its edges crossing is called a plane graph. The graphs shown in Figure 4.26(*a*) and are planar graphs.

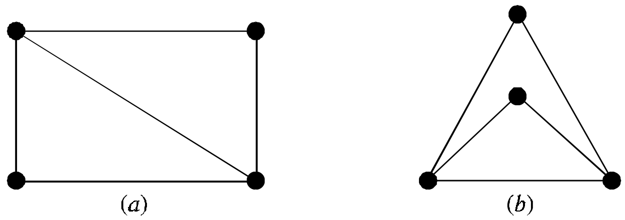


Fig. 4.26.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Note that if a graph G has been drawn with crossing edges, this does not mean that G is non planar, there may be another way to draw the graph without crossovers. Thus to declare that a graph G is non planar. We have to show that all possible geometric representations of G none can be embedded in a plane.

Equivalently, a graph G is planar is there if there exists a graph isomorphic to G that is embedded in a plane, otherwise G is non planar.

For example, the graph in Figure 4.27(*a*) is apparently non planar. However, the graph can be redrawn as shown in Figure (4.27)(*b*) so that edges don't cross, it is a planar graph, though its appearance is non coplanar.

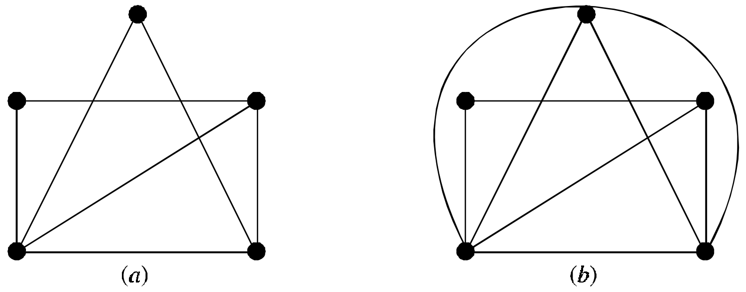


Fig. 4.27.

4.12 KURATOWSKI'S GRAPHS

For this we discuss two specific non-planar graphs, which are of fundamental importance, these are called Kuratowski's graphs. The complete graph with 5 vertices is the first of the two graphs of Kuratowski. The second is a regular, connected graph with 6 vertices and 9 edges.

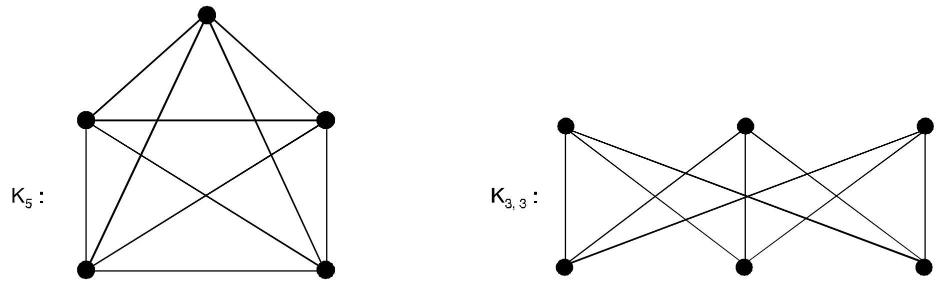


Fig. 4.28.

Observations

(*i*) Both are regular graphs

(*ii*) Both are non-planar graphs

(*iii*) Removal of one vertex or one edge makes the graph planar

(*iv*) (Kuratowski's) first graph is non-planar graph with smallest number of vertices and (Kuratowski's) second graph is non-planar graph with smallest number of edges. Thus both are simplest non-planar graphs.

The first and second graphs of Kuratowski are represented as K_5 and $K_{3,3}$. The letter K being for Kuratowski (a polish mathematician).

4.13 HOMEOMORPHIC GRAPHS

Two graphs are said to be homeomorphic if and only if each can be obtained from the same graph by adding vertices (necessarily of degree 2) to edges.

The graphs G_1 and G_2 in Figure (4.29) are homeomorphic since both are obtainable from the graph G in that figure by adding a vertex to one of its edges.

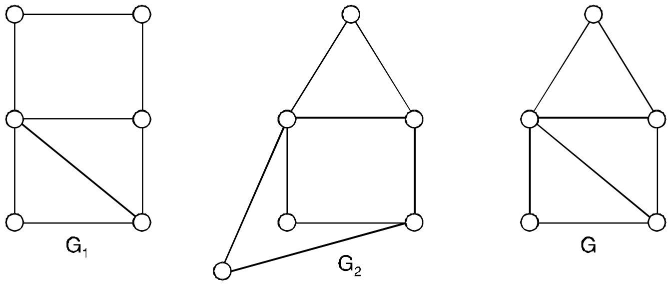


Fig. 4.29. Two homeomorphic graphs obtained from G by adding vertices to edges.

In Figure 4.30, we show two homeomorphic graphs, each obtained from K_5 by adding vertices to edges of K_5 (In each case, the vertices of K_5 are shown with solid dots).

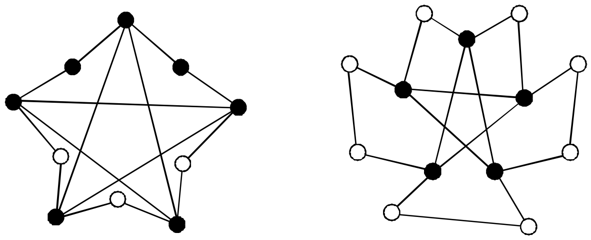


Fig. 4.30. Two homeomorphic graphs obtained from K_5.

4.14 REGION

A plane representation of a graph divides the plane into regions (also called windows, faces, or meshes) as shown in figure below. A region is characterized by the set of edges (or the set of vertices) forming its boundary.

Note that a region is not defined in a non-planar graph or even in a planar graph not embedded in a plane.

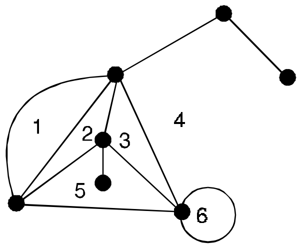


Fig. 4.31. Plane representation (the numbers stand for regions).

For example, the geometric graph in figure below does not have regions.

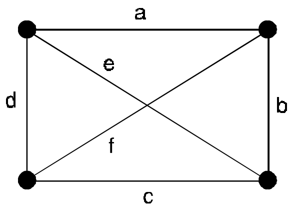


Fig. 4.32.

4.15 MAXIMAL PLANAR GRAPHS

A planar graph is maximal planar if no edge can be added without loosing planarity. Thus in any maximal planar graph with $p \geq 3$ vertices, the boundary of every region of G is a triangle for this maximal planar graphs (or plane graphs) are also refer to as triangulated planar graph (or plane graph).

4.16 SUBDIVISION GRAPHS

A subdivision of a graph is a graph obtained by inserting vertices (of degree 2) into the edges of G. For the graph G of the figure below, the graph H is a subdivision of G, while F is not a subdivision of G.

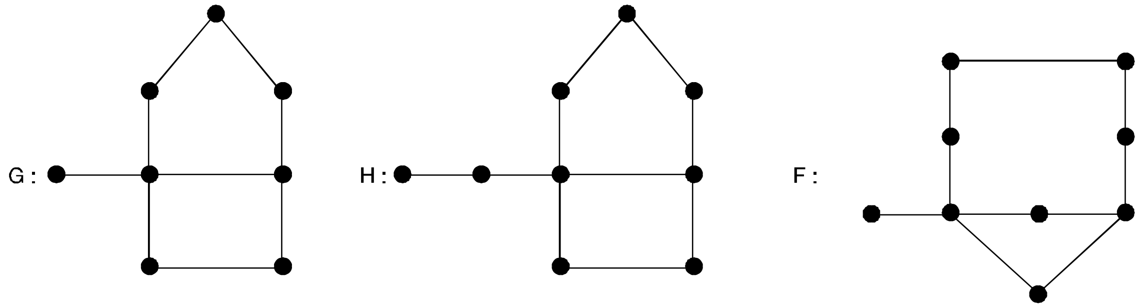


Fig. 4.33.

4.17 INNER VERTEX SET

A set of vertices of a planar graph G is called an inner vertex set I(G) of G. If G can be drawn on the plane in such a way that each vertex of I(G) lies only on the interior region and I(G) contains the minimum possible vertices of G. The number of vertices *i*(G) of I(G) is said to be the inner vertex number if they lie in interior region of G.

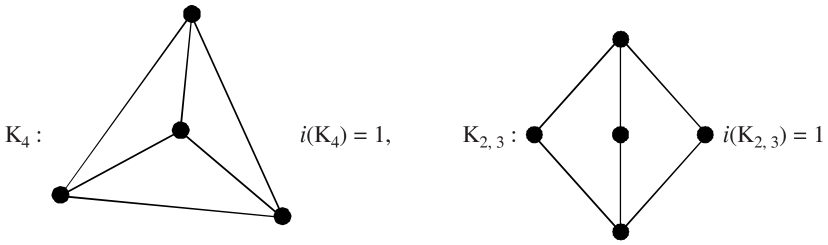


Fig. 4.34.

For any cycle C_p, $i(C_p) = 0$.

4.18 OUTER PLANAR GRAPHS

A planar graph is said to be outer planar if $i(G) = 0$. For example, cycles, trees, $K_4 - x$.

4.18.1. Maximal outer planar graph

An outer planar graph G is maximal outer planar if no edge can be added without losing outer planarity.

For example,

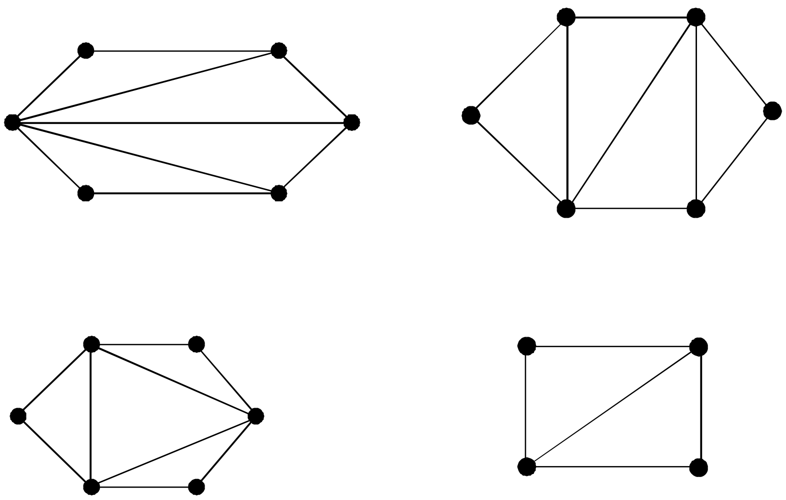
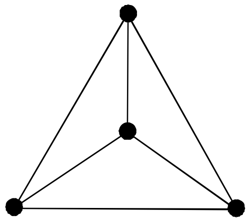


Fig. 4.35. Maximal outer planar graphs.

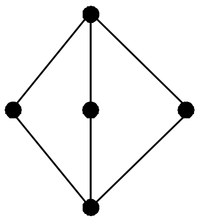
4.18.2. Minimally non-outer planar graphs

A planar graph G is said to be minimally non outer planar if $i(G) = 1$

For example, K_4 :



$K_{2, 3}$:



4.19 CROSSING NUMBER

The crossing number C(G) of a graph G is the minimum number of crossing of its edges among all drawings of G in the plane.

A graph is planar if and only if C(G) = 0. Since K_4 is planar $C(K_4) = 0$ for $p \leq 4$. On the other hand $C(K_5) = 1$. Also $K_{3, 3}$ is non planar and can be drawn with one crossing.

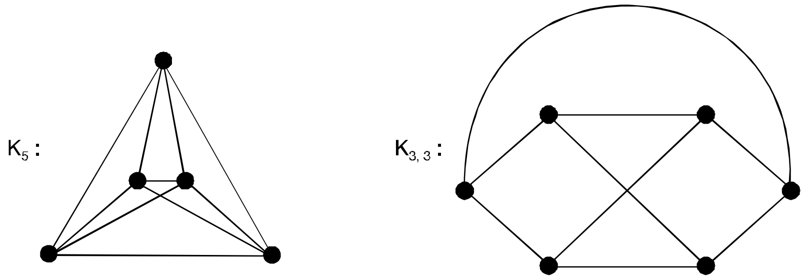


Fig. 4.36. K_5 and $K_{3, 3}$ are non planar graphs with one crossing.

4.20 BIPARTITE GRAPH

A graph G = (V, E) is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G. Obviously, a bipartite graph can have no loop.

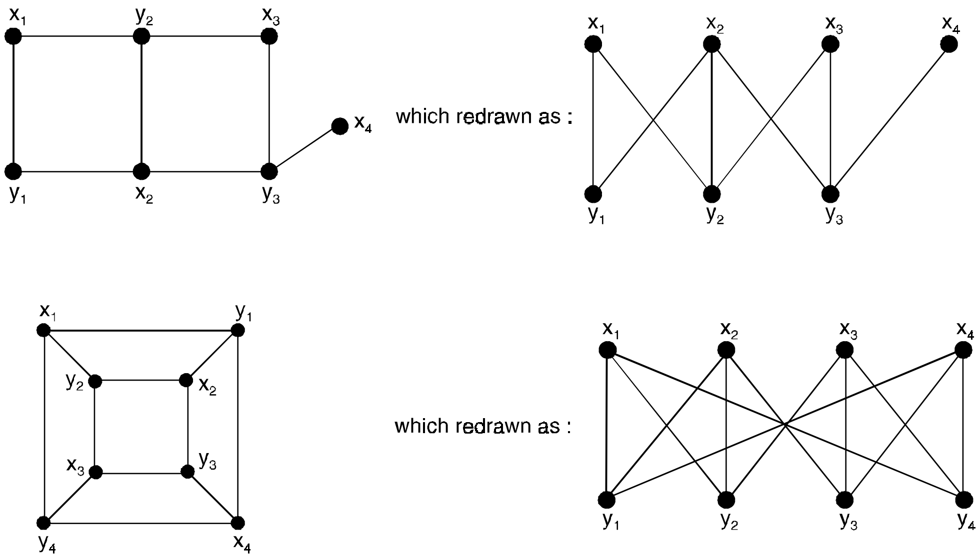


Fig. 4.37. Some bipartite graphs.

4.20.1. Complete bipartite graph

The complete bipartite graph on *m* and *n* vertices, denoted $K_{m, n}$ is the graph, whose vertex set is partitioned into sets V_1 with *m* vertices and V_2 with *n* vertices in which there is an edge between each pair of vertices V_1 and V_6. Where V_1 is in V_1 and V_2 is in V_2. The complete bipartite graphs $K_{2, 3}$, $K_{3, 3}$, $K_{3, 5}$ and $K_{2, 6}$ are shown in Figure below. Note that $K_{r, s}$ has $r + s$ vertices and rs edges.

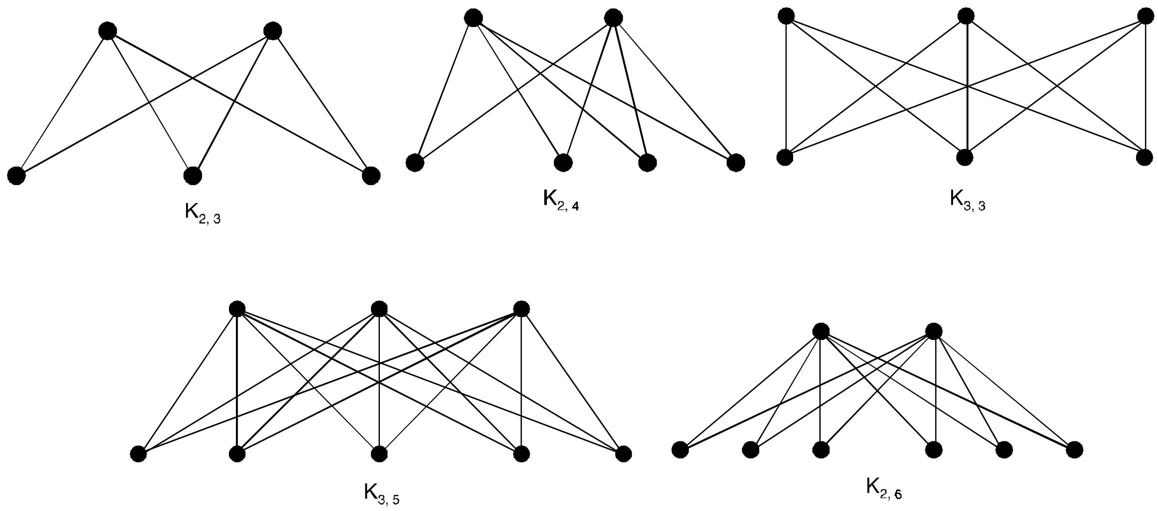


Fig. 4.38. Some complete bipartite graphs.

A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$.

Problem 4.14. *Show that C_6 is a bipartite graph.*

Solution. C_6 is a bipartite graph as shown in Figure below.

Since its vertex set can be partitioned into two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

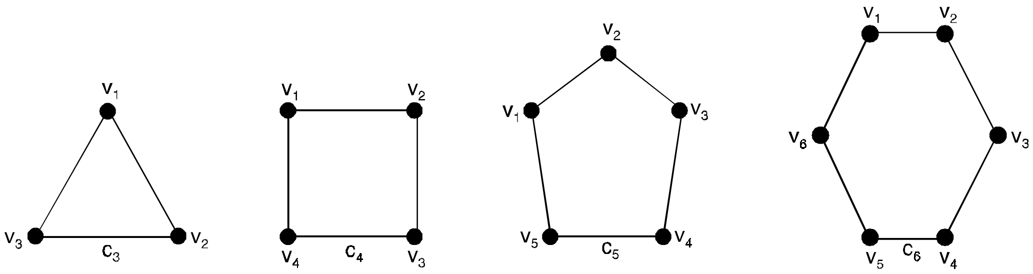


Fig. 4.39.

Problem 4.15. *Prove that a graph which contains a triangle cannot be bipartite.*

Solution. At least two of three vertices must lie in one of the bipartite sets, since these two are joined by two are joined by edge, the graph can not be bipartite.

Problem 4.16. *Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.*

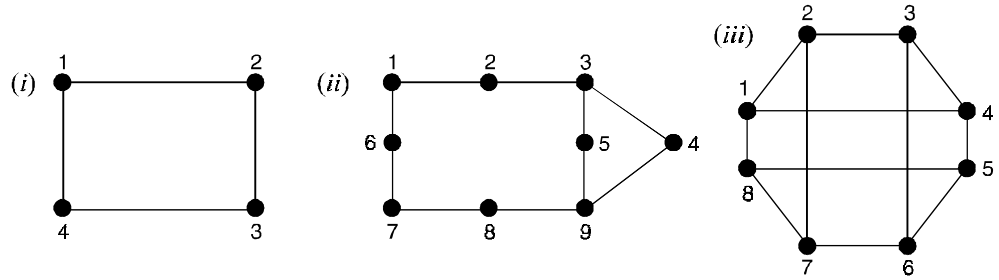


Fig. 4.40.

Solution. (*i*) The graph is not bipartite because it contains triangles (in fact two triangles).

(*ii*) This is bipartite and the bipartite sets are {1, 3, 7, 9} and {2, 4, 5, 6, 8}

(*iii*) This is bipartite and the bipartite sets are {1, 3, 5, 7} and {2, 4, 6, 8}.

4.21 EULER'S FORMULA

The basic results about planar graph known as Euler's formula is the basic computational tools for planar graph.

Euler's Formula

If a connected planar graph G has n vertices, e edges and r region, then $n - e + r = 2$.

Proof. We prove the theorem by induction on e, number of edges of G.

Basis of induction : If $e = 0$ then G must have just one vertex.

i.e., $n = 1$ and one infinite region, *i.e.,* $r = 1$

Then $n - e + r = 1 - 0 + 1 = 2$.

If $e = 1$ (though it is not necessary), then the number of vertices of G is either 1 or 2, the first possibility of occurring when the edge is a loop.

These two possibilities give rise to two regions and one region respectively, as shown in Figure (4.41) below.

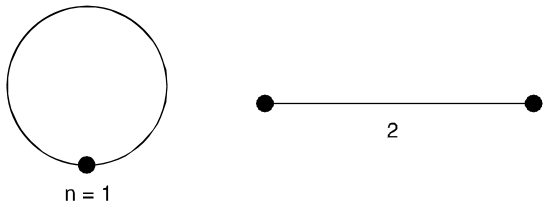


Figure. 4.41. Connected plane graphs with one edge.

In the case of loop, $n - e + r = 1 - 1 + 2 = 2$ and in case of non-loop, $n - e + r = 2 - 1 + 1 = 2$.

Hence the result is true.

Induction hypothesis :

Now, we suppose that the result is true for any connected plane graph G with $e - 1$ edges.

Induction step :

We add one new edge K to G to form a connected supergraph of G which is denoted by G + K.

There are following three possibilities.

(*i*) K is a loop, in which case a new region bounded by the loop is created but the number of vertices remains unchanged.

(*ii*) K joins two distinct vertices of G, in which case one of the region of G is split into two, so that number of regions is increased by 1, but the number of vertices remains unchanged.

(*iii*) K is incident with only one vertex of G on which case another vertex must be added, increasing the number of vertices by one, but leaving the number of regions unchanged.

If let n', e' and r' denote the number of vertices, edges and regions in G and n, e and r denote the same in G + K. Then

In case (*i*) $n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$.

In case (*ii*) $n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$

In case (*iii*) $n - e + r = (n' + 1) - (e' + 1) + r' = n' - e' + r'$.

But by our induction hypothesis, $n' - e' + r' = 2$.

Thus in each case $n - e + r = 2$.

Now any plane connected graph with e edges is of the form G + K, for some connected graph G with $e - 1$ edges and a new edge K.

Hence by mathematical induction the formula is true for all plane graphs.

Corollary (1)

If a plane graph has K components then $n - e + r = \text{K} + 1$.

The result follows on applying Euler's formula to each component separately, remembering not to count the infinite region more than once.

Corollary (2)

If G is connected simple planar graph with n (≥ 3) vertices and e edges, then $e \leq 3n - 6$.

Proof. Each region is bounded by atleast three edges (since the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree 2 or loops that could produce regions of degree 1, are permitted) and edges belong to exactly two regions.

$$2e \geq 3r$$

If we combine this with Euler's formula, $n - e + r = 2$, we get $3r = 6 - 3n + 3e \leq 2e$ which is equivalent to $e \leq 3n - 6$.

Corollary (3)

If G is connected simple planar graph with n (≥ 3) vertices and e edges and no circuits of length 3, then $e \leq 2n - 4$.

Proof. If the graph is planar, then the degree of each region is atleast 4.

Hence the total number of edges around all the regions is atleast $4r$.

Since every edge borders two regions, the total number of edges around all the regions is $2e$, so we established that $2e \geq 4r$, which is equivalent to $2r \leq e$.

If we combine this with Euler's formula $n - e + r = 2$, we get

$$2r = 4 - 2n + 2e \leq e$$

which is equivalent to $e \leq 2n - 4$.

Problem 4.17. *Show that the graph K_5 is not coplanar.*

Solution. Since K_5 is a simple graph, the smallest possible length for any cycle K_5 is three.

We shall suppose that the graph is planar.

The graph has 5 vertices and 10 edges so that $n = 5$, $e = 10$.

Now $3n - 6 = 3.5 - 6 = 9 < e$.

Thus the graph violates the inequality $e \le 3n - 6$ and hence it is not coplanar.

This may be noted that the inequality $e \le 3n - 6$ is only by a necessary condition but not a sufficient condition for the planarity of a graph.

For example, graph $K_{3,\,3}$ satisfies the inequality because $e = 9 \le 3.6 - 6 = 12$, yet the graph is non planar.

Problem 4.18. *Show that the graph $K_{3,\,3}$ is not coplanar.*

Solution. Since $K_{3,\,3}$ has no circuits of length 3 (it is bipartite) and has 6 vertices and 9 edges.

i.e., $n = 6$ and $e = 9$ so that $2n - 4 = 2.6 - 4 = 8$.

Hence the inequality $e \le 2n - 4$ does not satisfy and the graph is not coplanar.

Problem 4.19. *A connected plane graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane ?*

Solution. Here $n = 10$ and degree of each vertex is 3

$\Sigma \deg(v) = 3 \times 10 = 30$

But $\Sigma \deg(v) = 2e \quad \Rightarrow \quad 30 = 2e \quad \Rightarrow \quad e = 15$

By Euler's formula, we have $n - e + r = 2 \quad \Rightarrow \quad 10 - 15 + r = 2 \quad \Rightarrow \quad r = 7$.

Problem 4.20. *Show that K_n is a planar graph for $n \le 4$ and non-planar for $n \ge 5$.*

Solution. A K_4 graph can be drawn in the way as shown in the Figure (4.42). This does not contain any false crossing of edges.

Thus, it is a planar graph.

Graphs K_1, K_2 and K_3 are by construction a planar graph, since they do not contain a false crossing of edges.

K_5 is shown in the Figure (4.43)

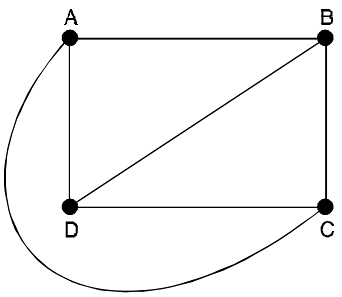


Fig. 4.42.

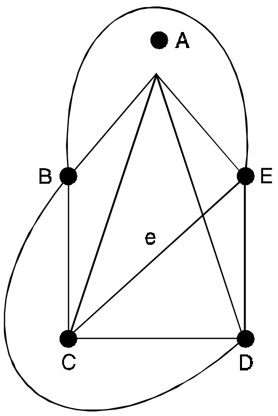


Fig. 4.43.

It is not possible to draw this graph on a 2-dimentional plane without false crossing of edges. Whatever way we adopt, at least one of the edges, say e, must cross the other for graph to be completed.

Hence K_5 is not a planar graph.

For any $n > 5$, K_n must contain a subgraph isomorphic to K_5.

Since K_5 is not planar, any graph containing K_5 as its one of the subgraph cannot be planar.

Theorem 4.11. *Show that $K_{3, 3}$ is a non-planar graph.*

Solution. Graph $K_{3, 3}$ is shown in the Figure (4.44) below.

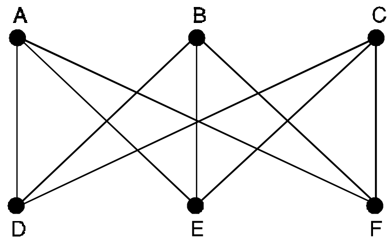


Fig. 4.44.

It is not possible to draw this graph such that there is no false crossing of edges. This is classic problem of designing direct lanes without intersection between any two houses, for three houses on each side of a road.

In this graph there exists an edge, say e, that cannot be drawn without crossing another edge.

Hence $K_{3, 3}$ is a non-planar graph.

It is easy to determine that the chromatic number of this graph is 2.

Theorem 4.12. *Sum of the degrees of all regions in a map is equal to twice the number of edges in the corresponding graph.*

Proof. As discussed earlier, a map can be drawn as a graph, where regions of the map is denoted by vertices in the graph and adjoining regions are connected by edges.

Degree of a region in a map is defined as the number of adjoining region.

Thus, degree of a region in a map is equal to the degree of the corresponding vertices in the graph.

We know that the sum of the degrees of all vertices in a graph is equal to the twice the number of edges in the graph.

Therefore, we have $2e = \Sigma\deg(R_i)$.

Problem 4.21. *Prove that K_4 and $K_{2, 2}$ are planar.*

Solution. In K_4, we have $v = 4$ and $e = 6$

Obviously, $6 \leq 3 * 4 - 6 = 6$

Thus this relation is satisfied for K_4.

For $K_{2, 2}$, we have $v = 4$ and $e = 4$.

Again in this case, the relation $e \leq 3v - 6$

i.e., $4 \leq 3 * 4 - 6 = 6$ is satisfied.

Hence both K_4 and $K_{2, 2}$ are planar.

Problem 4.22. *Determine the number of vertices, the number of edges, and the number of region in the graphs shown below. Then show that your answer satisfy Euler's theorem for connected planar graphs.*

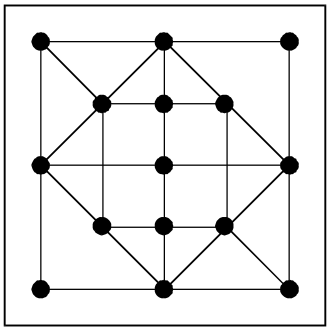


Fig. 4.45.

Solution. There are 17 vertices, 34 edges and 19 regions. So $v - e + r = 17 - 34 + 19 = 2$ which verifies Euler's theorem.

Problem 4.23. *If every region of a simple planar graph with n-vertices and e-edges embeded in a plane is bounded by k-edges then show that* $e = \dfrac{k(n-2)}{k-2}$.

Solution. Since every region is bounded by K-edges, then r-regions are bounded by Kr-edges. Also each edge is counted twice, once for two of its adjacent regions.

Hence we have $2e = \text{K}r \Rightarrow r = \dfrac{2e}{\text{K}}$ ...(1)

i.e., if G is a connected planar graph with n-vertices e-edges and r-regions, then $n - e + r = 2$.

From (1), we have

$$n - e + \frac{2e}{\text{K}} = 2$$

$$\Rightarrow \quad n\text{K} - e\text{K} + 2e = 2\text{K}$$

$$\Rightarrow \quad n\text{K} - 2\text{K} = e\text{K} - 2e$$

$$\Rightarrow \quad \text{K}(n-2) = e(\text{K}-2)$$

$$\Rightarrow \quad e = \frac{\text{K}(n-2)}{\text{K}-2}.$$

Problem 4.24. *Determine whether the graph G shown in Figure (4.46), is planar.*

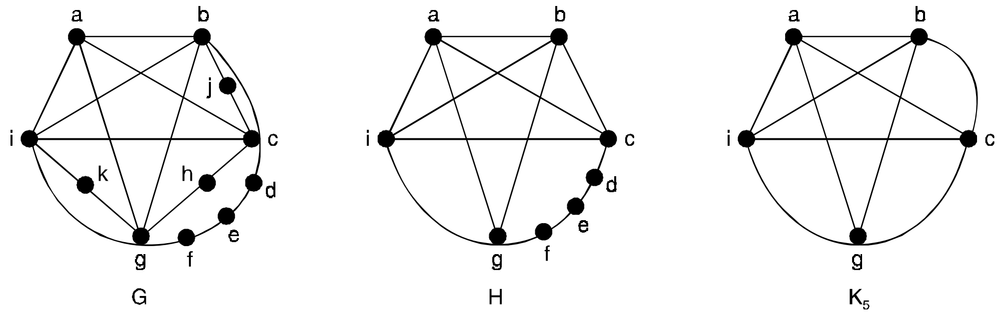


Fig. 4.46. The undirected graph G, a subgraph H homeomorphic to K_5 and K_5.

Solution. G has a subgraph H homeomorphic to K_5, H is obtained by deleting *h, j* and K and all edges incident with these vertices. H is homeomorphic to K_5 since it can be obtained from K_5 (with vertices *a, b, c*, *g* and *i*) by a sequence of elementary subdivisions, adding the vertices *d*, *e* and *f*.

Hence G is non planar.

Theorem 4.13. KURATOWSKI'S

$K_{3,3}$ and K_5 are non-planar.

Proof. Suppose first that $K_{3,3}$ is planar.

Since $K_{3,3}$ has a cycle $u \to v \to w \to x \to y \to z \to u$ of length 6, any plane drawing must contain this cycle drawn in the form of hexagon, as in Figure (4.47).

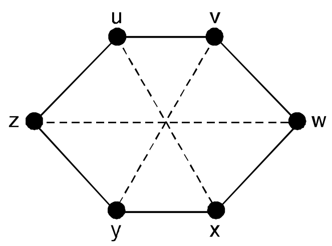


Fig. 4.47.

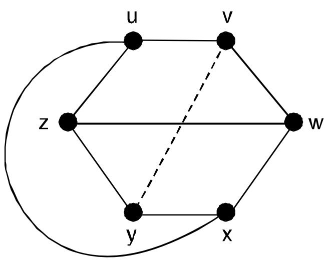


Fig. 4.48.

Now the edge *wz* must lie either wholly inside the hexagon or wholly outside it. We deal with the case in which *wz* lies inside the hexagon, the other case is similar.

Since the edge *ux* must not cross the edge *wz,* it must lie outside the hexagon ; the situation is now as in Figure (4.48).

It is then impossible to draw the edge *vy*, as it would cross either *ux* or *wz*.

This gives the required contradiction.

Now suppose that K_5 is planar.

Since K_5 has a cycle $v \to w \to x \to y \to z \to v$ of length 5, any plane drawing must contain this cycle drawn in the form of a pentagon as in Figure (4.49).

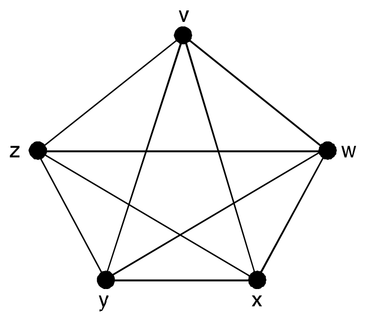


Fig. 4.49.

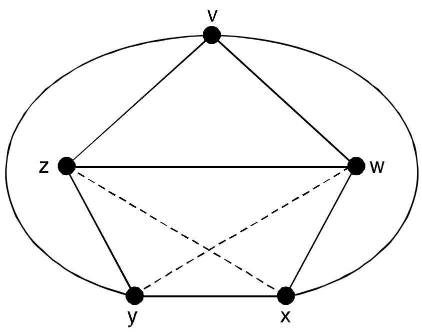


Fig. 4.50.

Now the edge *wz* must lie either wholly inside the pentagon or wholly outside it.

We deal with the case in which *wz* lies inside the pentagon, the other case is similar.

Since the edges *vx* and *vy* do not cross the edge *wz,* they must both lie outside the pentagon, the situation is now as in Figure (4.50)

But the edge *xz* cannot cross the edge *vy* and so must lie inside the pentagon.

Similarly the edge *wy* must lie inside the pentagon, and the edges *wy* and *xz* must then cross.

This gives the required contradiction.

Theorem 4.14. *Let G be a simple connected planar (p, q)-graph having at least K edges in a boundary of each region. Then* $(k-2)q \leq k(p-2)$.

Proof : Every edge on the boundary of G, lies in the boundaries of exactly two regions of G.

Further G may have some pendent edges which do not lie in a boundary of any region of G.

Thus, sum of lengths of all boundaries of G is less than twice the number of edges of G.

i.e., $kr \leq 2q$ ...(1)

But, G is a connected graph, therefore by Euler's formula

We have $r = 2 + q - p$ ...(2)

Substituting (2) in (1), we get

$$k(2 + q - p) \leq 2q$$

$$\Rightarrow \quad (k-2)q \leq k(p-2).$$

Problem 4.25. *Suppose G is a graph with 1000 vertices and 3000 edges. Is G planar ?*

Solution. A graph G is said to be planar if it satisfies the inequality. *i.e.,* $q \leq 3p - 6$

Here P = 1000, $q = 3000$ then

$$3000 \leq 3p - 6$$

i.e., $3000 \leq 3000 - 6$

or $3000 \leq 2994$ which is impossible.

Hence the given graph is not a planar.

Problem 4.26. *A connected graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ? How many faces are there ?*

Solution. By Handshaking lemma,

$$\sum_{i=1}^{n} \deg v_i = 2q$$

i.e., $2q = 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 28$

$\Rightarrow \quad q = 24$

Now by Euler's formula $p - q + r = 2$ or $9 - 14 + r = 2 \Rightarrow r = 7$

Hence there are 14 edges and 7 regions in the graph.

Problem 4.27. *Find a graph G with degree sequence (4, 4, 3, 3, 3, 3) such that (i) G is planar (ii) G is non planar.*

Solution. For (*i*) we have drawn a planar graph with six vertices with degree sequence 4, 4, 3, 3, 3, 3 as shown below.

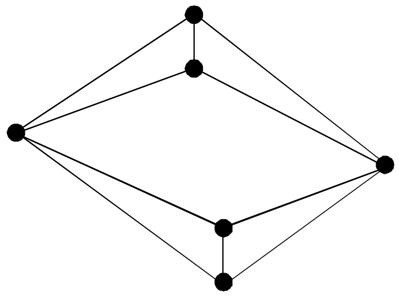


Fig. 4.51.

For (*ii*) By Handshaking lemma

$$\sum_{i=1}^{n} \deg v_i = 2q$$

i.e., $2q = 4 + 4 + 3 + 3 + 3 + 3$

$2q = 20$

$\Rightarrow \quad q = 10$

Hence the graph with P = 6, is said to be planar if it satisfies the inequality.

i.e., $q = 3p - 6$

i.e., $10 \leq 3 \times 6 - 6$

or $10 \leq 18 - 6$

$10 \leq 12$

Hence it is not possible to draw a non planar graph with given degree sequence 4, 4, 3, 3, 3, 3.

Problem 4.28. *Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a non-simple graph.*

Solution. Given $p = 6$, $q = 10$

Hence by Euler's formula for a planar graph

$$p - q + r = 2$$

$$6 - 10 + r = 2 \quad \Rightarrow \quad r = 6$$

Hence the graph should have 6 regions.

Simple and non-simple graphs with $p = 6$, $q = 10$ and $r = 6$ are shown below.

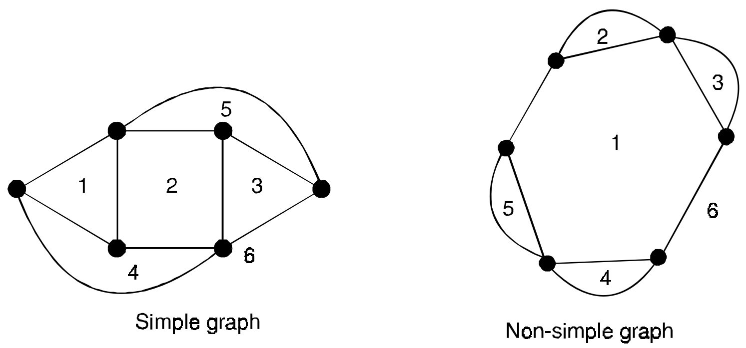


Fig. 4.52.

Problem 4.29. *Draw all planar graphs with five vertices, which are not isomorphic to each other.*

Solution. We have drawn all planar graphs with 5 vertices as shown below.

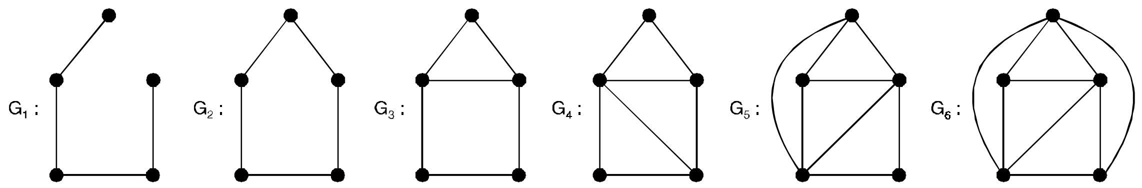


Fig. 4.53.

Problem 4.30. *How many edges must a planar graph have if it has 7 regions and 5 vertices. Draw one such graph.*

Solution. According to Euler's formula, in a planar graph G.

$$p - q + r = 2$$

Here $p = 5, r = 7, q = ?$

Since the graph is planar, therefore $5 - q + 7 = 2 \quad \Rightarrow \quad q = 10.$

Hence the given graph must have 10-edges.

Here we have drawn more than one graph as shown below.

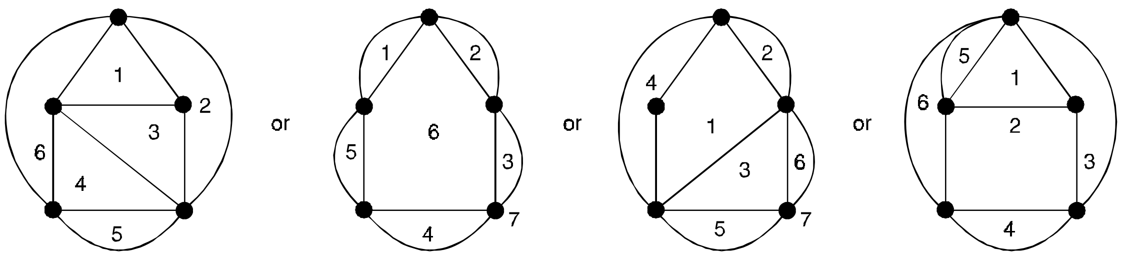
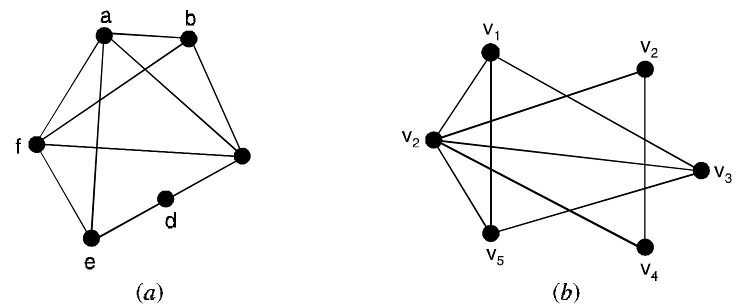


Fig. 4.54.

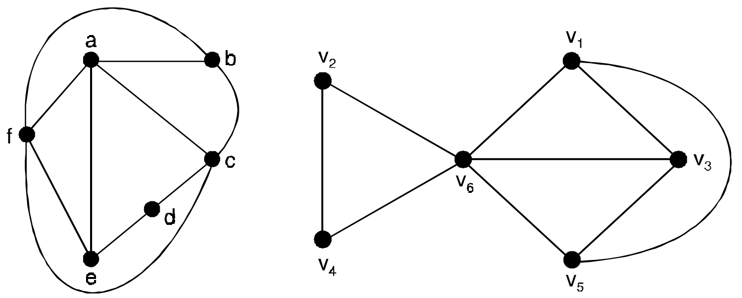
Problem 4.31. *By drawing the graph, show that the following graphs are planar graphs.*



(*a*) (*b*)

Fig. 4.55.

Solution. The graphs shown in Figure (2.28)(*a*, *b*) can be redrawn as planar graphs as follows see Figure (4.56) (*a*, *b*).



(*a*) (*b*)

Fig. 4.56.

Problem 4.32. *Show that the Petersen graph is non planar.*

Solution. Petersen graph is well known non planar graph. Since G has some similarity with K_5 because of 5-cycle, ABCDEA. However since K_5 has vertices of degree 4 only subdivision of K_5 will also have such vertices so G can not have only subdivision of K_5.

Since its vertices each have degree 3. So we look for a subgraph of G which is subdivision of the bipartite graph $K_{3,3}$.

The Petersen graph shown in Figure (4.57)(*a*) is non planar since it contains a subgraph homeomorphic to $K_{3,3}$ as shown in Figure (4.57)(*c*). Note that the Petersen graph does not contain a subgraph homeomorphic to K_5,

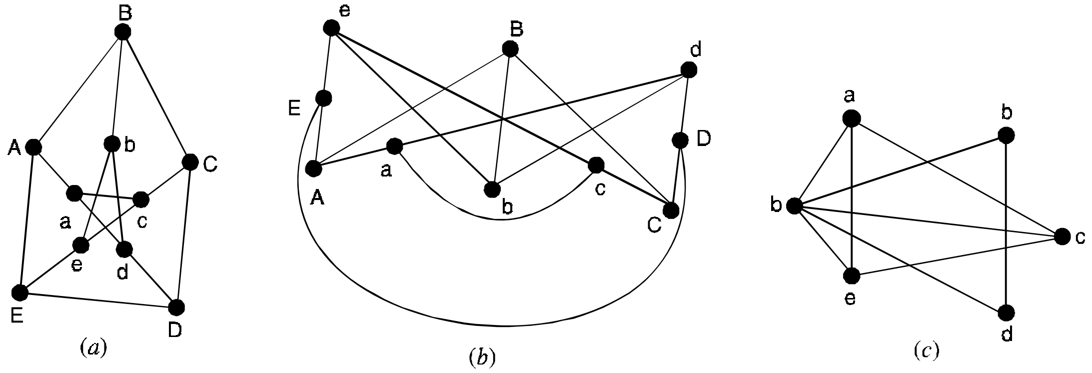


Fig. 4.57.

Problem 4.33. *Find a smallest planar graph that is regular of degree 4.*

Solution. For the graph with two vertices, which is complete, then degree of each vertex is one. For the next smallest graphs are with vertices 3 and 4, if they are complete then degree of each vertex is 2 and 3.

The next graph is with 5 vertices. If degree of each vertex is 4, then it is complete graph with 5 vertices K_5 which is non planar. For the next graph with 6 vertices, if it complete then degree of each vertex is P – 1. *i.e.,* 5. To make this graph 4 regular or regular of degree 4. Remove any 3 non adjacent edges from K_6 we get $K_6 - 3x$ where x is an edge of G, as shown in Figure (4.58), which is regular of degree 4.

K_6–$3x$:

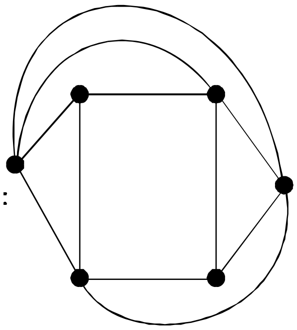


Fig. 4.58.

4.22. THREE UTILITY PROBLEM

There are three homes H_1, H_2 and H_3 each to be connected to each of three utilities Water (W), Gas(G) and Electricity(E) by means of conduits. Is it possible to make such connections without any crossovers of the conduits ?

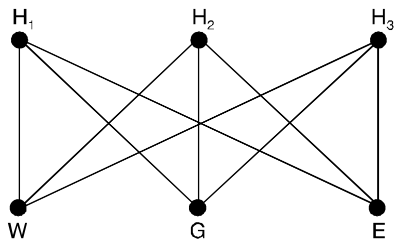


Fig. 4.59.

The problem can be represented by a graph shown in Figure the conduits are shown as edges while the houses and utility supply centers are vertices.

The above graph is a complete bipartite graph $K_{3,3}$ which is a non planar graph. Hence it is not possible to draw without crossover. Therefore it is not possible to make the connection without any crossover of the conduits.

Problem 4.34. *Is the Petersen graph, shown in Figure below, planar ?*

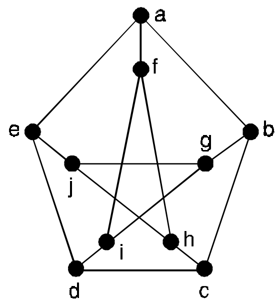


Fig. 4.60. Petersen graph

Solution. The subgraph H of the Petersen graph obtained by deleting *b* and the three edges that have *b* as an end point, shown in Figure (4.61) below, is homeomorphic to $K_{3,3}$ with vertex sets $\{f, d, j\}$

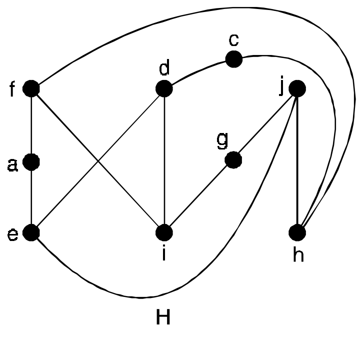


Fig. 4.61.

and $\{e, i, h\}$, since it can be obtained by a sequence of elementary subdivisions, deleting $\{d, h\}$ and adding $\{c, h\}$ and $\{c, d\}$, deleting $\{e, f\}$ and adding $\{a, e\}$ and $\{a, f\}$ and deleting $\{i, j\}$ and adding $\{g, i\}$ and $\{g, j\}$.

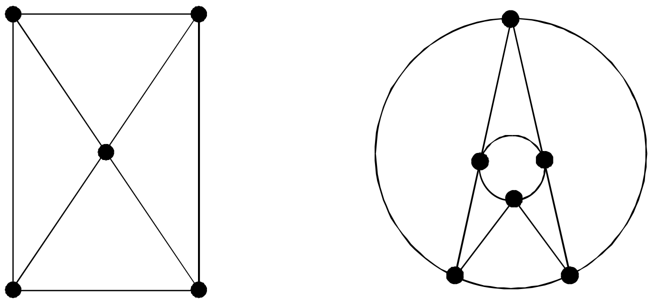
Hence the Petersen graph is not planar.

Problem 4.35. *Show that the following graphs are planar :*

(*i*) *Graph of order 5 and size 8* (*ii*) *Graph of order 6 and size 12.*

Solution. To show that a graph is planar, it is enough if we draw one plane diagram representing the graph in which no two edges cross each other.

Figure (4.62) (*a*) and (*b*) show that the given graphs are planar.



(*a*) (*b*)

Fig. 4.62.

Problem 4.36. *Verify that the following two graphs are homeomorphic but not isomorphic.*

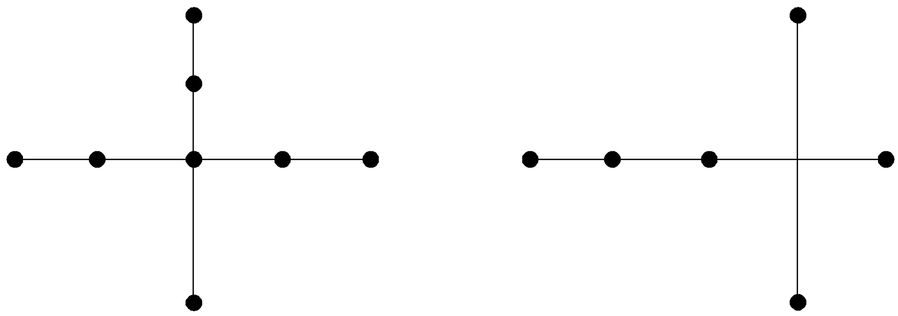


Fig. 4.63.

Solution. Each graph can be obtained from the other by adding or removing appropriate vertices.

Therefore, they are homeomorphic.

That they are not isomorphic is evident if we observe that the incident relationship is not identical.

Problem 4.37. *Show that if a planar graph G of order n and size m has r regions and K components, then $n - m + r = k + 1$.*

Solution. Let $H_1, H_2, \ldots\ldots H_k$ be the K components of G.

Let the number of vertices, the number of edges and the number of non-exterior regions in H_i be n_i, m_i, r_i respectively, $i = 1, 2, \ldots\ldots k$.

The exterior region is the same for all components.

Therefore, $\Sigma n_i = n, \quad \Sigma m_i = m, \quad \Sigma r_i = r - 1$

If the exterior region is not considered, then the Euler's formula applied to H_i yields

$$n_i - m_i + r_i = 1$$

On summation (from $i = 1$ to $i = k$) this yields

$$n - m + (r - 1) = k$$

$$\Rightarrow \quad n - m + r = k + 1$$

Problem 4.38. *Let G be a connected simple planar (n, m) graph in which every region is bounded by at least k edges. Show that* $m \le \dfrac{k(n-2)}{(k-2)}$.

Solution. Since every region in G is bounded by at least k edges, we have $2m \ge kr$ ...(1)

Where r is the number of regions

Substituting for r from the Euler's formula in (1), we get

$$2m \ge k(m - n + 2)$$

$$\Rightarrow \quad k(n - 2) \ge km - 2m$$

$$\Rightarrow \quad m \le \frac{k(n-2)}{(k-2)}$$

Problem 4.39. *Let G be a simple connected planar graph with fewer than 12 regions, in which each vertex has degree at least 3. Prove that G has a region bounded by at most four edges.*

Solution. Suppose every region in G bounded by at least 5 edges.

Then, if G has n vertices and m edges,

we have, $\quad 2m \ge 5r$ ...(1)

Since each vertex has degree atleast 3, the sum of the degrees of the vertices is greater than or equal to $3n$. By virtue of the handshaking property, this means that

$$2m \ge 3n \quad \text{...(2)}$$

By Euler's formula, we have

$$r = m - n + 2$$

$$\ge m - \left(\frac{2}{3}\right)m + 2 \qquad (\because \quad (2))$$

$$= \frac{m}{3} + 2 \ge \frac{5}{6}r + 2 \qquad (\because \quad (1))$$

This yields $6r \ge 5r + 12$, $r \ge 12$.

This is a contradiction, because G has fewer than 12 regions.

Hence, some region in G is bounded by atmost four edges.

Problem 4.40. *Show that these does not exist a connected simple planar graph with $m = 7$ edges and with degree $\delta = 3$.*

Solution. Suppose there is a graph G of the desired type.

Then, for this graph, the inequality $\delta \le \left(\dfrac{2m}{n}\right)$ gives $3n \le 14$.

On the other hand, $7 \leq 3n - 6$ or $3n \geq 13$.

Thus, we have $13 \leq 3n \leq 14$ which is not possible (because n has to be a positive integer).

Hence the graph of the desired type does not exist.

Problem 4.41. *Show that every simple connected planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.*

Solution. Suppose every vertex of G has degree greater than or equal to 5.

Then, if $d_1, d_2, d_3, \ldots\ldots d_n$ are the degrees of n vertices of G, we have $d_1 \geq 5, d_2 \geq 5, \ldots\ldots d_n \geq 5$.

So that $d_1 + d_2 + \ldots\ldots + d_n \geq 5n$.

or $2m \geq 5n$, by handshaking property,

or
$$\frac{5n}{2} \leq m \quad \ldots(1)$$

On the other hand, $m \leq 3n - 6$

Thus, we have, in view of (1)

$$\frac{5n}{2} \leq 3n - 6 \quad \text{or} \quad n \geq 12.$$

Thus, if every vertex of G has degree ≥ 5, then G must have at least 12 vertices.

Hence, if G has less than 12 vertices, it must have a vertex of degree < 5.

Problem 4.42. *Show that the condition m ≤ 3n – 6 is not a sufficient condition for a connected simple graph with n vertices and m edges to be planar.*

Solution. Consider the graph $K_{3,\,3}$ which is simple and connected and which has $n = 6$ vertices and $m = 9$ edges.

We check that, for this graph, $m \leq 3n - 6$.

But the graph is non-planar.

Problem 4.43. *What is the minimum number of vertices necessary for a simple connected graph with 11 edges to be planar ?*

Solution. For a simple connected planar (n, m) graph,

We have, $m \leq 3n - 6$

or
$$n \geq \frac{1}{3}(m + 6)$$

When $m = 11$, we get $n \geq \frac{17}{3}$.

Thus, the required minimum number of vertices is 6.

Problem 4.44. *Verify Euler's formula for the graph shown in Figure (4.64).*

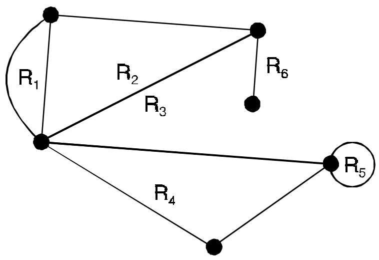


Fig. 4.64.

Solution. The graph has $n = 6$ vertices, $m = 10$ edges and $r = 6$ regions.

Therefore $n - m + r = 6 - 10 + 6 = 2$

Thus, Euler's formula is verified.

Problem 4.45. *What is the maximum number of edges possible in a simple connected planar graph with eight vertices ?*

Solution. When $n = 8$,

$$m \leq 3n - 6 = 18$$

Thus, the maximum number of edges possible is 18.

Theorem 4.15. *A graph is planar if and only if each of its blocks is planar.*

Theorem 4.16. *Every 2-connected plane graph can be embedded in the plane so that any specified face is the exterior.*

Proof. Let f be a non exterior face of a plane block G. Embed G on a sphere and call some point interior to f the North pole.

Consider a plane tangent to the sphere at the South pole and project G onto that plane from the North pole.

The result is a plane graph isomorphic to G in which f is the exterior face.

Corollary :

Every planar graph can be embedded in the plane so that a prescribed line is an edge of the exterior region.

Theorem 4.17. *Every maximal planar graph with $P \geq 4$ points is 3-connected.*

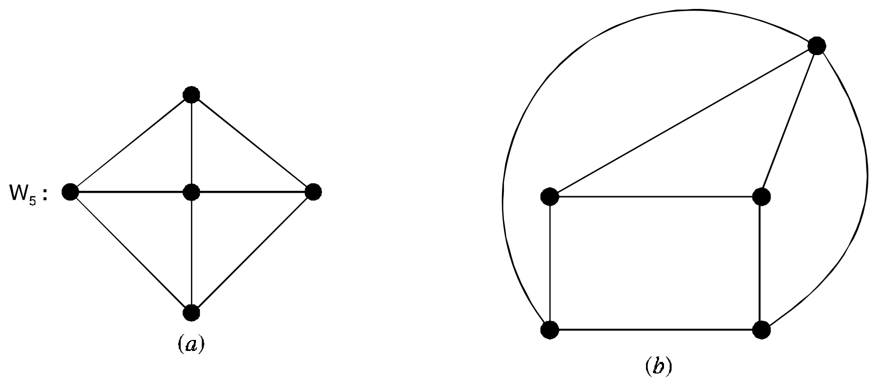


Fig. 4.65. Plane wheels.

There are five ways of embedding the 3-connected wheel W_5 in the plane : one looks like Figure (4.65)(*a*) and the other four look like Figure (4.65)(*b*).

However, there is only one way of embedding W_5 on a sphere, an observation which holds for all 3-connected graphs.

Theorem 4.18. *Every 3-connected planar graph is uniquely embeddable on the sphere.*

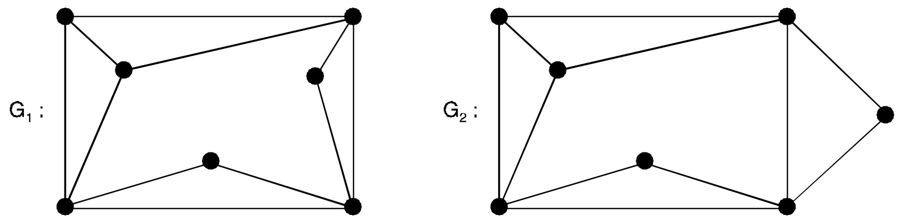


Fig. 4.66. Two plane embeddings of a 2-connected graph.

To show the necessity of 3-connectedness, consider the isomorphic graphs G_1 and G_2 of connectivity 2 shown in Figure above.

The graph G_1 is embedded on the sphere so that none of its regions are bounded by five edges while G_2 has two regions bounded by five edges.

Theorem 4.19. *A graph is the 1-skeleton of a convex 3-dimensional polyhedron if and only if it is planar and 3-connected.*

Theorem 4.20. *Every planar graph is isomorphic with a plane graph in which all edges are straight segments.*

Theorem 4.21. *A graph G is outer planar if and only if each of its blocks is outerplanar.*

Theorem 4.22. *Let G be a maximal outerplane graph with P ≥ 3 vertices all lying on the exterior face. Then G has P – 2 interior faces.*

Proof. Obviously the result holds for P = 3.

Suppose it is true for P = n and let G have P = n + 1 vertices and m interior faces.

Clearly G must have a vertex v of degree 2 on its exterior face.

In forming G – v we reduce the number of interior faces by 1 so that $m - 1 = n - 2$.

Thus $m = n - 1 = P - 2$, the number of interior faces of G.

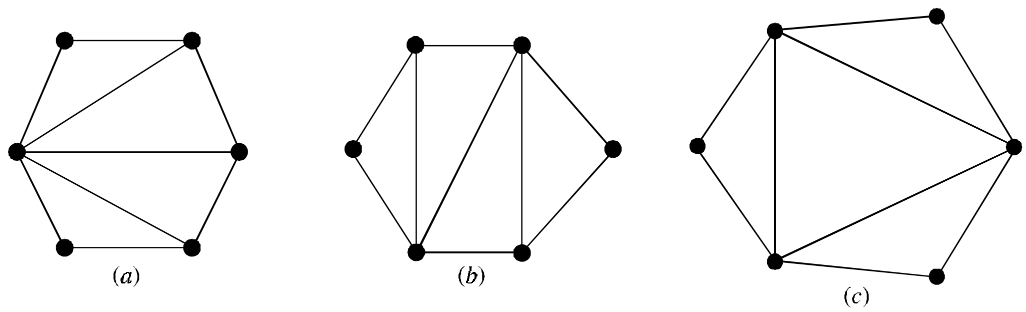


Fig. 4.67. Three maximal outerplanar graphs.

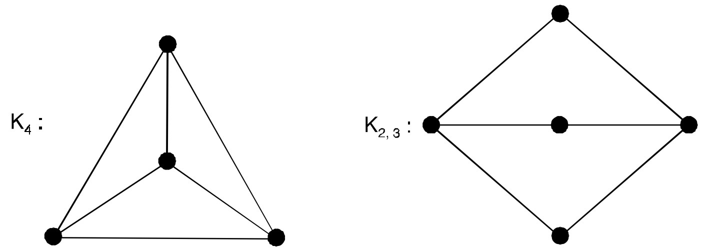


Fig. 4.68. The forbidden graphs for outer planarity.

Corollary :

Every maximal out planar graph G with P points has

(*a*) 2P – 3 lines

(*b*) at least three points of degree not exceeding 3.

(*c*) at least two points of degree 2.

(*d*) K(G) = 2.

All plane embeddings of K_4 and $K_{2,3}$ are of the forms shown in Figure (4.68) above, in which each has a vertex inside the exterior cycle.

Therefore, neither of these graphs is outer planar.

Theorem 4.23. *A graph is outer planar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$.*

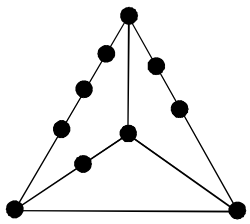


Fig. 4.69. A homeomorph of K_4.

Theorem 4.24. *Every planar graph with atleast nine points has a non planar complement, and nine is the smallest such number.*

Theorem 4.25. *Every outerplanar graph with atleast seven points has a non outer planar complement, and seven is the smallest such number.*

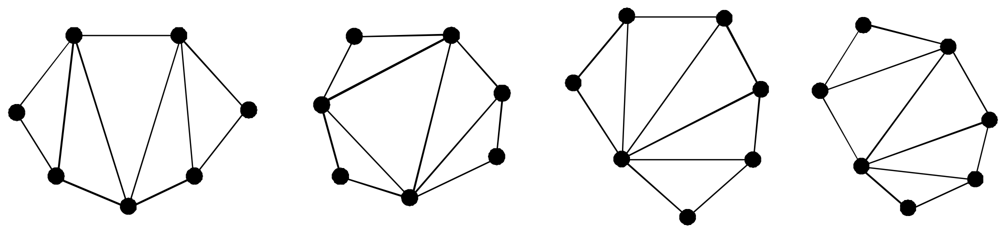


Fig. 4.70. The four maximal outer planar graphs with seven points.

Proof. To prove the first part, it is sufficient to verify that the complement of every maximal outerplanar graph with seven points is not outer planar.

This holds because there are exactly four maximal outer planar graphs with P = 7. (See Figure above) and the complement of each is readily seen to be non outer planar.

The minimality follows from the fact that the (maximal) outer planar graph of Figure below, with six points has an outer planar complement.

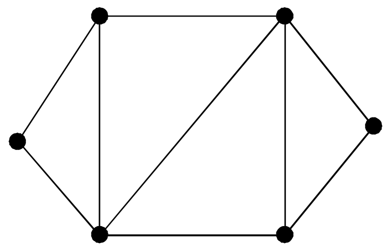


Fig. 4.71.

Lemma 1.

There is a cycle in F containing u_0 and v_0.

Proof. Assume that there is no cycle in F containing u_0 and v_0.

Then u_0 and v_0 lie in different blocks of F.

Hence, there exists a cut point W of F lying on every $u_0 - v_0$ path.

We form the graph F_0 by adding to F the lines wu_0 and wv_0 if they are not already present in F.

In the graph F_0, u_0 and v_0 still lie in different blocks, say B_1 and B_2, which necessarily have the point W in common. Certainly, each of B_1 and B_2 has fewer lives than G, so either B_1 is planar or it contains a subgraph homeomorphic to K_5 or $K_{3, 3}$.

If, however, the insertion of wu_0 produces a subgraph H of B_1 homeomorphic to K_5 or $K_{3, 3}$, then the subgraph of G obtained by replacing wu_0 by a path from u_0 to W which begins with x_0 is necessarily homeomorphic to H and so to K_5 or $K_{3, 3}$, but this is a contradiction.

Hence, B_1 and similarly B_2 is planar. Both B_1 and B_2 can be drawn in the plane so that the lines wu_0 and wv_0 bound the exterior region.

Hence it is possible to embed the graph F_0 in the plane with both wu_0 and wv_0 on the exterior region.

Inserting x_0 cannot then destroy the planarity of F_0. Since G is a subgraph of $F_0 + x_0$, G is planar, this contradiction shows that there is a cycle in F containing u_0 and v_0.

Let F be embedded in the plane in such a way that a cycle Z containing u_0 and v_0 has a maximum number of regions interior to it.

Orient the edges of Z in a cyclic fashion, and let Z[*u, v*] denote the oriented path from *u* to *v* along Z.

If *v* does not immediately follow *u* to *z*, we also write Z(*u, v*) to indicate the subgraph of Z[*u, v*] obtained by removing *u* and *v*.

By the exterior of cycle Z, we mean the subgraph of F induced by the vertices lying outside Z, and the components of this subgraph are called the exterior components of Z.

By an outer piece of Z, we mean a connected subgraph of F induced by all edges incident with atleast one vertex in some exterior component or by an edge (if any) exterior to Z meeting two vertices of Z. In a like manner, we define the interior of cycle Z, interior component, and inner piece.

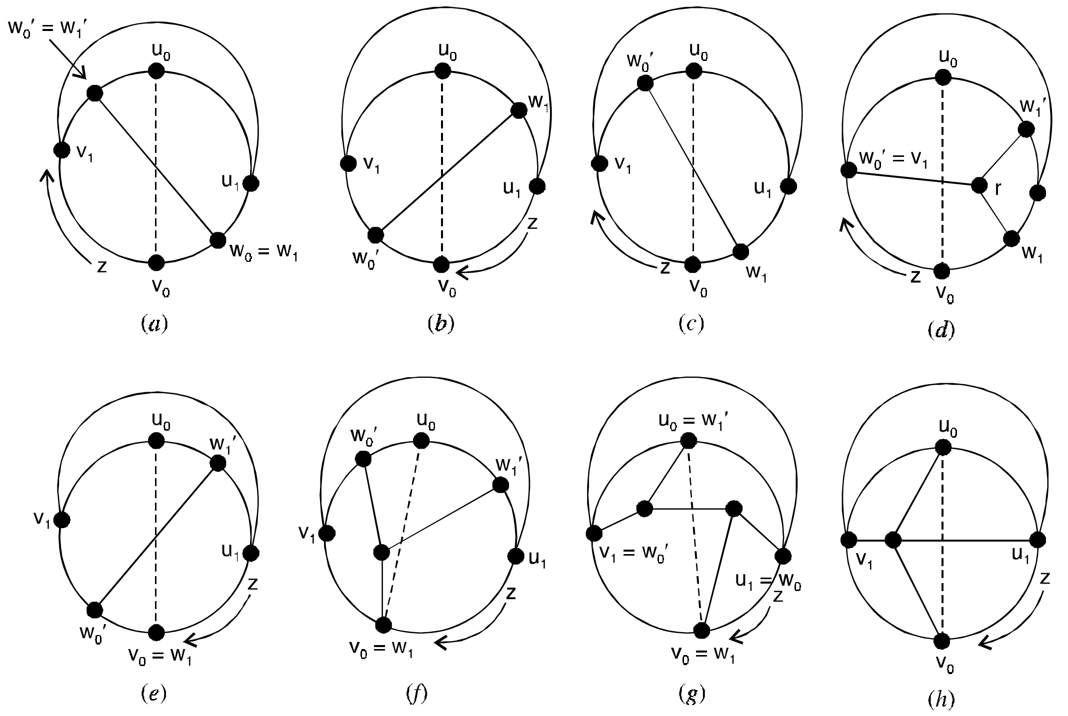


Fig. 4.72. Separating cycle Z illustrating lemma.

An outer or inner piece is called $u - v$ separating if it meets both Z(u, v) and Z(v, u).

Clearly, an outer or inner piece cannot be $u - v$ separating if u and v are adjacent on Z.

Since F is connected, each outer piece must meet Z, and because F has no cut vertices, each outer piece must have atleast two vertices in common with Z.

No outer piece can meet Z(u_0, v_0) or Z(v_0, u_0) in more than one vertex, for otherwise there would exist a cycle containing u_0 and v_0 with more interior regions than Z.

For the same region, no outer piece can meet u_0 or v_0.

Hence every outer piece meets Z in exactly two vertices and is $u_0 - v_0$ separating.

Furthermore, since x_0 cannot be added to F in planar fashion, there is at least one $u_0 - v_0$ separating inner piece.

Lemma 2.

There exists a $u_0 - v_0$ separating outer piece meeting Z(u_0, v_0), say at u_1, and Z(v_0, u_0), say at v_1, such that there is an inner piece which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating.

Proof. Suppose, to the contrary, that the lemma does not hold. It will be helpful in understanding this proof to refer to Figure (4.73).

We order the $u_0 - v_0$ separating inner pieces for the purpose of relocating them in the plane. Consider any $u_0 - v_0$ separating inner piece I_1 which is nearest to u_0 in the sense of encountering points of this inner piece on moving along Z from u_0. Continuing out from u_0, we can index the $u_0 - v_0$ separating inner pieces I_2, I_3 and so on.

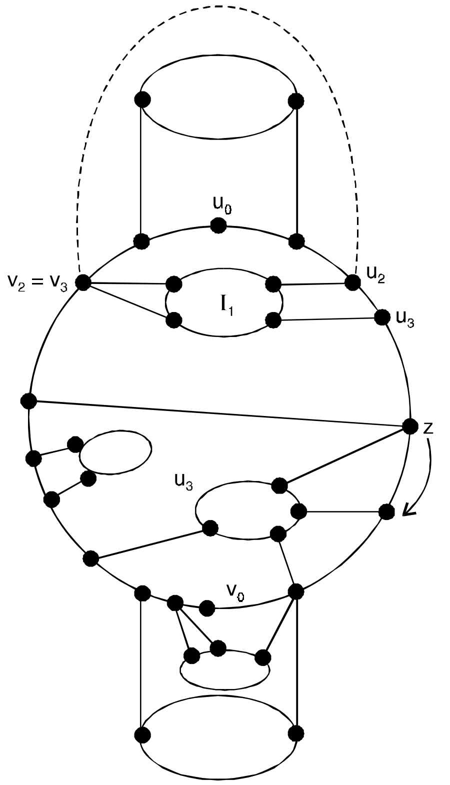


Fig. 4.73. The possibilities for non planar graphs.

Let u_2 and u_3 be the first and last points of I_1 meeting $Z(u_0, v_0)$ and v_2 and v_3 be the first and last vertices of I_1 meeting $Z(v_0, u_0)$.

Every outer piece necessarily has both its common vertices with Z on either $Z[v_3, u_2]$ or $Z[u_2, v_2]$, for otherwise, there would exist an outer piece meeting $Z(u_0, v_0)$ at u_1 and $Z(v_0, u_0)$ at v_1 and an inner piece which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating, contrary to the supposition that the lemma is false.

Therefore, a curve C joining v_3 and u_2 can be drawn in the exterior region so that it meets no edge of F (see Figure (4.73).

Thus, I_1 can be transferred outside of C in a planar manner.

Similarly, the remaining $u_0 - v_0$ separating inner pieces can be transferred outside of Z, in order, so that the resulting graph is plane.

However, the edge x_0 can then be added without destroying the planarity of F, but this is a contradiction, completing the lemma.

4.23. KURATOWSKI'S THEOREM

A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Proof. Let H be the inner piece guaranteeed by lemma (2) which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating. In addition, let w_0, w_0', w_1 and w_1' be vertices at which H meets $Z(u_0, v_0)$, $Z(v_0, u_0)$, $Z(u_1, v_1)$ and $Z(v_1, u_1)$ respectively.

There are now four cases to consider, depending on the relative position on Z of these four vertices.

Case 1. One of the vertices w_1 and w_1' is on $Z(u_0, v_0)$ and the other is on $Z(v_0, u_0)$.

We can then take, say, $w_0 = w_1$ and $w_0' = w_1'$, in which case G contains a subgraph homeomorphic to $K_{3,3}$ as indicated in Figure (4.72)(*a*) in which the two sets of vertices are indicated by open and closed dots.

Case 2. Both vertices w_1 and w_1' are on either $Z(u_0, v_0)$ or $Z(v_0, u_0)$.

Without loss of generality we assume the first situation. There are two possibilities : either $v_1 \neq w_0'$ or $v_1 = w_0'$.

If $v_1 \neq w_0'$, then G contains a subgraph homeomorphic to $K_{3,3}$ as shown in Figure (4.72)(*b* or *c*), dependending on whether w_0' lies on $Z(u_1, v_1)$ or $Z(v_1, u_1)$ respectively.

If $v_1 = w_0'$ (see Figure 4.72), then H contains a vertex r from which there exist disjoint paths to w_1, w_1' and v_1, all of whose vertices (except w_1, w_1' and v_1) belong to H.

In this case also, G contains a subgraph homeomorphic to $K_{3,3}$.

Case 3. $w_1 = v_0$ and $w_1' \neq u_0$.

Without loss of generality, let w_1' be on $Z(u_0, v_0)$. Once again G contains a subgraph homeomorphic to $K_{3,3}$.

If w_0' is on (v_0, v_1), then G has a subgraph $K_{3,3}$ as shown in Figure 4.72(*e*).

If, on the other hand, w_0' is on $Z(v_1, u_0)$, there is a $K_{3,3}$ as indicated in Figure 2.44(*f*).

This Figure is easily modified to show G contains $K_{3,3}$ if $w_0' = v_1$.

Case 4. $w_1 = v_0$ and $w_1' = u_0$.

Here we assume $w_0 = u_1$ and $w_0' = v_1$, for otherwise we are in a situation covered by one of the first 3 cases.

We distinguish between two subcases.

Let P_0 be a shortest path in H from u_0 to v_0, and let P_1 be such a path from u_1 to v_1,

The paths P_0 and P_1 must intersect.

If P_0 and P_1 have more than one vertex in common, then G contains a subgraph homeomorphic to $K_{3,3}$ as shown in Figure 4.72(*g*).

Otherwise, G contains a subgraph homeomorphic to K_5 as in Figure 4.72(*h*).

Since these are all possible cases, the theorem has been proved.

Theorem 4.26. *A graph is planar if and only if it does not have a subgraph contractible to K_5 or $K_{3,3}$.*

4.24 DETECTION OF PLANARITY OF A GRAPH :

If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We take the following simplifying steps :

Elementary Reduction :

Step 1 : Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set.

$$G = \{G_1, G_2, \ldots\ldots G_k\}$$

where each G_i is a non separable block of G.

Then we have to test each G_i for planarity.

Step 2 : Since addition or removal of self-loops does not affect planarity, remove all self-loops.

Step 3 : Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

Step 4 : Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series.

Repeated application of step 3 and 4 will usually reduce a graph drastically.

For example, Figure (4.75) illustrates the series-parallel reduction of the graph of Figure (2.45).

Let the non separable connected graph G_i be reduced to a new graph H_i after the repeated application of step 3 and 4. What will graph H_i look like ?

Graph H_i is

1. A single edge, or
2. A complete graph of four vertices, or
3. A non separable, simple graph with $n \geq 5$ and $e \geq 7$.

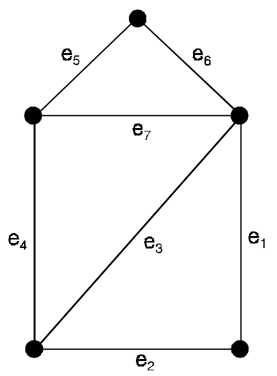


Fig. 4.74.

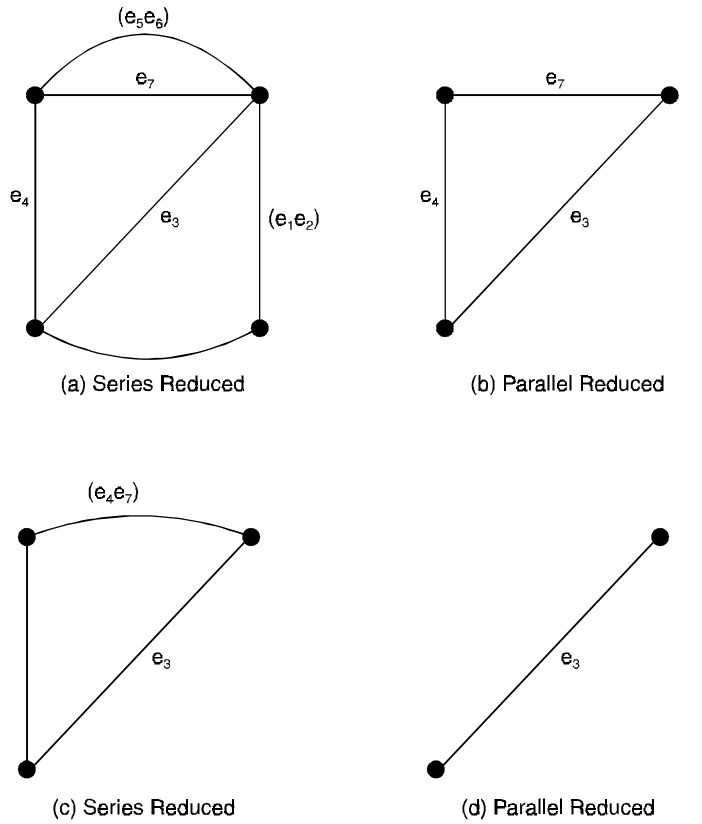


Fig. 4.75. Series-parallel reduction of the graph in Figure 4.74

Problem 4.46. *Check the planarity of the following graph by the method of elementary deduction.*

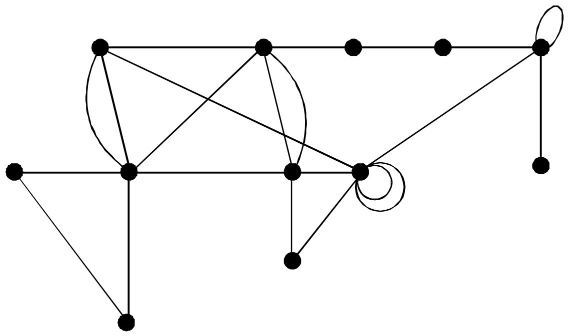


Fig. 4.76.

Solution. Step 1 : Does not apply, because the graph is connected.

Step 2 : Separating blocks of G

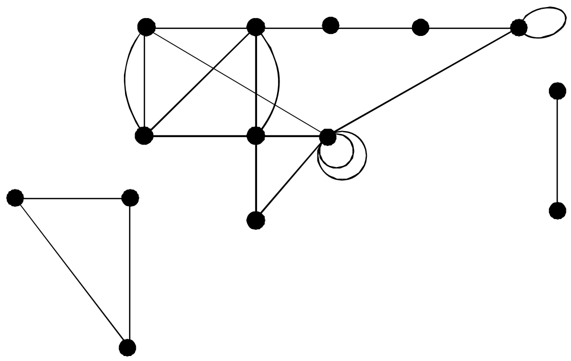


Fig. 4.77.

Step 3 : Removing self-loops and parallel edges

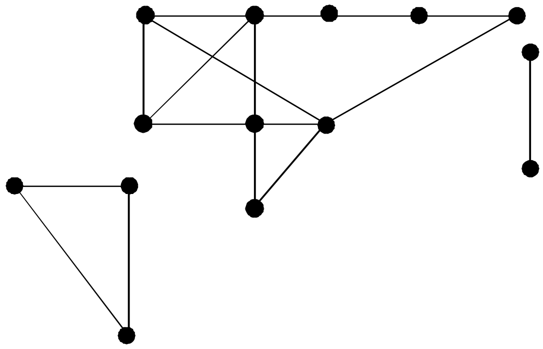


Fig. 4.78.

Step 4 : Merging the series edges.

The final graph contains three components. Largest component contains 7 vertices. Remaining two is triangle and an edge hence they are planar. The largest component contains no subgraph isomorphic to K_5 or $K_{3, 3}$ and hence it is planar.

Thus the given graph is planar.

Problem 4.47. *Check the planarity of the following graph by the method of elementary reduction.*

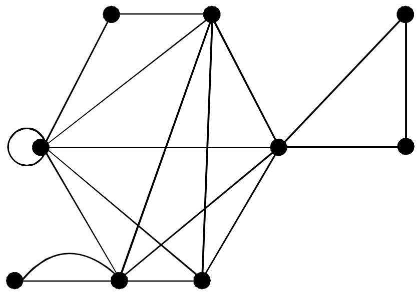


Fig. 4.79.

Solution. The elementary reduction of the given graph G consists of the following stages :

Step 1 : Splitting G into blocks. This splitting is shown below :

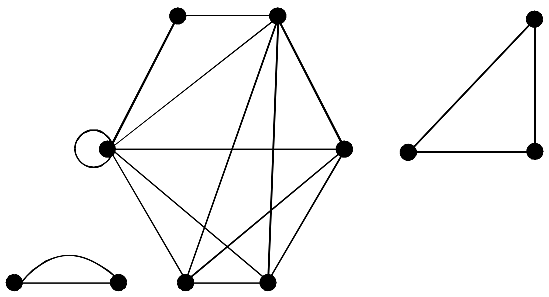


Fig. 4.80.

Step 2 : Removing self-loops and eliminating multiple edges. The resulting graph is as shown below :

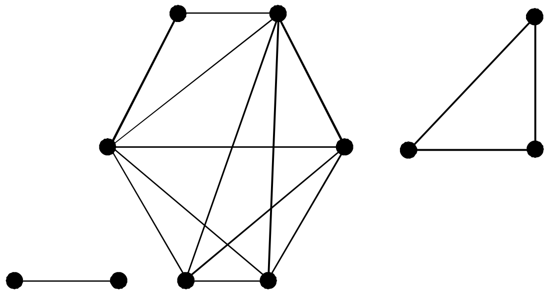


Fig. 4.81.

Step 3 : Merging the edges incident on vertices of degree 2. The resulting graph is as shown below :

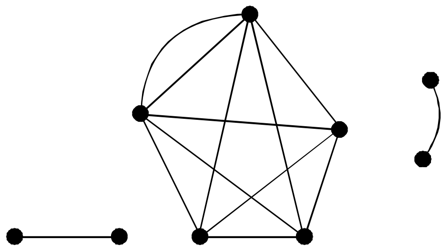


Fig. 4.82.

Step 4 : Eliminating parallel edges. The resulting graph is shown below :

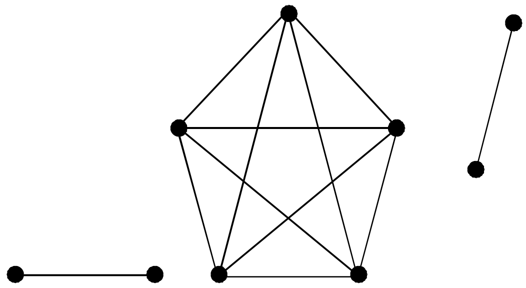


Fig. 4.83.

The reduction is now complete. The final reduced graph (shown in Figure above) has three blocks, of which the first and the third are obviously planar. The second one is evidently the complete graph K_5, which is non planar.

Thus, the given graph contains K_5 as a subgraph and is therefore non planar.

Problem 4.48. *Carryout the elementary reduction process for the following graph :*

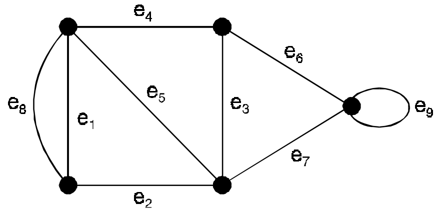


Fig. 4.84.

Solution. The given graph G is a single non separable block. Therefore, the set A of step 1 contains only G. As per step 2, we have to remove the self loops. In the graph, there is one self-loop consisting of the edge e_9. Let us remove it.

As per step 3, we have to remove one of the two parallel edges from each vertex pair having such edges. In the given graph, e_1, e_8 are parallel edges. Let us remove e_8 from the graph.

The graph left-out after the first three steps is as shown below :

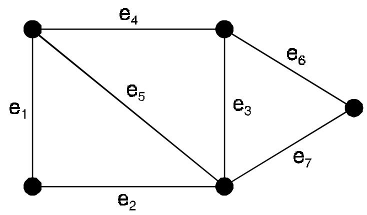


Fig. 4.85.

As per step 4, we have to eliminate the vertices of degree 2 by merging the edges incident on these vertices.

Thus, we merge (*i*) the edges e_1 and e_2 into an edge e_{10} (say) and (*ii*) the edges e_6 and e_7 into an edge e_{11} (say).

The resulting graph will be as shown below :

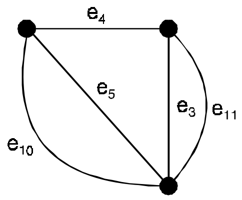


Fig. 4.86.

As per step 3, let us remove one of the parallel edges e_5 and e_{10} and one of the parallel edges e_3 and e_{11}. The graph got by removing e_{10} and e_{11} will be as shown below :

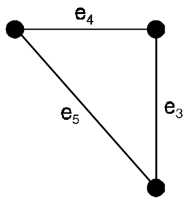


Fig. 4.87.

As per step 4, we merge the edges e_3 and e_4 into an edge e_{12} (say) to get the following graph.

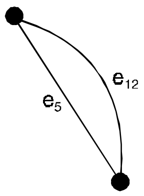


Fig. 4.88.

As per step 3, we remove one of the two parallel edges, say e_{12}. Thus, we get the following graph :

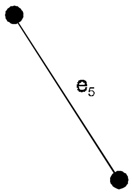


Fig. 4.89.

This graph is the final graph obtained by the process of elementary reduction applied to the graph in Figure (4.89). This final graph which is a single edge is evidently a planar graph.

Therefore, the graph in Figure (4.89) is also planar.

4.25 DUAL OF A PLANAR GRAPH

Consider the plane representation of a graph in Figure (4.90)(*a*) with six regions of faces F_1, F_2, F_3, F_4, F_5 and F_6.

Let us place six points P_1, P_2, ... P_6, one in each of the regions, as shown in Figure (4.90)(*b*).

Next let us join these six points according to the following procedure :

(*i*) If two regions F_i and F_j are adjacent (*i.e.,* have a common edge), draw a line joining points P_i and P_j that intersects the common edge between F_i and F_j exactly once.

(*ii*) If there is more than one edge common between F_i and F_j, draw one line between points P_i and P_j for each of the common edges.

(*iii*) For an edge *e* lying entirely in one region, say F_k, draw a self-loop at point P_k intersecting *e* exactly once.

By this procedure we obtained a new graph G* (in broken lines in Figure (2.60)(*c*) consisting of six vertices, P_1, P_2, ... P_6 and of edges joining these vertices. Such a graph G* is called **dual** (a geometrical dual) of G

Clearly, there is a one-to-one correspondence between the edges of graph G and its dual G*—one edge of G* intersecting one edge of G. Some simple observations that can be made about the relationship between a planar graph G and its dual G* are :

(*i*) An edge forming a self-loop in G yields a pendant edge in G*.

(*ii*) A pendant edge in G yields a self-loop in G*.

(*iii*) Edges that are in series in G produce parallel edges in G*.

(*iv*) Parallel edges in G produce edges in series in G*.

(*v*) Remarks (*i*)-(*iv*) are the result of the general observation that the number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex P_i in G*.

(*vi*) Graph G* is also embedded in the plane and is therefore planar.

(*vii*) Considering the process of drawing a dual G* from G, it is evident that G is a dual of G* (see Fig. (4.90) (*c*)). Therefore, instead of calling G* a dual of G, we usually say that G and G* are dual graphs.

(*viii*) If *n, e, f, r* and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n^*, e^*, f^*, r^* and μ^* are the corresponding numbers in dual graph G*, then

$$n^* = f,\ e^* = e,\ f^* = n.$$

Using the above relationship, one can immediately get $r^* = \mu$, $\mu^* = r$.

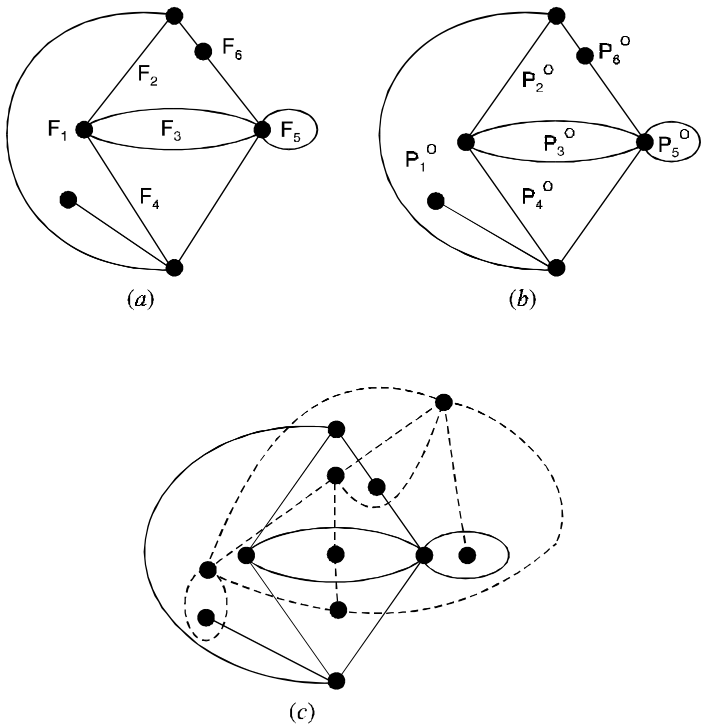


Fig. 4.90. Construction of a dual graph.

4.25.1. Uniqueness of the dual

Given a planar graph G, we can construct more than one geometric dual of G. All the duals so constructed have one important property. This property is stated in the following result :

All geometric duals of a planar graph G are 2-isomorphic, and every graph 2-isomorphic to a geometric dual of G is also a geometric dual of G.

4.25.2. Double dual

Given a planar graph G, suppose we construct its geometric dual G* and the geometric dual G** of G*.

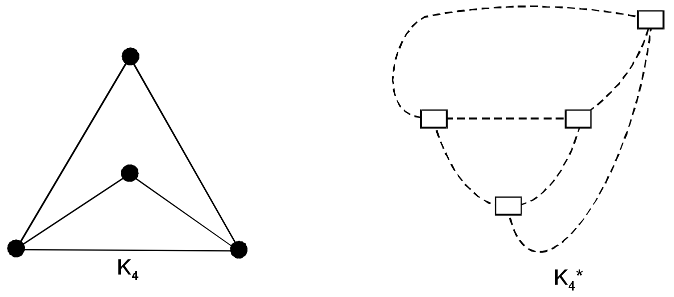
Then G** is called a double geometric dual of G.

If G is a planar graph, then G** and G are 2-isomorphic.

4.25.3. Self-dual graphs

A planar graph G is said to be self-dual if G is isomorphic to its geometric dual G*, *i.e.,* if $G \approx G^*$.

Consider the complete graph K_4 of four vertices shown in Figure (4.91)(*a*). Its geometric dual K_4^* can be constructed. This is shown in Figure (4.91)(*b*).



(*a*) (*b*)

Fig. 4.91.

We observe that K_4* has four vertices and six edges. Also, every two vertices of K_4* are joined by an edge. This means that K_4* also represents the complete graph of four vertices. As such, K_4 and K_4* are isomorphic. In other words, K_4 is a self-dual graph.

4.25.4. Dual of a subgraph

Let G be a planar graph and G* be its geometric dual. Let *e* be an edge in G and *e** be its dual in G*. Consider the subgraph G – *e* got by deleting *e* from G. Then, the geometric dual of G – *e* can be constructed as explained in the two possible cases.

Case (1) :

Suppose *e* is on a boundary common to two regions in G.

Then the removal of *e* from G will merge these two regions into one.

Then the two corresponding vertices in G* get merged into one, and the edge *e** gets deleted from G*.

Thus, in this case, the dual of G – *e* can be obtained from G* by deleting the edge *e** and then fusing the two end vertices of *e** in G* – *e**.

Case (2) :

Suppose *e* is not on a boundary common to two regions in G.

Then *e* is a pendant edge and *e** is a self-loop.

The dual of G – *e* is now the same as G* – *e**.

Thus, the geometric dual of G – *e* can be constructed for all choices of the edge *e* of G.

Since every subgraph H of a graph is of the form G – *s* where *s* is a set edges of G.

4.25.5. Dual of a homeomorphic graph

Let G be a planar graph and G* be its geometric dual.

Let *e* be an edge in G and *e** be its dual in G*.

Suppose we create an additional vertex in G by introducing a vertex of degree 2 in the edge *e*. This will simply add an edge parallel to *e** in G*. If we merge two edges in series in G then one of the corresponding parallel edges in G* will be eliminated. The dual of any graph homeomorphic to G can be obtained from G*.

4.25.6. Abstract dual

Given two graphs G_1 and G_2, we say that G_1 and G_2 are abstract duals of each other if there is a one-to-one correspondence between the edges in G_1 and the edges in G_2, with the property that a set of edges in G_1 forms a circuit in G_1 if and only if the corresponding set of edges in G_2 forms a cut-set in G_2.

Consider the graphs G_1 and G_2 shown in Figure (4.92).

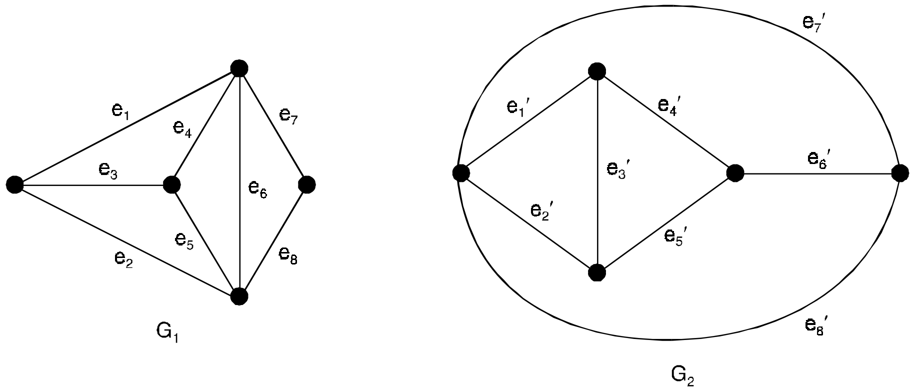


Fig. 4.92.

We observe that there is a one-to-one correspondence between the edges in G_1 and the edges in G_2 with the edge e_i in G_1 corresponding to the edge e_i' in G_2, $i = 1, 2, ... 8$.

Further, note that a set of edges in G_1 which forms a circuit in G_1 corresponds to a set of edges in G_2 which forms a cut sets in G_2.

For example, $\{e_6, e_7, e_8\}$ is a circuit in G_1 and $\{e_6', e_7', e_8')$ is a cut-set in G_2.

Accordingly, G_1 and G_2 are abstract duals of each other.

4.25.7. Combinatorial dual

Given two planar graphs G_1 and G_2, we say that they are combinatorial duals of each other if there is a one-to-one correspondence between the edges of G_1 and G_2 such that if H_1 is any subgraph of G_1 and H_2 is the corresponding subgraph of G_2, then

Rank of $(G_2 - H_2)$ = Rank of G_2 – Nullity of H_1

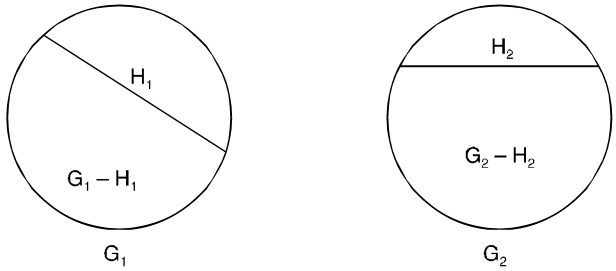


Fig. 4.93.

Consider the graph G_1 and G_2 shown in Figure (4.92) above, and their subgraphs H_1 and H_2 shown in Figure (4.94)(*a*, *b*).

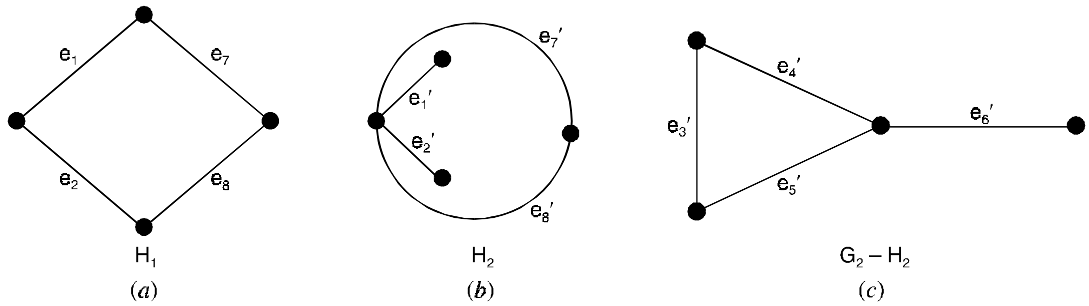


Fig. 4.94.

Note that there is one-to-one correspondence between the edges of G_1 and G_2 and that the subgraphs H_1 and H_2 correspond to each other.

The graph of $G_2 - H_2$ is shown in Figure (4.94)(*c*).

This graph is disconnected and has two components.

Rank of $G_2 = 5 - 1 = 4$, Rank of $H_1 = 4 - 1 = 3$

Nullity of $H_1 = 4 - 3 = 1$

Rank of $(G_2 - H_2) = 5 - 2 = 3$.

$\Rightarrow$ Rank of $(G_2 - H_2) = 3 =$ Rank of G_2 – Nullity of H_1.

Hence, G_1 and G_2 are combinatorial duals of each other.

Theorem 4.27. *If G is a plane connected graph, then G** is isomorphic to G.*

Proof. The result follows immediately, since the construction that gives rise to G* from G can be reversed to give G from G*,

For example, in Figure (4.95), the graph G is the dual of the graph G*

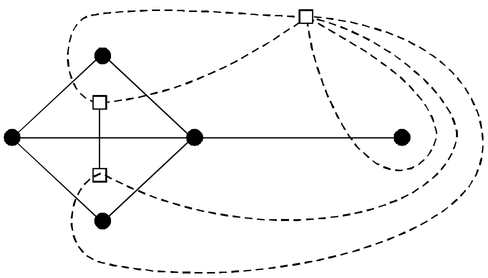


Fig. 4.95.

We need to check only that a face of G* cannot contain more than one vertex of G (it certainly contains at least one) and this follows immediately from the relations $n^{**} = f^* = n$, where n^{**} is the number of vertices of G**.

Theorem 4.28. *Let G be a planar graph and let G* be a geometric dual of G. Then a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G* forms a cutset in G*.*

Proof. We can assume that G is a connected plane graph. If C is a cycle in G, then C encloses one or more finite faces C, and thus contains in its interior a non-empty set S of vertices of G*.

It follows immediately that choose edges of G* that cross the edges of C form a cutset of G* whose removal disconnects G* into two subgraphs, one with vertex set S and the other containing those vertices that do not lie in S (see Figure 4.96).

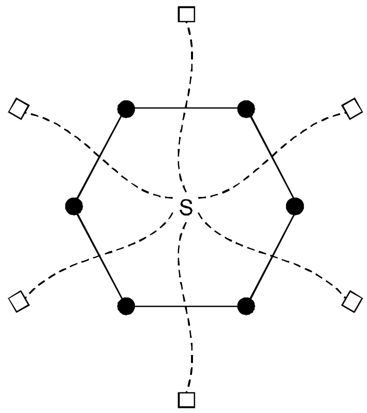


Fig. 4.96.

Corollary : A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G* forms a cycle in G*.

Theorem 4.29. *If G* is an abstract dual of G, then G is an abstract dual of G*.*

Proof. Let C be a cutset of G and let C* denote the corresponding set of edges of G*.

We show that C* is a cycle of C*.

C has an even number of edges in common with any cycle of G, and so C* has an even number of edges in common with any cut set of G*.

C* is either a cycle in G* or an edge-disjoint union of at least two cycles.

But the second possibility cannot occur, since we can show similarly that cycles in C* correspond to edge-disjoint unions of cut sets in G, and so C would be an edge-disjoint union of at least two cutsets, rather than a single cutset.

Theorem 4.30. *A graph is planar if and only if it has an abstract dual.*

Proof. It is sufficient to prove that if G is a graph with an abstract dual G*, then G is planar. The proof is in four steps.

(*i*) We note first that if an edge *e* is removed from G, then the abstract dual of the remaining graph may be obtained from G* by contracting the corresponding edge *e**.

On repeating this procedure, we deduce that, if G has an abstract dual, then so does any subgraph of G.

(*ii*) We next observe that if G has an abstract dual, and G′ is homeomorphic to G, then G′ also has an abstract dual.

This follows from the fact that the insertion or removal in G of a vertex of degree 2 results in the addition or deletion of a multiple edge in G*.

(*iii*) The third step is to show that neither K_5 nor $K_{3,3}$ has an abstract dual.

If G* is a dual of $K_{3,3}$ then since $K_{3,3}$ contains only cycles of length 4 or 6 and no cutsets with two edges, G* contains no multiple edges and each vertex of G* has degree at least 4.

Hence G* be have at least five vertices, and thus atleast $\frac{(5\times 4)}{2} = 10$ edges, which is a contradiction.

The argument for K_5 is similar and is omitted.

(*iv*) Suppose, now, that G is a non-planar graph with an abstract dual G*.

Then, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 to $K_{3,3}$.

It follows from (*i*) and (*ii*) that H, and hence also K_5 or $K_{3,3}$, must have an abstract dual, contradicting (*iii*).

Theorem 4.31. *Let G be a connected planar graph with n vertices, m edges and r regions, and let its geometric dual G* have n* vertices, m* edges and r* regions. Then n* = r, m* = m, r* = n.*

Further, if ρ *and* ρ^* *are the ranks and* μ *and* μ^* *are the nullities of G and G* respectively, then* $\rho^* = \mu$ *and* $\mu^* = \rho$.

Proof. Every region of G yields exactly one vertex of G* and G* has no other vertex.

Hence the number of regions in G is precisely equal to the number of vertices of G*,

i.e., $r = n^*$. ...(1)

Corresponding to every edge e of G, there is exactly one edge e^* of G* that crosses e exactly once, and G* has no other edge.

Thus G and G* have the same number of edges,

i.e., $m = m^*$ ...(2)

Now, the Euler's formula applied to G* and G yields

$$\begin{aligned} r^* &= m^* - n^* + 2 \\ &= m - r + 2 \\ &= n \end{aligned}$$

Since G and G* are connected, we have

$$\rho = n - 1, \quad \mu = m - n + 1$$
$$\rho^* = n^* - 1, \quad \mu^* = m^* - n^* + 1$$

These together with the results (1) and (2) and the Euler's formula yield

$$\begin{aligned} \rho^* &= n^* - 1 = r - 1 = (m - n + 2) - 1 \\ &= m - n + 1 = \mu \\ \mu^* &= m^* - n^* + 1 = m - r + 1 \\ &= m - (m - n + 2) + 1 = n - 1 = \rho. \end{aligned}$$

Theorem 4.32. *A graph has a dual if and only if it is planar.*

Proof. Suppose that a graph G is planar.

Then G has a geometric dual in G*.

Since G* is a geometric dual, it is a dual.

Thus G has a dual.

Conversely, suppose G has a dual.

Assume that G is non planar. Then by Kuratowski's theorem, G contains K_5 and $K_{3,3}$ or a graph homeomorphic to either of these as a subgraph.

But K_5 and $K_{3,3}$ have no duals and therefore a graph homeomorphic to either of these also has no dual.

Thus, G contains a subgraph which has no dual.

Hence G has no dual. This is a contradiction.

Hence G is planar if it has a dual.

Problem 4.49. *If G is a 3-connected planar graph, prove that its geometric dual is a simple graph.*

Solution. If G is 3-connected, then G has no vertices of degree 1 or 2.

Therefore, G* has no self-loops or multiple edges. That is, G* is simple.

Problem 4.50. *Show that a connected planar self-dual graph G with n vertices should have 2n – 2 edges.*

Solution. Since the graph G is self-dual, we have $n = n^*$. But $n^* = r$,

Therefore, in G, $n = r$,

The Euler's formula now gives $n = m - n + 2$

or $$m = 2n - 2.$$

Problem 4.51. *Show that a set of edges in a connected planar graph G forms a spanning tree of G if and only if the set of duals of the remaining edges forms a spanning tree of a geometric dual of G.*

Solution. Consider a connected planar graph G with n vertices and m edges.

Let T be a spanning tree of G. This is a set of $n - 1$ edges. The remaining edges are $m - (n - 1)$ in number.

The duals of these edges are also $m - (n - 1)$ in number.

The set T* of these duals belong to G*.

Since G* has $m - n + 2$ vertices, the set T* which consists of $m - n + 1$ vertices is a spanning tree of G*.

This proves the first part of the required result.

By reversing the roles of G and G* in the above argument, we get the second proof.

Problem 4.52. *Show that there is no planar graph with five regions such that there is an edge between every pair of regions.*

Solution. Suppose there is a planar graph G having the desired property.

Then, the geometric dual G* of G will have five vertices such that there is an edge between every pair of vertices.

This means that G* is the graph K_5.

Therefore, G* is non planar.

This is a contradiction because G* has to be planar. (like G).

Hence, a planar graph of the desired type does not exist.

Problem 4.53. *Disprove that the geometric dual of the geometric dual of a planar graph G is the same as the abstract dual of the abstract dual of G.*

Solution. Consider the disconnected graph G with two components, each of which is a triangle as shown in Figure (4.97)(*a*).

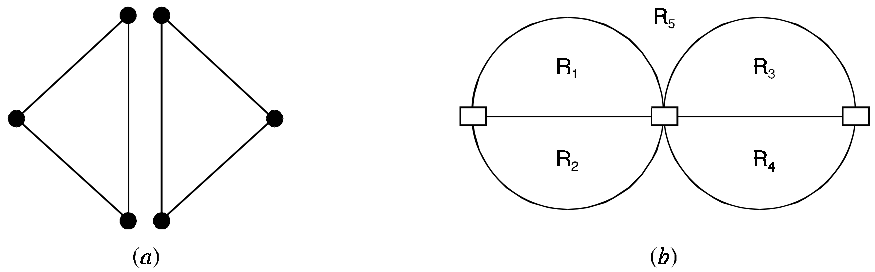


Fig. 4.97.

The geometric dual G* is shown in Figure (4.97)(*b*), we observe that G* has five regions.

Therefore, the geometric dual G** of G* has five vertices.

On the other hand, if G′ is the abstract dual of G, then G is the abstract dual of G′.

Hence, G is the abstract dual of G. *i.e.,* G = G″.

Since G has six vertices, it follows that G″ cannot be the same as G** (which has five vertices).

The above counter example disproves that the geometric dual of the geometric dual is the same as the abstract dual of the abstract dual.

Problem 4.54. *Let G be a connected planar graph. Prove that G is bipartite if and only if its dual is on Euler graph.*

Solution. If G is bipartite, then each circuit of G has even length.

Therefore, each cutset of its dual G′ has an even number of edges.

In particular, each vertex of G′ has even degree.

Therefore G′ is an Euler graph.

Theorem 4.33. *Let G be a plane connected graph. Then G is isomorphic to its double dual G**.*

Proof. Let f be any face of the dual G* contains atleast one vertex of G, namely its corresponding vertex v.

In fact this is the only vertex of G that f contains since by theorem.

i.e., a connected graph G with n-vertices, e-edges, f-faces and n^*, e^*, f^* denotes the vertices, edges and faces of G* then $n^* = f$, $e^* = e$, $f^* = n$, the number of faces of G* is the same as the number of vertices of G.

Hence in the construction of double dual G**, we may choose the vertex v to be the vertex in G** corresponding to face f of G*.

This choice gives our required result.

4.26 KÖNIGSBERG'S BRIDGE PROBLEM

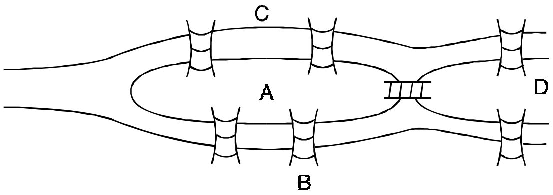


Fig. 4.98.

There were two islands linked to each other to the bank of the Pregel river by seven bridges as shown above.

The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.

One can easily try to solve this problem, but all attempts must be unsuccessful. In proving that, the problem is unsolvable. Euler replaced each land area by a vertex and each bridge by an edge joining the corresponding vertices, there by producing a 'graph' as shown below :

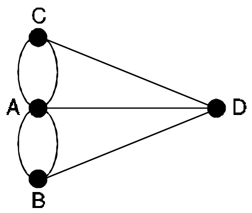


Fig. 4.99.

4.27 REPRESENTATION OF GRAPHS

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of nodes and edges is reasonably small.

Two types of representation are given below :

4.27.1. Matrix representation

The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

4.27.2. Adjacency matrix

4.27.2(*a*) Representation of undirected graph

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n matrix A = $\{a_{ij}\}$ whose elements are given by a_{ij} = 1, if there is an edge between ith and jth vertices, and

= O, if there is no edge between them.

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices.

Hence, there are as many as n ! different adjacency matrices for a graph with n vertices, since there are n ! different ordering of n vertices.

However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

There are a number of observations that one can make about the adjacency matrix A of a graph G are :

Observations :

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j

(*ii*) The entries along the principal diagonal of A all zeros if and only if the graph has no self loops. A self loop at the vertex corresponding to $a_{ij} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i vertex v_j.

(*v*) If G be a graph with n vertices $v_1, v_2, \ldots\ldots v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots\ldots + A^{n-1}$$

Then G is a connected graph if B has no zero entries of the main diagonal.

This result can be also used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_1 is represented by a 1 at the (i, j)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (i, j)th entry equals the number of edges these are associated to $\{v_i - v_j\}$.

All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

4.27.2(*b*) Representation of directed graph

The adjacency matrix of a diagonal D, with n vertices is the matrix $A = \{a_{ij}\}_{n \times n}$ in which

$$a_{ij} = 1 \text{ if arc } \{v_i - v_j\} \text{ is in D}$$
$$= 0 \text{ otherwise.}$$

One can make a number of observations about the adjacency matrix of a diagonal.

Observations

(*i*) A is not necessary symmetric, since there may not be an edges from v_i to v_j when there is an edge from v_i to v_j.

(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(*iv*) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j entries of A^T. A shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices.

In the adjacency matrix for a directed multigraph a_{ij} equals the number of edges that are associated to (v_i, v_j).

4.27.3. Incidence matrix

4.27.3(*a*) Representation of undirected graph

Consider a undirected graph G = (V, E) which has n vertices and m edges all labelled. The incidence matrix B = $\{b_{ij}\}$, is then $n \times m$ matrix,

where $b_{ij} = 1$ when edge e_j is incident with v_i

$= 0$ otherwise

We can make a number of observations about the incidence matrix B of G.

Observations :

(*i*) Each column of B comprises exactly two unit entries.

(*ii*) A row with all O entries corresponds to an isolated vertex.

(*iii*) A row with a single unit entry corresponds to a pendent vertex.

(*iv*) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(*v*) The permutation of any two rows (any two columns) of B corresponds to a labelling of the vertices (edges) of G.

(*vi*) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(*vii*) If G is connected with n vertices then the rank of B is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

4.27.3(*b*) Representation of directed graph

The incidence matrix B = $\{b_{ij}\}$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

$b_{ij} = 1$ if arc j is directed away from a vertex v_i

$= -1$ if arc j directed towards vertex v_i

$= 0$ otherwise.

4.27.4. Linked representation

In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a case has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex.

Problem 4.55. *Use adjacency matrix to represent the graphs shown in Figure below*

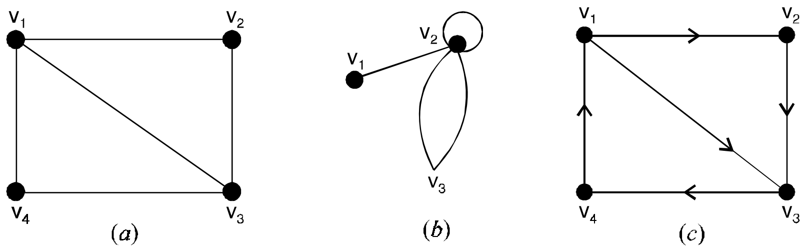


Fig. 4.100

Solution. We order the vertices in Figure (4.100)(*a*) as v_1, v_2, v_3 and v_4.

Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We order the vertices in Figure (4.100)(*b*) as v_1, v_2 and v_3. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Taking the order of the vertices in Figure(4.100)(*c*) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4.56. *Draw the undirected graph represented by adjacency matrix A given by*

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5.

Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is drawn in Figure below.

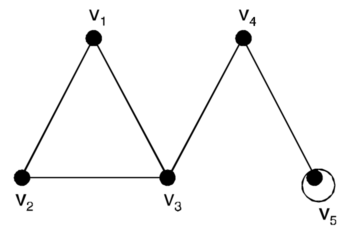


Fig. 4.101

Problem 4.57. *Draw the digraph G corresponding to adjacency matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is shown in Figure below.

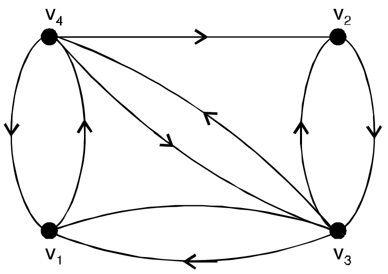


Fig. 4.102

Problem 4.58. *Draw the undirected graph G corresponding to adjacency matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Solution. Since the given adjacency matrix is square matrix of order 4, G has four vertices v_1, v_2, v_3 and v_4. Draw n edges from v_i to v_j where $a_{ij} = n$.

Also draw n loop at v_i where $a_{ij} = n$.

The required graph is shown in Figure below.

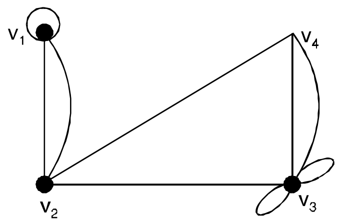


Fig. 4.103

Problem 4.59. *Show that the graphs G and G ′ are isomorphic*

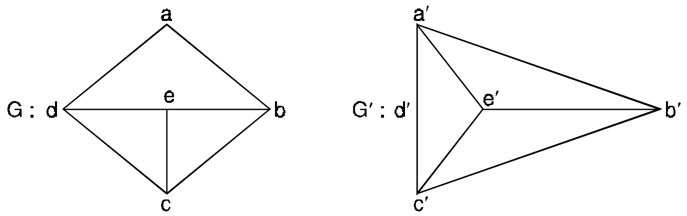


Fig. 4.104

Solution. Consider the map $f: G \to G'$ define as $f(a) = d'$, $f(b) = a'$, $f(c) = b'$, $f(d) = c'$ and $f(e) = e'$.

The adjacency matrix of G for the ordering *a*, *b*, *c*, *d* and *e* is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G′ for the ordering *d*′, *a*′, *b*′, *c*′ and *e*′ is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array}\begin{array}{c} \begin{array}{ccccc} d' & a' & b' & c' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e., A(G) = A(G′)

∴ G and G′ are isomorphic.

Problem 4.60. *Find the incidence matrix to represent the graph shown in Figure below :*

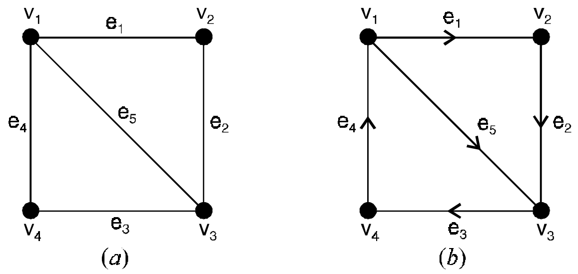


Fig. 4.105

Solution. The incidence matrix of Figure (*a*) is obtained by entering for row *v* and column *e* is 1 if *e* is incident on *v* and 0 otherwise. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the graph of Figure (*b*) is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Problem 4.61. *Use an adjacency matrix to represent the graph shown in Figure below :*

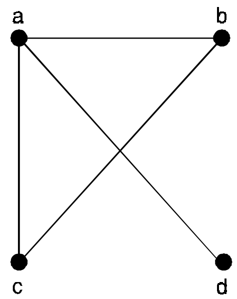


Fig. 4.106

Solution. We order the vertices as *a*, *b*, *c*, *d*.

The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4.62. *Draw a graph with the adjacency matrix*

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Solution. A graph with this adjacency matrix is shown in Figure below :

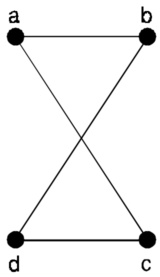


Fig. 4.107

Problem 4.63. *Use an adjacency matrix to represent the pseudograph shown in Figure below :*

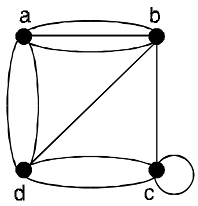


Fig. 4.108

Solution. The adjacency matrix using the ordering of vertices *a*, *b*, *c*, *d* is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Problem 4.64. *Represent the graph shown in Figure below, with an incidence matrix.*

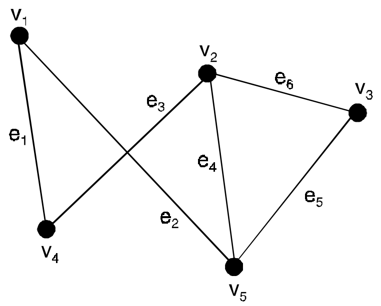


Fig. 4.109

Solution. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Problem 4.65. *Represent the Pseudograph shown in Figure below, using an incidence matrix.*

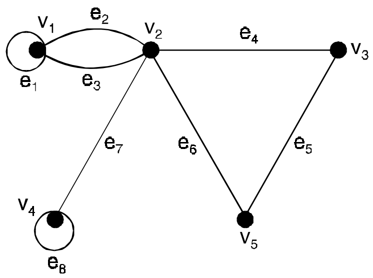


Fig. 4.110

Solution. The incidence matrix for this graph is :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problem 4.66. *Write adjacency structure for the graphs shown in Figure.*

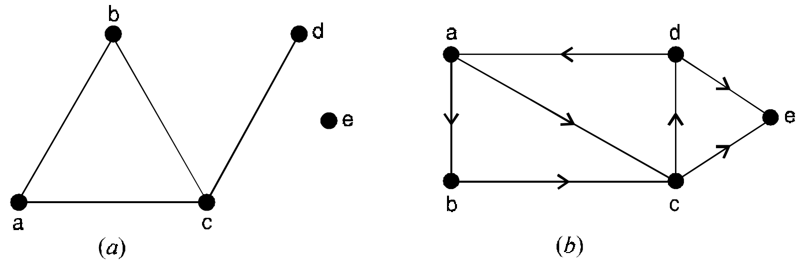


Fig. 4.111

Solution. The adjacency structure representation is given in the table for Figure (*a*).

Here the symbol ϕ is used to denote the empty list.

Vertex	*Adjacency list*
a	*b*, *c*
b	*a*, *c*
c	*a*, *b*, *d*
d	*e*
e	ϕ

The adjacency structure representation is given in the table for the directed graph shown in Figure (*b*).

Vertex	*Adjacency list*
a	*b*, *c*
b	*c*
c	*d*
d	*a*, *e*
e	*c*

Problem Set 4.1

1. Define :

(*i*) Cutset and cut vertex
(*ii*) Edge connectivity
(*iii*) Vertex connectivity
(*iv*) Strongly connected and weakly connected

2. Write a short note on ‘‘Transport Networks’’.

3. State and prove Max-Flow-Min-Cut Theorem.

4. Define :

(*i*) Combinatorial and Geometric graphs

(*ii*) Planar graphs
(*iii*) Kuratoski's graphs
(*iv*) Homeomorphic graphs

5. Define :
(*i*) Maximal planar graphs (*ii*) Subdivision graphs
(*iii*) Inner vertex set (*iv*) Outer planar graphs

6. Define :
(*i*) Crossing Number (*ii*) Bipartite graph
(*iii*) Complete Bipartite graph

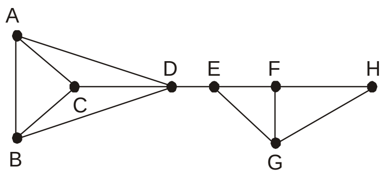
7. State and prove Euler's formula for planar graph.

8. Explain detection of planarity of a graph.

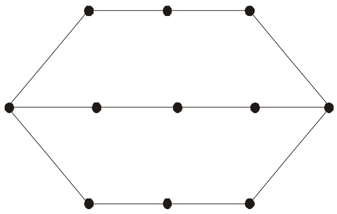
9. Define :
(*i*) Dual of a planar graph (*ii*) Dual of a subgraph
(*iii*) Double dual (*iv*) Self-dual graphs
(*v*) Dual of a homeomorphic graph (*vi*) Abstract dual
(*vii*) Combinatorial dual.

10. Define :
(*i*) Adjacency matrix (*ii*) Incidence matrix

11. Find the V(G), E(G) and deg(G) for the graph of the figure below :



(*i*)



(*ii*)

12. Prove that in any graph G, V(G) ≤ E(G) ≤ deg (G).

13. Let *f* be a flow in network N = (V, E). If C = (P, $\overline{P}$) is any cut in N then prove that val (*f*) cannot exceed C(P, $\overline{P}$).

14. If *f* is a flow in a transport network N = (V, E) then prove that the value of the flow the source *a* is equal to the value of the flow into the sink Z.

15. Prove that the value of any flow in a given transport network is less than or equal to the capacity of any cut in the network.

16. Prove that, in any directed network, the value of an (s, t) flow never exceeds the capacity of any (s, t)-cut.

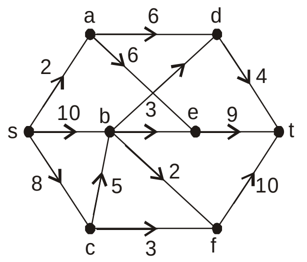
17. If F is any (s, t)-flow and (S, T) is any (s, t)-cut then show that val(F), $\sum_{u \in S, v \in T} (f_{uv} - f_{vu})$.

18. Suppose there exists some (s, t)-flow F and some (s, t)-cut {S, T} such that the value of F equals the capacity of {S, T}. Then show that val(F) is the maximum value of any flow and cap(S, T) is the minimum capacity of any cut.

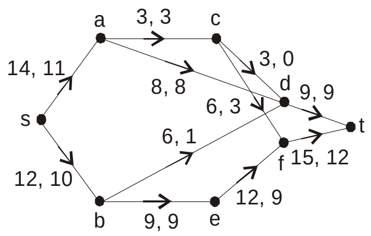
19. Prove that, in any network, the value of any maximum flow is equal to the capacity of any minimum cut.

20. Why does the procedure just described of adding an amount q to the forward arcs of a chain and subtracting the same amount from the backward arcs serve conservation of flow at each vertex ?

21. Find the maximum flow in the directed network shown in figure below and prove that it is a maximum.



22. Find the capacity of the (s, t)-cut defined by S = {s, a, b, d} and T = {c, e, f, t}.



23. Show that C_6 is a bipartite graph.

24. Prove tha a graph which contains a triangle cannot be bipartite.

25. If a connected planar graph G has n vertices, e edges and r region then show that $n - e + r = 2$.

26. If G is connected simple planar graph with ($n \geq 3$) vertices and e edges then show that $e \leq 3n - 6$.

27. If G is connected simple planar graph with ($n \leq 3$) vertices and e edges and no circuits of length 3 then show that $e \leq 2n - 4$.

28. Show that the graph K_5 is not coplanar.

29. Show that the graph $K_{3,\,3}$ is not coplanar.

30. Show that K_n is a planar graph for $n \leq 4$ and non-planar for $n \geq 5$.

31. A connected plane graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane ?

32. Show that $K_{3,\,3}$ is a non-planar graph.

33. Prove that K_4 and $K_{2,\,2}$ are planar.

34. If every region of a simple planar graph with n-vertices and e-edges embeded in a plane is bounded by K-edges then show that $e = \dfrac{K(n-2)}{K-2}$.

35. Show that $K_{3, 3}$ and K_5 are non-planar.

36. Suppose G is a graph with 1000 vertices and 3000 edges. Is G planar ?

37. Let G be a simple connected planar (P, q)-graph having atleast K edges in a boundary of each region, then show that $(K - 2)q \le K(P - 2)$.

38. A connected graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ? How many faces are there ?

39. Find a graph G with degree sequence (4, 4, 3, 3, 3, 3) such that (*i*) G is planar and (*ii*) G is non-planar.

40. How many edges must a planar graph have if it has 7 regions and 5 vertices.

41. Show that the petersen graph is non-planar.

42. Find a smallest planar graph that is regular of degree 4.

43. Show that the following graphs are planar (*i*) graph of order 5 and size 8 (*ii*) graph of order 6 and size 12.

44. If a planar graph G of order n and size m has r regions and K components then show that $n - m + r = K + 1$.

45. Let G be a connected simple planar (n, m) graph in which every region is bounded by atleast K edges then show that $m \le \dfrac{K(n-2)}{K-2}$.

46. Show that these does not exist a connected simple planar graph with $m = 7$ edges and with degree $\delta = 3$.

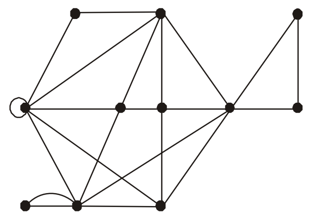
47. Let G be a simple connected planar graph with fewer than 12 regions, in which each vertex has degree atleast 3. Prove that G has a region bounded by atmost four edges.

48. Show that every simple connected planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.

49. Let G be a maximal outerplane graph with $P \ge 3$ vertices all lying on the exterior face, then prove that G has P–2 interior faces.

50. Prove that, A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3, 3}$.

51. Check the planarity of the following graph by the method of elementary reduction.



52. If G is a plane connected graph then prove that G** is isomorphic to G.

53. Prove that, A graph is planar if and only if it has an abstract dual

54. If G* is an abstract dual of G then prove that G is an abstract dual of G*.

55. Let G be a planar graph and let G* be a geometric dual of G, then show that a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G* forms a cutset in G*.

56. If P and P* are the ranks and μ and μ^* are the nullities of G and G* respectively then show that $\rho^* = \mu$ and $\mu^* = \rho$.

57. Show that, A graph has a dual if and only if it is planar.

58. Show that there is no planar graph with five regions such that there is an edge between every pair of regions.

59. If G is a 3-connected planar graph prove that its geometric dual is a simple graph.

60. Show that a set of edges in a connected planar graph G forms a spanning tree of G if and only if the set of duals of the remaining edges forms a spanning tree of a geometric dual of G.

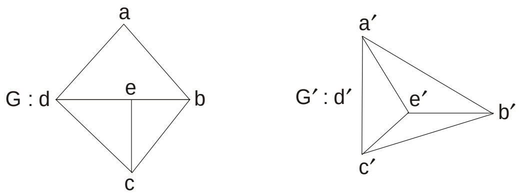
61. Let G be a plane connected graph then prove that G is a isomorphic to its double dual G**.

62. Let G be a connected planar graph, prove that G is bipartite if and only if its dual is on Euler graph.

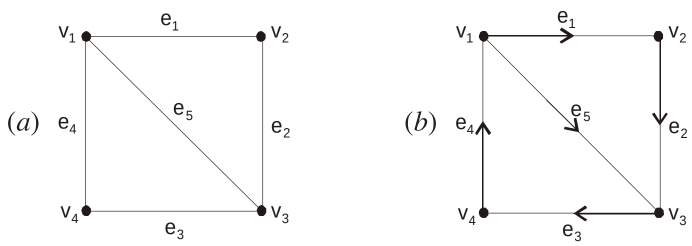
63. Draw the undirected graph represented by adjacency matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

64. Show that the graphs G and G′ are isomorphic.



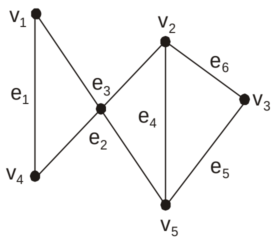
65. Find the incidence matrix to represent the graph shown in figure below.



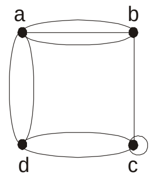
66. Draw a graph with the adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ with respect to the ordering of vertices *a*, *b*, *c*, *d*.

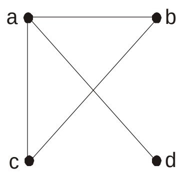
67. Represent the graph shown in figure below, with an incidence matrix.



68. Use an adjacency matrix to represent the pseudograph shown in figure below.



69. Use an adjacency matrix to represent the graph in figure below.



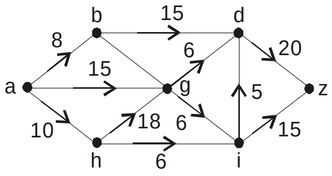
70. D is prove that the geometric dual of the geometrical dual of a planar graph G is the same as the abstract dual of the abstract dual of G.

71. Find a maximal (s, t)-flow. Verify your answer by fnding an (s, t)-cut whose capacity equals the value of the flow.

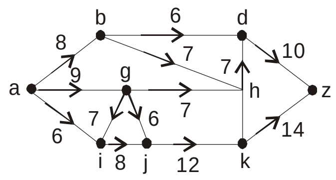
72. Using the concept of flow in a transport network, construct a directed multigraph G = (V, E) with V = {*u*, *v*, *w*, *x*, *y*} and id(*u*) = 1 ; *od*(*u*) = 3 ; id(*v*) = 3 ; od(*w*) = 3 ; od(*w*) = 4 ; id(*x*) = 5 ; od(*x*) = 4 ; and id(*y*) = 4 ; od(*y*) = 2.

Problem Set 4.2

1. Find a maximum flow and the corresponding minimum cut for each transport network shown in figure below.



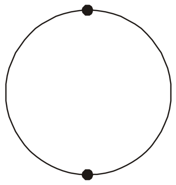
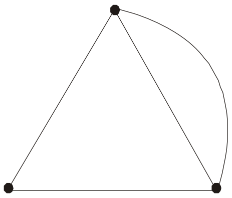
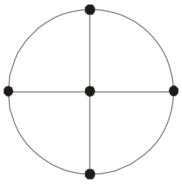
(*i*)



(*ii*)

2. Show that a simple graph with n vertices and more than $\left[\frac{n^2}{4}\right]$ edges cannot be 2-chromatic.

3. Show that the following graphs are self dual

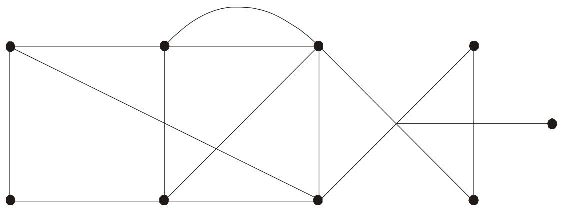
(*i*)  (*ii*)  (*iii*) 

4. Prove that 5-connected planar graph has atleast 12 vertices.

5. Prove that a self loop free planar graph G is 2-connected if and only if G* is 2-connected.

6. Let G be a planar connected graph. Prove that G s bipartite if and only if G* is an Euler graph.

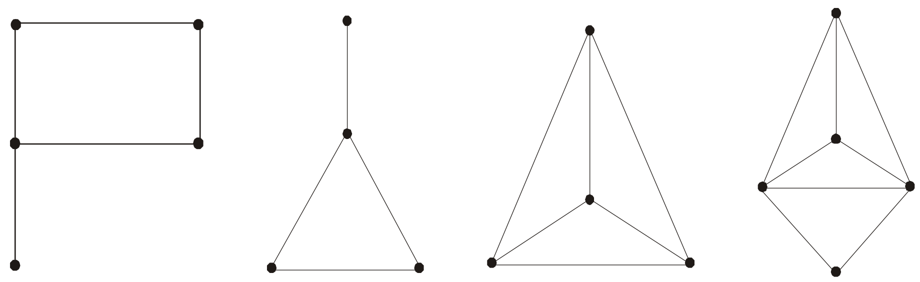
7. By using the method of elementary reduction, show that the following graph is planar.



8. What is the minimum number of vertices necessary for a simple connected graph with 7 edges to be planar ?

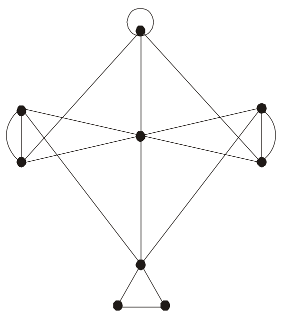
9. Prove or disprove that in a graph of order n and size m, $\chi(G) \leq 1 + \frac{2m}{n}$.

10. Find the chromatic numbers of the following



(*i*) (*ii*) (*iii*) (*iv*)

11. By the method of elementary reduction show that the following graph is non planar.



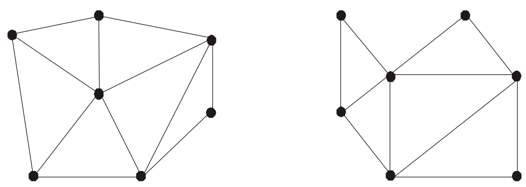
12. Show that a simple planar connected graph with less than 30 edges must have a vertex of degree ≤ 4.

13. Prove that the sum of the degree of the regions of a planar graph is equal to twice the number of edges in the graph.

14. Let G be a connected planar graph with more than two vertices. If G has exactly n_K vertices of degree K and $\Delta(G) = P$, show that $5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geq n_7 + 2n_8 + \ldots\ldots + (P - 6)n_P + 12$.

15. Verify the Euler's formula for the graphs W_8, $K_{1,5}$ and $K_{2,7}$.

16. Verify the Euler's formula for the graphs shown below



17. Show that every graph with four or fewer vertices is planar.

18. Show that the graphs $K_{1,S}$ for $S \geq 1$ and $K_{2,3}$ for $S \geq 2$ are planar.

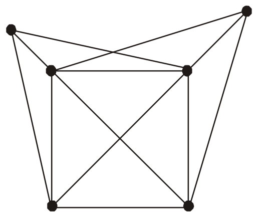
19. Let G be a simple connected graph with atleast 11 vertices prove that either G or its complement $\overline{G}$ must be non-planar.

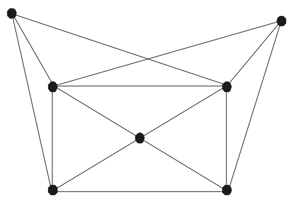
20. Prove that every subgraph of a planar graph is planar.

21. Prove that K_5 is the non planar graph with the smallest number of vertices.

22. Prove that $K_{3,3}$ is the non planar graph with the smallest number of edges.

23. Find its chromtic number and explain why this piece of information is consistent with the four color problem.





24. Show that any graph homeomorphic to K_5 or $K_{3,3}$ is obtainable from K_5 or $K_{3,3}$ respectively, by addition of vertices to edges.

25. Suppose a graph G_1 with V_1 vertices and E_1 edges is homeomorphic to a graph G_2 with V_2 vertices and E_2 edges prove that $E_2 - E_1 = E_1 - V_1$.

26. Show that $K_{2,2}$ is homeomorphic to K_3.

27. For which n is K_n planar ?

28. Let G be a connected planar bipartite graph with E edges and $V \geq 3$ vertices. Prove that $E \leq 2V - 4$.

29. If G is a connected plane with atleast three vertices such that no boundary of a region is a triangle, prove that $E \leq 2V - 4$.

30. Let G be a connected plane graph for which $E = 3V - 6$ show that every region of G is a triangle.

31. Give an example of a connected planar graph for which $E = 3V - 6$.

32. If G is a connected plane graph with $V \geq 3$ vertices and R regions. Show that $R \leq 2V - 4$.

33. Verify Euler's formula $V - E + F = 2$ for each of the five platanoic solids.

34. Verify that $V - E + R = 2$, $N \leq 2E$ and $E \leq 3V - 6$.

35. Label the regions defined by your plane graph and list the edges which form the boundary of each region.

36. Show that the graph is planar by drawing an isomorphic plane graph with straight edges.

37. Let G be a connected graph with V vertices ; E edges ; and $E \leq V + 2$, show that G is planar.

38. Show that any planar graph all of whose vertices have degree atleast 5 must have atleast 12 vertices.

39. Find a planar graph each of whose vertices has degree atleast 5.

40. Let G be a graph and let H be obtained from G by adjoining a new vertex of degree 1 to some vertex of G. Is it possible for G and H to be Homeomorphic ? Explain :

41. Let G be a connected graph with V_1 vertices and E_1 edges and let H be a subgraph with V_2 vertices and E_2 edges. Show that $E_2 - V_2 = E_1 - V_1$.

42. Prove that if G is a planar graph with n connected components, each components having at least three vertices then $E \le 3V - 6n$.

43. Prove that if G is a planar graph with n connected components then it is always true that $E \le 3V - 3n$.

44. A connected planar graph G has 20 vertices. Prove that G has atmost 54 edges.

45. A connected planar graph G has 20 vertices, seven of which have degree 1. Prove that G has at most 40 edges.

46. Prove that every planar graph with $V \ge 2$ vertices has at least two vertices of degree $d \le 5$.

47. What is $\chi(K_{14})$ and what is $\chi(K_{5,14})$? Why.

48. Let G_1 and G_2 by cycles with 38 and 107 edges, respectively. What is $\chi(G_1)$? What is $\chi(G_2)$? Explain.

49. Let $n \ge 4$ be a natural number. Let G be the graph which consists of the union of K_{n-3} and a 5-cycle C together with all possible edges between the vertices of these graphs. Show that $\chi(G) = n$, yet G does not have K_n as a subgraph.

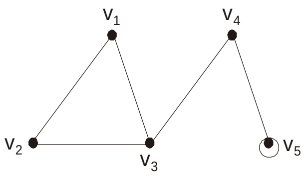
50. Find a formula for $V - E + R$ which applies to planar graphs which are not necessarily connected.

Answers 4.1

21. Maximum flow : 14 **22.** 21 **31.** $r = 7$

38. 14 edges and 7 regions **40.** 10 edges

63.



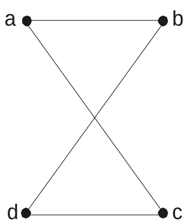
65. (*a*)

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

66.

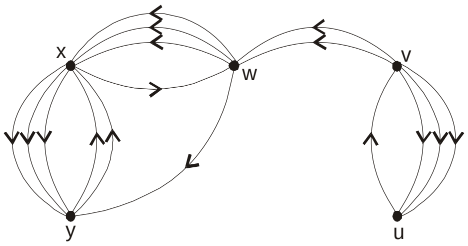


67.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

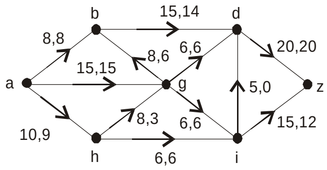
68. $\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$ **69.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

72.



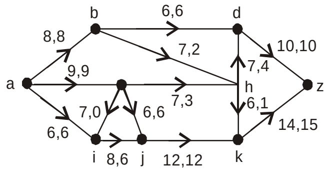
Answers 4.2

1. (*i*)



The maximum flow is 32 which is C(P, $\overline{P}$) for P = {*a*, *b*, *d*, *g*, *h*} and $\overline{P}$ = {*i*, *z*}.

(*ii*)

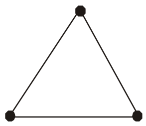


The maximum flow is 23, which is C{P, $\overline{P}$ } for P = {*a*} and $\overline{P}$ = {*b*, *g*, *i*, *j*, *d*, *h*, *k*, *z*}

10. (*i*) 2 (*ii*) 3 (*iii*) 4 (*iv*) 4

27. K_n is planar if and only if $n \le 4$.

31.



E = 3(3) – 6 = 3 ($\because$ E = 3V – 6)

here V = 3

32. We know that E ≤ 3V – 6, substituting E = V + R – 2
we obtain V + R – 2 ≤ 3V – 6 or R ≤ 2V – 4 as required.

33.

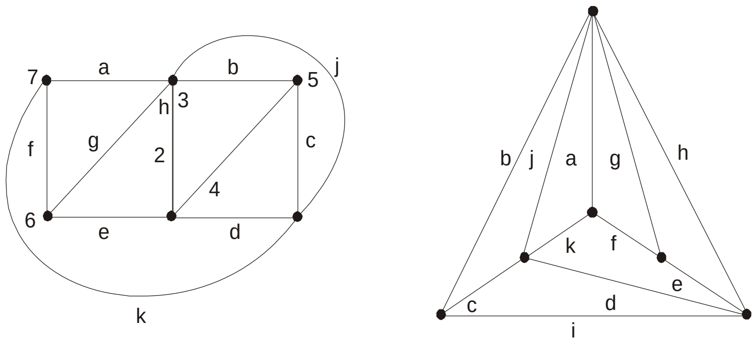
Solid	V	E	F	V – E + F
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	201	30	12	2
icosahedron	12	30	20	2

34. E = 11, V = 6, R = 7, N = 22, So V – E + R = 6 – 11 + 7 = 22

N = 22 ≤ 22 = 2E and E = 11 ≤ 12 ≤ 3V – 6

35. There are seven regions, numered 1, 2, ...7, with boundaries *afg, ghe, hbi, icd, bjc, fedk,* and *ajk* respectively.

36. We draw the graph quickly as a planar and then after some thinking, as a planar graph with straight edges.



47. For any n, $\chi(K_n) = n$ and for any m, n, $\chi(K_{m,n}) = 2$.

Thus $\chi(K_{14}) = 14$ and $\chi(K_{5,14}) = 2$.

50. Letting x denote the number of connected components of G, we have V – E + R = 1 + x, for each component C, $V_C - R_C = 2$, Adding, we get

$\Sigma V_C - \Sigma E_C + ER_C = 2x$. We have $\Sigma V_C = V$ and $\Sigma E_C = E$, but $\Sigma R_C = R + (x - 1)$ since the exterior region is common to all components.

Thus, $V - E + R + x - 1 = 2x$; $V - E' + R' = x + 1$.

Unit-5

Coloring, Digraphs and Enumeration

5.1 GRAPH COLORING

5.1.1. Coloring problem

Suppose that you are given a graph G with n vertices and are asked to paint its vertices such that no two adjacent vertices have the same color. What is the minimum number of colors that you would require. This constitutes a coloring problem.

5.1.2. Partitioning problem

Having painted the vertices, you can group them into different sets—one set consisting of all red vertices, another of blue, and so forth. This is a partitioning problem.

For example, finding a spanning tree in a connected graph is equivalent to partitioning the edges into two sets—one set consisting of the edges included in the spanning tree, and the other consisting of the remaining edges. Identification of a Hamiltonian circuit (if it exists) is another partitioning of set of edges in a given graph.

5.1.3. Properly coloring of a graph

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the proper coloring (or simply coloring) of a graph.

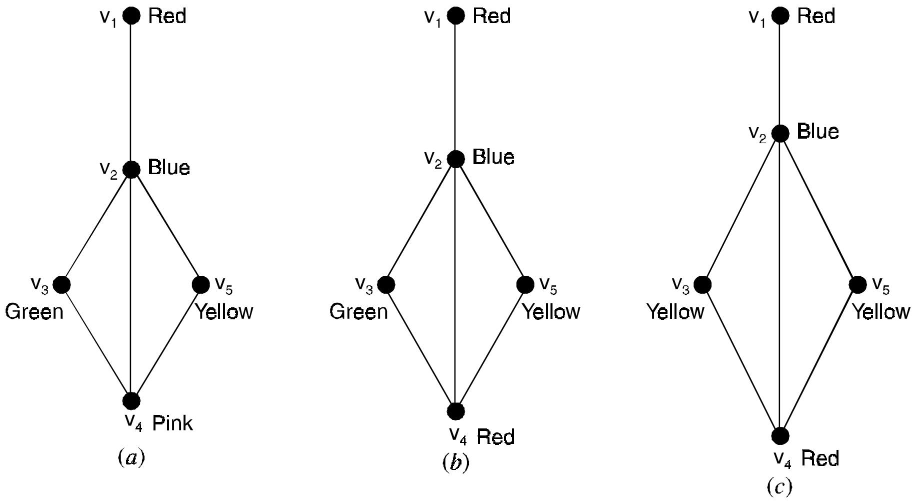


Fig. 5.1. Proper colors of a graph.

A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.

Usually a given graph can be properly colored in many different ways. Figure 5.1 shows three different proper coloring of a graph.

The K-colorings of the graph G is a coloring of graph G using K-colors. If the graph G has coloring, then the graph G is said to be K-colorable.

5.1.4. Chromatic number

A graph G is said to be K-colorable if we can properly color it with K (number of) colors.

A graph G which is n-colorable but not (K – 1)-colorable is called a K-chromatic graph.

In other words, a K-chromatic graph is a graph that can be properly colored with K-colors but not with less than K colors.

If a graph G is K-chromatic, then K is called chromatic number of the graph G. Thus the chromatic number of a graph is the smallest number of colors with which the graph can be properly colored.The chromatic number of a graph G is usually denoted by $\chi(G)$.

We list a few rules that may be helpful :

1. $\chi(G) \le |V|$, where $|V|$ is the number of vertices of G.

2. A triangle always requires three colors, that is $\chi(K_3) = 3$; more generally, $\chi(K_n) = n$, where K_n is the complete graph on n vertices.

3. If some subgraph of G requires K colors then $\chi(G) \ge K$.

4. If degree $(v) = d$, then atmost d colors are required to color the vertices adjacent to v.

5. $\chi(G) = \text{maximum } \{\chi(C)/C \text{ is a connected component of } G\}$

6. Every K-chromatic graph has at least K vertices v such that degree $(v) \ge k - 1$.

7. For any graph G, $\chi(G) \le 1 + \Delta(G)$, where $\Delta(G)$ is the largest degree of any vertex of G.

8. When building a K-coloring of a graph G, we may delete all vertices of degree less than K (along with their incident edges).

In general, when attempting to build a K-coloring of a graph, it is desirable to start by K-coloring a complete subgraph of K vertices and then successively finding vertics adjacent to K – 1 different colors, thereby forcing the color choice of such vertices.

9. These are equivalent :

(*i*) A graph G is 2-colorable (*ii*) G is bipartite

(*iii*) Every cycle of G has even length.

10. If $\delta(G)$ is the minimum degree of any vertex of G, then $\chi(G) \ge \frac{|V|}{|V|} - \delta(G)$ where $|V|$ is the number of vertices of G.

5.1.5. K-Critical graph

If the chromatic number denoted by (G) = K, and (G – v) is less than equal to K – 1 for each vertex v of G, then

5.2 CHROMATIC POLYNOMIAL

A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ of fewer colors. Let C_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways, there are $c_i \binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms, that is,

$$P_n(\lambda) = \sum_{i=1}^{n} C_i \binom{\lambda}{i}$$

$$= C_1 \frac{\lambda}{1!} + C_2 \frac{\lambda(\lambda-1)}{2!} + C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \ldots\ldots$$

$$\ldots + C_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots\ldots(\lambda-n+1)}{n!}$$

Each C_i has to be evaluated individually for the given graph.

For example, any graph with even one edge requires at least two colors for proper coloring, and therefore $C_1 = 0$.

A graph with n vertices and using n different colors can be properly colored in n ! ways.

that is, $C_n = n$!.

Problem 5.1. *Find the chromatic polynomial of the graph given in Figure 5.2..*

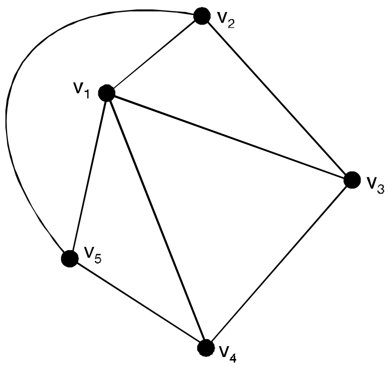


Fig. 5.2. A 3-chromatic graph.

Solution. $P_5(\lambda) = C_1\lambda + C_2 \dfrac{\lambda(\lambda-1)}{2!} + C_3 \dfrac{\lambda(\lambda-1)(\lambda-2)}{3!}$

$$+ C_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + C_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

Since the graph in Figure 5.2 has a triangle, it will require at least three different colors for proper colorings.

Therefore, $C_1 = C_2 = 0$ and $C_5 = 5\,!$

Moreover, to evaluate C_3, suppose that we have three colors x, y and z.

These three colors can be assigned properly to vertices v_1, v_2 amd v_3 in 3 ! = 6 different ways.

Having done that, we have no more choices left, because vertex v_5 must have the same color as v_3 and v_4 must have the same color as v_2.

Therefore, $C_3 = 6$.

Similarly, with four colors, v_1, v_2 and v_3 can be properly colored in 4 • 6 = 24 different ways.

The fourth color can be assigned to v_4 or v_5, thus providing two choices.

The fifth vertex provides no additional choice.

Therefore, $C_4 = 24 \bullet 2 = 48$.

Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in Figure 5.2.

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

The presence of factors $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.

Problem 5.2. *Find the chromatic polynomial and chromatic number for the graph $K_{3,3}$.*

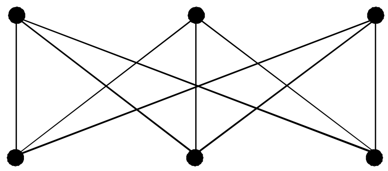


Fig. 5.3.

Solution. Chromatic polynomial for $K_{3,3}$ is given by $\lambda(\lambda-1)^5$.

Thus chromatic number of this graph is 2.

Since $\lambda(\lambda-1)^5 > 0$ first when $\lambda = 2$.

Here, only two distinct colors are required to color $K_{3,3}$.

The vertices a, b and c may have one color, as they are not adjacent.

Similarly, vertices d, e and f can be colored in proper way using one color.

But a vertex from $\{a, b, c\}$ and a vertex from $\{d, e, f\}$ both cannot have the same color.

In fact every chromatic number of any bipartite graph is always 2.

Problem 5.3. *Find the chromatic polynomial and hence the chromatic number for the graph shown below.*

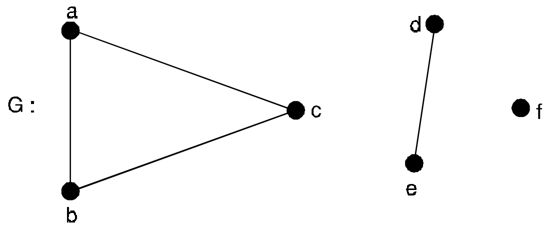


Fig. 5.4.

Solution. Since G is made up of components of G_1, G_2 and G_3 where $G_1 = K_3$, G_2 is a linear graph and G_3 is an isolated vertex.

Now G_1 can be colored in $\lambda(\lambda - 1)(\lambda - 2)$ ways, G_2 can be colored in $\lambda(\lambda - 1)$ ways and G_3 is λ ways.

Therefore, by the rule of product G can be colored be

$$\lambda(\lambda - 1)(\lambda - 2)\lambda(\lambda - 1)\lambda = \lambda^3(\lambda - 1)^2(\lambda - 2).$$

5.3 DECOMPOSITION THEOREM

If G = (V, E) is a connected graph and $e = \{a, b\} \in E$, then $P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$.

Where G_e denotes the subgraph of G obtained by deleting e from G without removing vertices a and b.

i.e., $G_e = G - e$ and G_e' is a second subgraph of G obtained from G_e' by coloring the vertices a and b.

Proof. Let $e = \{a, b\}$. The number of ways to properly color the vertices in $G_e = G - e$ with (atmost) λ colors in $P(G_e, \lambda)$.

Those colorings where end points a and b of e have different colors are proper colorings of G.

The colorings of G_e that are not proper colorigns of G occur when a and b have the same color.

But each of these colorings corresponds with a proper coloring for G_e'.

This partition of the $P(G_e, \lambda)$ proper colorings of G_e into two disjoint subsets results in the equation

$$P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$$

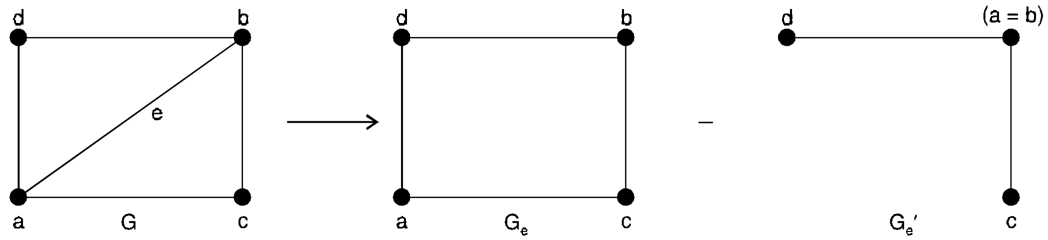


Fig. 5.5.

Problem 5.4. *Using decomposition theorem find the chromatic polynomial and hence the chromatic number for the graph given below in Figure 5.6.*

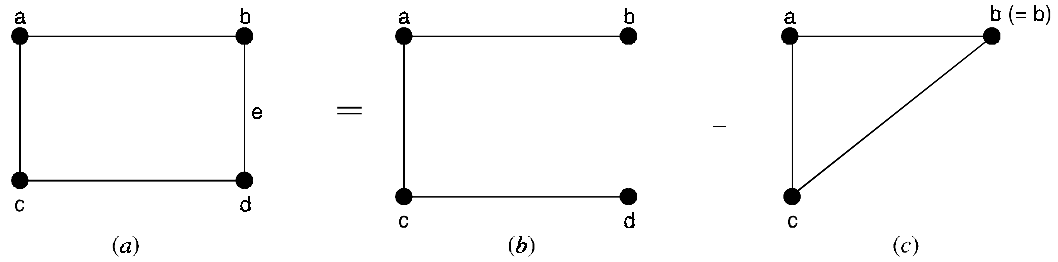


Fig. 5.6.

Solution. Deleting the edge e from G, we get G_2 as shown in Figure 5.6(b). Then the chromatic polynomial of G_e is

$$P(G_e, \lambda) = \lambda(\lambda - 1)(\lambda - 2)$$

By coloring the endpoints of e, *i.e.,* a and b, we get $G_e{}'$ as shown in Figure 5.6(c). Then the chromatic polynomial of G_e is

$$P(G_e{}', \lambda) = \lambda(\lambda - 1)^3.$$

Hence, by decomposition theorem, the chromatic polynomial of G is

$$\begin{aligned} P(G, \lambda) &= \lambda(\lambda - 1)^3 - \lambda(\lambda - 1)(\lambda - 2) \\ &= \lambda(\lambda - 1)\, [(\lambda - 1)^2(\lambda - 2)] \\ &= \lambda(\lambda - 1) - (\lambda^3 - 3\lambda + 3)\, \lambda^4 \\ &= 4\lambda^3 + 6\lambda - 3\lambda. \end{aligned}$$

Theorem 5.1. *For each graph G, the constant term in P(G, λ) is 0.*

Proof. For each graph G, $\lambda(G) > 0$ because $V \neq \phi$.

If $P(G, \lambda)$ has constant term a, then $P(G, 0) = a \neq 0$.

This implies that there are a ways to color G properly with 0 colors, a contradiction.

Theorem 5.2. *Let G = (V, E) with* $|E| > 0$*. Then the sum of the coefficients in P(G, λ) is 0.*

Proof. Since $|E| \geq 1$, we have $\lambda(G) \geq 2$, so we cannot properly color G with only one color.

Consequently, $P(G, 1) = 0$ = the sum of the coefficients in $P(G, \lambda)$.

Problem 5.5. *Explain why each of the following polynomials cannot be a chromatic polynomial*

(i) $\lambda^3 + 5\lambda^2 - 3\lambda + 5 = 0$

(ii) $\lambda^4 + 3\lambda^3 - 3\lambda^2 = 0.$

Solution. (i) The polynomial cannot be a chromatic polynomial since the constant term is 5, not 0.

(ii) The polynomial cannot be a chromatic polynomial since the sum of the coefficient is 1, not 0.

Theorem 5.3. *(Vizing) If G is a simple graph with maximum vertex degree* Δ *then* $\Delta \leq \chi'(G) \leq \Delta + 1$.

Theorem 5.4. *Let* $\Delta(G)$ *be the maximum of the degrees of the vertices of a graph G. Then* $\chi(G) \leq 1 + \Delta(G)$.

Proof. The proof is by induction on V, the number of vertices of the graph.

When V = 1, $\Delta(G) = 0$ and $\chi(G) = 1$, so the result clearly holds.

Now let K be an integer $K \geq 1$, and assume that the result holds for all graphs with V = K vertices.

Suppose G is a graph with K + 1 vertices.

Let v be any vertex of G and let $G_0 = \frac{G}{\{v\}}$ be the subgraph with v (and all edges incident with it) deleted.

Note that $\Delta(G_0) \leq \Delta(G)$. Now G_0 can be be colored with $\chi(G_0)$ colors.

Since G_0 has K vertices, we can use the induction hypothesis to conclude that $\chi(G_0) \leq 1 + \Delta(G_0)$.

Thus, $\chi(G_0) \leq 1 + \Delta(G)$, so go can be colored with atmost $1 + \Delta(G)$ colors.

Since there are atmost $\Delta(G)$ vertices adjacent to v, one of the variable $1 + \Delta(G)$ colors remains for v.

Thus, G can be colored with atmost $1 + \Delta(G)$ colors.

Theorem 5.5. (Kempe, Heawood). *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. We must prove that any planar graph with V vertices has a 5-coloring.

Again we use induction on V and note that if V = 1, the result is clear.

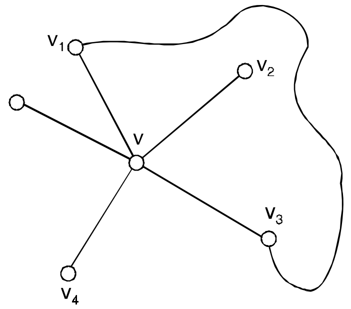


Fig. 5.7.

Let $K \geq 1$ be an integer and suppose that any planar graph with K vertices has a 5-coloring.

Let G be a planar graph with K + 1 vertices and assume that G has been drawn as a plane graph with straight edges. We describe how to obtain a 5-coloring of G.

First, G contains a vertex v of degree atmost 5.

Let $G_0 = \frac{G}{\{v\}}$ be the subgraph obtained by deleting v (and all edges with which it is incident).

By the induction hypothesis, G_0 has a 5-coloring.

For convenience, label the five colors 1, 2, 3, 4 and 5.

If one of these colors was not used to color the vertices adjacent to v, then it can be used for v and G has been 5-colored.

Thus, we assume that v has degree 5 and that each of the color 1 through 5 appears on the vertices adjacent to v.

In clockwise order, label these vertices v_1, v_2, ... v_5 and assume that v_i is colored with color i (see Figure 5.7).

We show how to recolor certain vertices of G_0 so that a color becomes available for v.

There are two possibilities :

Case 1 : There is no path in G_0 from v_1 to v_3 through vertices all of which are colored 1 or 3.

In this situation, let H be the subgraph of G consisting of the vertices and edges of all paths through vertices colored 1 or 3 which start at v_1.

By assumption, v_3 is not in H. Also, any vertex which is not in H but which is adjacent to a vertex of H is colored neither 1 nor 3.

Therefore, interchanging colors 1 and 3 throughout H produces another 5-coloring of G_0.

In this new 5-coloring both v_1 and v_3 acquire color 3, so we are now free to give color 1 to v, thus obtaining a 5-coloring of G.

Case 2 : There is a path P in G_0 from v_1 to v_3 through vertices all of which are colored 1 or 3.

In this case, the path P, followed by v and v_1, gives a circuit in G which does not enclose both v_2 and v_4. Thus, any path from v_2 to v_4 must cross P and, since G is a plane graph, such a crossing can occur only at a vertex of P.

It follows that there is no path in G_0 from v_2 to v_4 which uses just colors 2 and 4.

Now we are in the situation described in case (1), where we have already shown that a 5-coloring for G exists.

Problem 5.6. *$\chi(K_n) = n$, $\chi(K_{m,\,n}) = 2$, why ?*

Solution. It takes n colors to color K_n because any two vertices of K_n are adjacent. $\chi(K_n) = n$.

On the otherhand, $\chi(K_{m,\,n}) = 2$, coloring the vertices of each bipartition set the same color produces a 2-coloring of $K_{m,\,n}$.

Problem 5.7. *What is the chromatic number of the graph in Figure 5.8.*

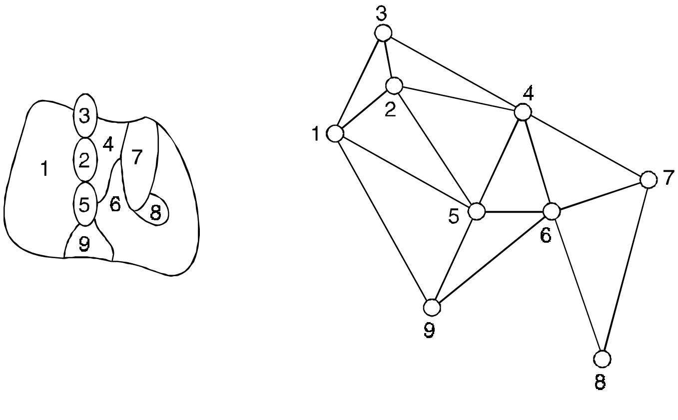


Fig. 5.8. A map and an associated planar graph.

Solution. A way to 4-color the associated graph, was given in the text. From this, we deduce that $\chi(G)\ {}^3\leq 4$.

To see that $\chi(G) = 4$, we investigate the consequences of using fewer than four colors.

Vertices 1, 2, 3 from a triangle, so three different colors are needed for these.

Suppose we use red, blue and green, respectively, as before.

To avoid a fourth color, vertex 4 has to be colored red and vertex 5 green.

Thus, vertex 6 has to be blue.

Since vertex 9 is adjacent to vertices 1, 5 and 6 of colors red, green and blue, respectively. Vertex 9 requires a fourth color.

Problem 5.8. *Show that $\chi(G) = 4$ for the graph of G of Figure 5.9.*

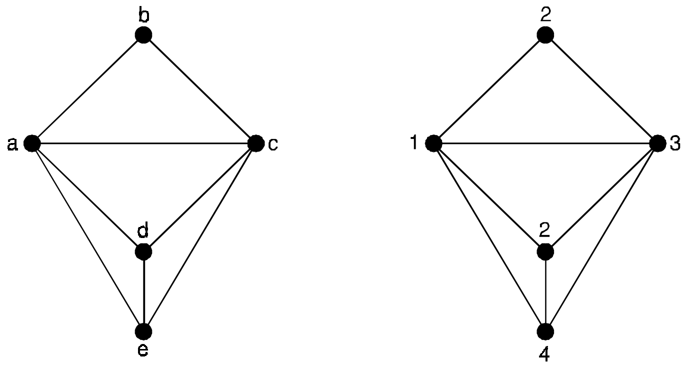


Fig. 5.9.

Solution. Clearly the triangle *abc* requires three colors, assign the colors 1, 2 and 3 to *a*, *b* and *c* respectively.

Then since *d* is adjacent to *a* and *c*, *d* must be assigned a color different from the colors for *a* and *c*, color *d* is color 2.

But then *e* must be assigned a color different from 2 since *e* is adjacent to *d*.

Likewise *e* must be assigned a color different from 1 or 3 because *e* is adjacent to *a* and to *c*.

Hence a fourth color must be assigned to *e*.

Thus, the 4-coloring exhibited incidates $\chi(G) \leq 4$.

But, at the same time, we have argued that $\chi(G)$ cannot be less than 4.

Hence $\chi(G) = 4$.

Theorem 5.6. *The minimum number of hours for the schedule of committee meetings in our scheduling problem is $\chi(G_0)$.*

Proof. Suppose $\chi(G_0) = K$ and suppose that the colors used in coloring G_0 are 1, 2, ... K.

First we assert that all committees can be scheduled in K one-hour time periods.

In order to see this, consider all those vertices colored 1, say, and the committees corresponding to these vertices.

Since no two vertices colored 1 are adjacent, no two such committees contain the same member.

Hence, all these committees can be scheduled to meet at the same time.

Thus, all committees corresponding to same-colored vertices can meet at the same time.

Therefore, all committees can be scheduled to meet during K time periods.

Next, we show that all committees cannot be scheduled in less than K hours. We prove this by contradiction.

Suppose that we can schedule the committees in m one-hour time periods, where $m < K$.

We can then give G_0 an m-coloring by coloring with the same color all vertices which correspond to committees meeting at the same time.

To see that this is, infact, a legitimate m-coloring of G_0, consider two adjacent vertices.

These vertices correspond to two committees containing one or more common members.

Hence, these committees meet at different times, and thus the vertices are colored differently.

However, an m-coloring of G_0 gives a contradiction since we have $\chi(G_0) = K$.

Problem 5.9. *Suppose $\chi(G) = 1$ for some graph G. What do you know about G ?*

Solution. If G has an edge, its end vertices must be colored differently, so $\chi(G) \geq 2$.

Thus $\chi(G) = 1$ if and only if G has no edges.

Problem 5.10. *Any two cycles are homeomorphic. Why ?*

Solution. Any cycle can be obtained from a 3-cycle by adding vertices to edges.

Problem 5.11. *Find the number N defined in this proof for the graph of Figure (5.10). Verify that $N \leq 2E$. Give an example of an edge which is counted just once.*

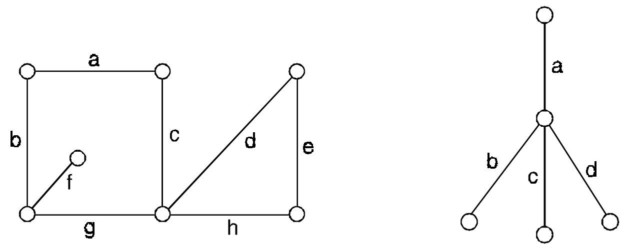


Fig. 5.10.

Solution. The boundaries of the regions are gievn :

$\{d, e, h\}$, $\{a, b, f, g, c\}$ and $\{a, b, g, c, d, e, h\}$

$N = 3 + 5 + 7 = 15 \leq 16 = 2\,E.$

Edge f is counted only once.

Problem 5.12. *Show that, Euler's theorem is not necessarily true if ''connected'' is omitted from its statement.*

Solution.

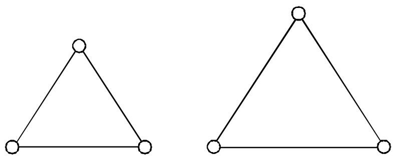


Fig. 5.11.

In the graph shown, $V - E + R = 6 - 6 + 3 = 3.$

Problem 5.13. *Consider the plane graph shown on the left of Figure 5.12, below :*

(*a*) *How many regions are there ?*

(*b*) *List the edges which form the boundary of each region.*

(*c*) *Which region is exterior ?*

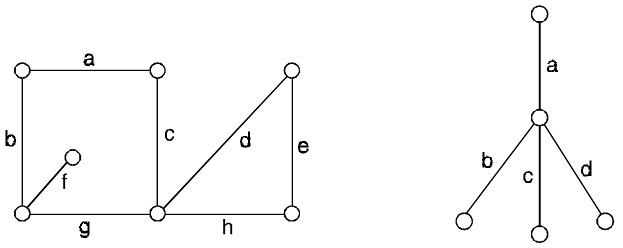


Figure 5.12.

Solution. The graph on the left of Figure 5.12 has three regions whose boundaries are {*d, e, h*}, {*a, b, f, g, c*} and {*a, b, g, c, d, e, h*}, the last region is exterior.

The graph on the right is a tree, it determines only one region, the exterior one, with boundary {*a, b, c, d*}.

5.1.3 Scheduling Final Exams

How can the final exams at a university be scheduled so that no student has two exams at the same time ?

Solution. This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.

For instance, suppose there are seven finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students : 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7.

In Figure 5.13, the graph associated with this set of classes is shown.

A scheduling consists of a coloring of this graph.

Since the chromatic number of this graph is 4, four times slots are needed.

A coloring of the graph using four colors and the associated schedule are shown in Figure 5.14.

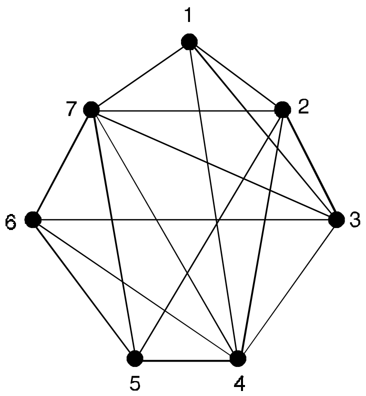


Fig. 5.13.

The graph representing the scheduling of final exams

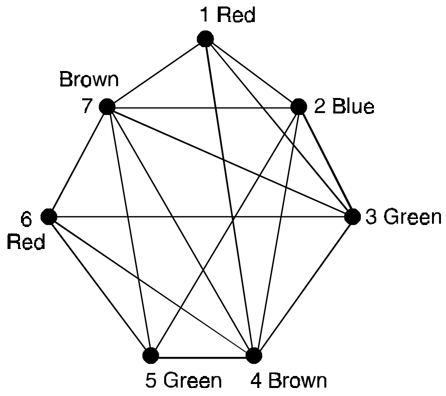


Fig. 5.14.

Using a coloring to schedule final exams.

Time period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

5.3.2 Frequency Assignments

Television channels 2 through 13 are assigned to stations in New Delhi so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring ?

Solution. Construct a graph by assigning a vertex to each station.

Two vertices are connected by an edge if they are located within 150 miles of each other.

An assignment of channels corresponds to a coloring of the graph. Where each color represents a different channel.

5.3.3 Index Registers

In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed ?

Solution. This problem can be addressed using a graph coloring model.

To set up the model, let each vertex of a graph represent a variable in the loop.

There is an edge between two vertices if the variables they represent must be stored in index registers at the same time during the execution of the loop.

Thus, the chromatic number of the graph gives the number of index registers needed, since different registers must be assigned to variables when the vertices respresenting these variables are adjacent in the graph.

Problem 5.14. *What is the chromatic number of the graph C_n ?*

Solution. We will first consider some individual cases.

To begin, let $n = 6$. Pick a vertex and color it red.

Proceed clockwise in the planar depiction of C_6 shown in Figure (5.15).

It is necessary to assign a second color, say blue, to the next vertex reached.

Continue in the clockwise direction, the third vertex can be colored red, the fourth vertex blue, and the fifth vertex red.

Finally, the sixth vertex, which is adjacent to the first, can be colored blue.

Hence, the chromatic number of C_6 is 2. Figure (5.15) displays the coloring constructed here.

Next, let $n = 5$ and consider C_5. Pick a vertex and color it red.

Proceeding clockwise, it is necessary to assign a second color, say blue, to the next vertex reached.

Continuing in the clockwise direction, the third vertex can be colored red, and the fourth vertex can be colored blue.

The fifth vertex cannot be colored either red or blue, since it is adjacent to the fourth vertex and the first vertex.

Consequently, a third color is required for this vertex.

Note that we would have also needed three colors if we had colored vertices in the counter clockwise direction.

Thus, the chromatic number of C_5 is 3. A coloring of C_5 using three colors is displayed in Figure (5.15).

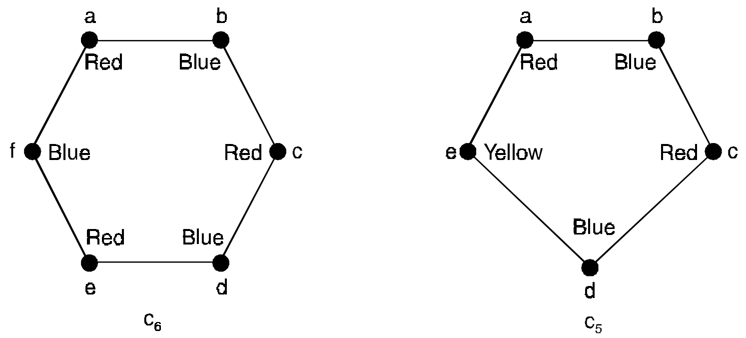


Fig. 5.15. Colorings of C_5 and C_6.

In general, two colors are needed to colors C_n when n is even. To construct such a coloring, simply pick a vertex and color it red.

Proceeding around the graph in a clockwise direction (using a planar representation of the graph) coloring the second vertex blue, the third vertex red, and so on.

The nth vertex can be colored blue, since the two vertices adjacent to it, namely the $(n - 1)$st and the first vertices, are both colored red.

When n is odd and $n > 1$, the chromatic number of C_n is 3.

To see this, pick an initial vertex. To use only two colors, it is necessary to alternate colors as the graph is traversed in a clockwise direction.

However, the nth vertex reached is adjacent to two vertices of different colors, namely, the first and $(n - 1)$st.

Hence, a third color must be used.

Problem 5.15. *What is the chromatic number of the complete bipartite graph $K_{m, n}$, where m and n are positive integers ?*

Solution. The number of colors needed may seem to depend on m and n.

However, only two colors are needed. Color the set of m vertices with one color and the set of n vertices with a second color.

Since edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color.

A coloring of $K_{3, 4}$ with two colors is displayed in Figure 5.16.

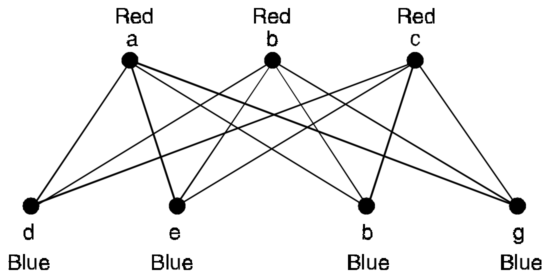


Fig. 5.16. A coloring of $K_{3, 4}$.

Problem 5.16. *What is the chromatic number of K_n ?*

Solution. A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. Is there a coloring using fewer colors ? The answer is no. No two vertices can be assigned the same color, since every two vertices of this graph are adjacent.

Hence, the chromatic number of $K_n = n$.

A coloring of K_5 using five colors is shown in Figure (5.17).

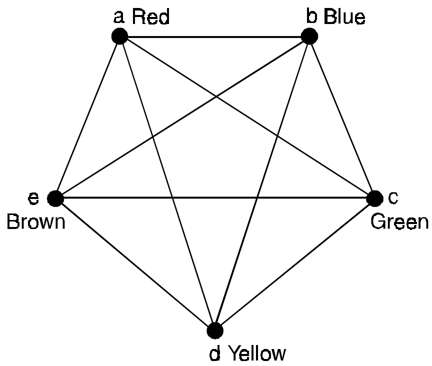


Fig. 5.17. A coloring of K_5.

Problem 5.17. *What is the chromatic numbers of the graphs G and H shown in Figure (5.18).*

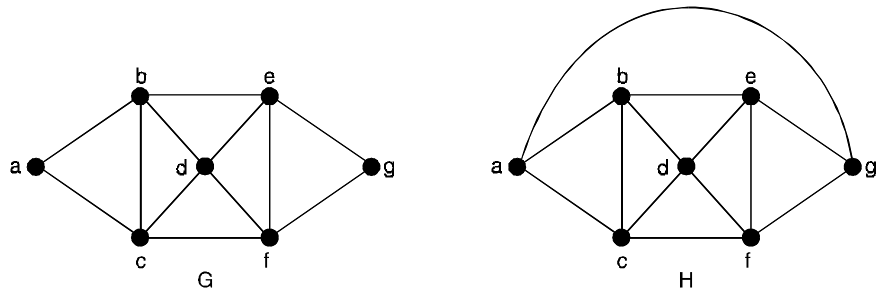


Fig. 5.18. The simple graphs G and H.

Solution. The chromatic number of G is at least three, since the vertices *a*, *b* and *c* must be assigned different colors.

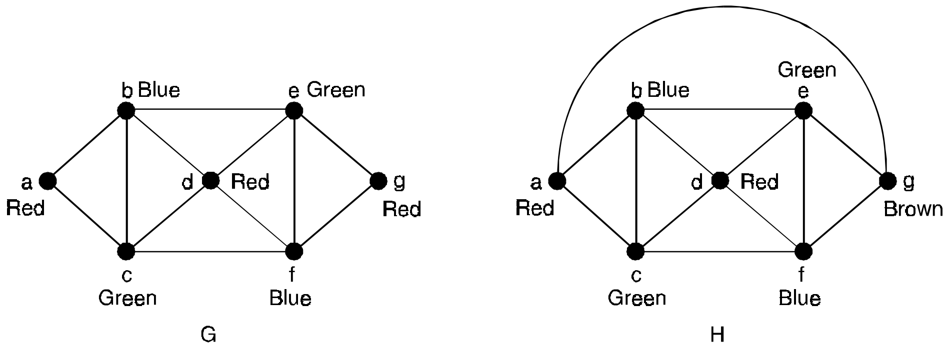


Fig. 5.19. Colorings of the graphs G and H.

To see if G can be colored with three colors, assign red to *a*, blue to *b*, and green to *c*. Then, *d* can (and must) be colored red since it is adjacent to *b* and *c*.

Furthermore, *e* can (and must) be colored green since it is adjacent only to vertices colored red and blue, and *f* can (and must) be colored blue since it is adjacent only to vertices colored red and green.

Finally, *g* can (and must) be colored red since it is adjacent only to vertices colored blue and green.

This produces a coloring of G using exactly three colors. Figure 5.19 displays such a coloring.

The graph H is made up of the graph G with an edge connecting *a* and *g*.

Any attempt to color H using three colors must follow the same reasoning as that used to color G, except at the last stage, when all vertices other than *g* have been colored.

Then, since *g* is adjacent (in H) to vertices colored red, blue, and green, a fourth color, say brown, needs to be used.

Hence, H has a chromatic number equal to 4.

A coloring of H is shown in Figure (5.19).

Problem 5.18. *Suppose that in one particular semester, there are students taking each of the following combinations of courses.*

** Mathematics, English, Biology, Chemistry*

** Mathematics, English, Computer Science, Geography*

** Biology, Psychology, Geography, Spanish*

** Biology, Computer Science, History, French*

** English, Psychology, History, Computer Science*

** Psychology, Chemistry, Computer Science, French*

** Psychology, Geography, History, Spanish.*

What is the minimum number of examination periods required for exams in the ten courses specified so that students taking any of the given combinations of courses have no conflicts ?

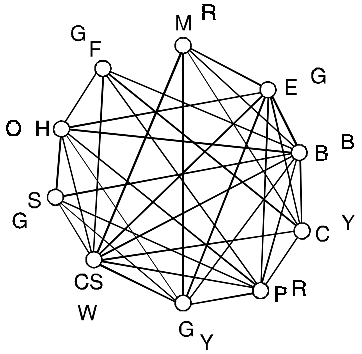
Find a possible schedule which uses this minimum number of periods.

Solution. In order to picture the situation, we draw a graph with ten vertices labeled M, E, B, ... corresponding to Mathematics, English, Biology and so on, and join two vertices with an edge if exams in the corresponding subjects must not be scheduled together.

The minimum number of examination periods is evidently the chromatic number of this graph. What is this ? Since the graph contains K_5 (with vertices M, E, B, G, CS), at least five different colors are needed. (The exams in the subjects which these vertices represent must be scheduled at different times). Five colors are not enough, however, since P and H are adjacent to each other and to each of E, B, G and CS.

The chromatic number of the graph is, infact 6.

In Figure (5.20), we show a 6-coloring and the corresponding exam schedule.



Period 1	Mathematics, Psychology
Period 2	English, Spanish, French
Period 3	Biology
Period 4	Chemistry, Geography
Period 5	Computer Science
Period 6	History

Fig. 5.20.

Theorem 5.7. *A graph G is bipartite if and only if it does not contain a odd cycle.*

Proof. Let G be bipartite. Then the vertex set G can be partitioned into two subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

Suppose G contains a cycle. Let v be a vertex of this cycle. Then to trace the cycle starting from v we have to travel on the edges of G.

The edges of G are the only edges between V_1 and V_2.

Thus starting from v to come back to v along the cycle of G we have to travel exactly even number of times between V_1 and V_2.

That is, the number of edges in C is even, that is, the length of C is even.

Conversely, without loss of generality we assume G is connected.

Let G does not contain a odd cycle. Choose a vertex x of G. Color the vertex by the Color Black. Color all the vertices that are at odd distances from x with the color Red. Color all the vertices that are at even distances from x with color Black. Since every distance is either a odd or even (but not both), every vertex of G is now colored.

We now show that the graph G is now properly colored. Suppose G is not properly colored, the G contains two adjacent vertices say u and v, colored with the same color. Then distance from the vertex x to both the vertices u and v is odd.

Let P_1 and P_2 be shortest paths from x to u and x to v respectively.

Let y be the last vertex common to P_1 and P_2 (*i.e.,* the path from y to u and path from y to v along P_1 and P_2 are disjoint). Then $d(x, y)$ along P_1 is same along P_2 (since both P_1 and P_2 are shortest paths).

Otherwise, if the $d(x, y)$ along P_1 is smaller than that on P_2, then the path from x to y along P_1 with the path from y to v along P_2 is shorter than P_2, which is a contradiction to the fact that P_2 is shortest.

Let $d(x, u) = m$ and $d(x, v) = n$, then both m and n are odd numbers or both are even numbers (since u and v are colored with same color).

Then $d(y, u)$ and $d(y, v)$ are both either odd or even and hence the sum is even.

Hence, the circuit formed due to these paths together with the edge uv is of odd length, which is a contradiction.

Thus we conclude that the coloring is proper.

Now consider the set V_1 of all vertices of G colored by Black and the set V_2 of all the vertices of G colored by the color Red.

These sets are the partition of G such that no two vertices in the same set are adjacent.

Hence G is bipartite.

Theorem 5.8. *A graph of n vertices is a complete graph if and only if its chromatic polynomial is*

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1).$$

Proof. With λ colors, there are λ different ways of coloring any selected vertex of a graph.

A second vertex can be colored properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, ..., and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other.

That is, if and only if the graph is complete.

Theorem 5.9. *Let a and b be two non adjacent vertices in a graph G. Let G′ be a graph obtained by adding an edge between a and b. Let G″ be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then*

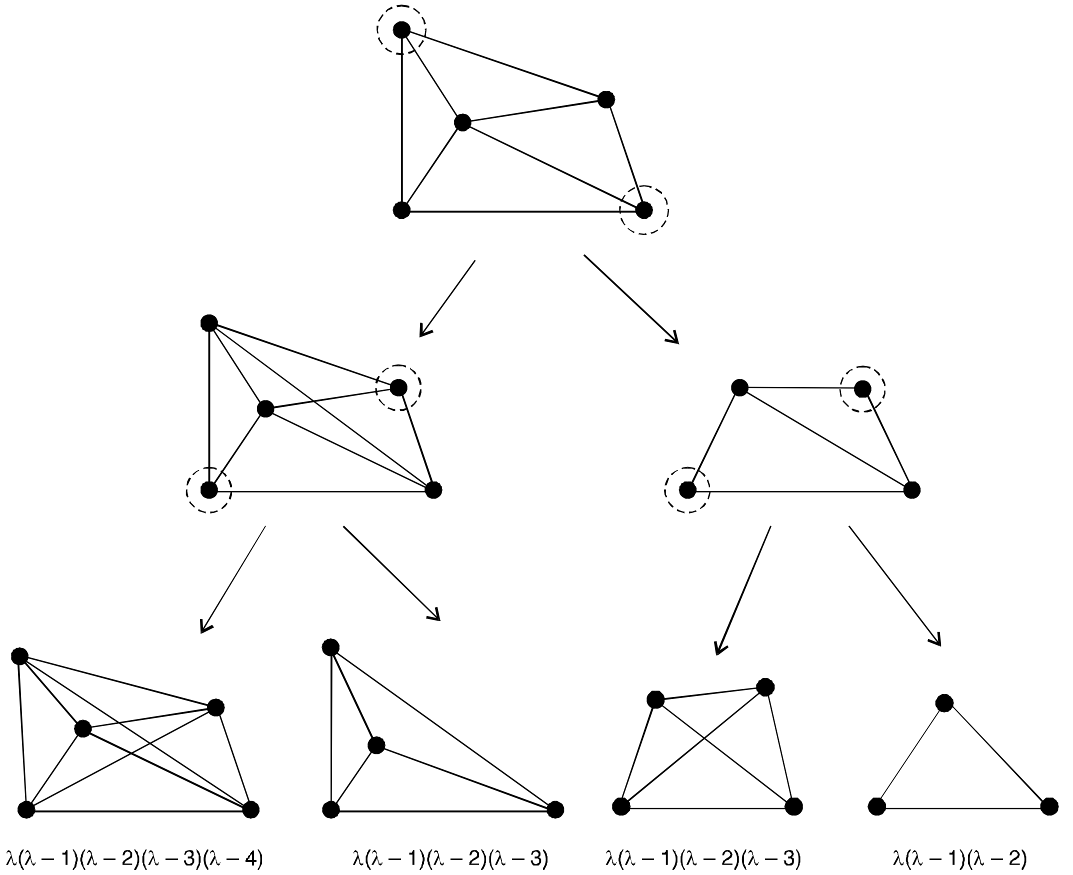
$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

Proof. The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors.

Since the number of ways of properly coloring G such that a and b have different colors = number of ways of properly coloring G′, and

Number of ways of properly coloring G such that a and b have the same color = number of ways of properly coloring G″.

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''$$



$$P_5(\lambda) \text{ of } G = \lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$$
$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$$

Fig. 5.21. Evaluation of a chromatic polynomial.

Theorem 5.10. *A graph is bicolorable if and only if it has no odd cycles.*

Theorem 5.11. *For any graph G, $\chi(G) \leq 1 + \max \delta(G')$,*
Where the maximum is taken over all induced subgraphs G′ of G.

Proof. The result is obvious for totally disconnected graphs.
Let G be an arbitrary n-chromatic graph, $n \geq 2$.
Let H be any smallest induced subgraph such that $\chi(H) = n$
The graph H therefore has the property that

$$\chi(H - v) = n - 1 \text{ for all its points } v.$$

It follows that deg $v \geq n - 1$ so that $\delta(H) \geq n - 1$ and hence

$$n - 1 \leq \delta(H) \leq \max \delta(H') \leq \max \delta(G')$$

The first maximum taken over all induced subgraphs H′ of H and the second over all induced subgraphs G′ of G.

This implies that

$$\chi(G) = n < 1 + \max \delta(G')$$

Corollary : For any graph G, the chromatic number is atmost one greater than the maximum degree $\chi \leq 1 + \Delta$.

Theorem 5.12. If $\Delta(G) = n \geq 2$, then G is n-colorable unless, or

(*i*) $n = 2$ and G has a component which is an odd cycle, or

(*ii*) $n > 2$ and K_{n+1} is a component of G.

Theorem 5.13. *For any graph G,* $\dfrac{P}{\beta_0} \leq \chi \leq P - \beta_0 + 1$.

Proof. If $\chi(G) = n$, then V can be partitioned into n color classes $V_1, V_2, ... V_n$, each of which, as noted above, is an independent set of points.

If $|V_i| = P_i$, then every $P_i \leq \beta_0$ so that

$$P = \Sigma P_i \leq n\,\beta_0$$

To verify the upper bound, let S be a maximal independent set containing β_0 points.

It is clear that $\chi(G - S) \geq \chi(G) - 1$.

Since G – S has $P - \beta_0$ points, $\chi(G - S) \leq P - \beta_0$

Therefore, $\chi(G) \leq \chi(G - S) + 1 \leq P - \beta_0 + 1$.

Theorem 5.14. *For every two positive integers m and n, there exists an n-chromatic graph whose girth exceeds m.*

Theorem 5.15. *For any graph G, the sum and product of* χ *and* $\bar{\chi}$ *satisfy the inequalities :*

$$2\sqrt{P} \leq \chi + \bar{\chi} \leq P + 1,$$

$$P \leq \chi\bar{\chi} \leq \left(\frac{P+1}{2}\right)^2$$

Proof. Let G be n-chromatic and let $V_1, V_2, ... V_n$, be the color classes of G, where $|V_i| = P_i$

Then of course $\Sigma P_i = P$ and $\max P_i \geq \dfrac{P}{n}$.

Since each V_i induces a complete subgraph of $\overline{G}$

$\bar{\chi} \geq \max P_i \geq \dfrac{P}{n}$ so that $\chi\bar{\chi} \geq P$.

Since the geometric mean, it follows that $\chi + \bar{\chi} \geq 2\sqrt{P}$.

This establishes both lower bounds.

To show that $\chi + \bar{\chi} \leq P + 1$, we use induction on P, noting that equality holds when P = 1.

We thus assume that $\chi(G) + \bar{\chi}(G) \leq P$ for all graphs G having P – 1 points.

Let H and $\bar{H}$ be complementary graphs with P points, and let v be a point of H.

Then $G = H - v$ and $\bar{G} + \bar{H} - v$ are complementary graphs with P – 1 points.

Let the degree of v in H be d so that the degree of v in $\bar{H}$ is $P - d - 1$.
It is obvious that

$$\chi(H) \leq \chi(G) + 1 \text{ and } \bar{\chi}(H) \leq \bar{\chi}(G) + 1$$

If either

$$\chi(H) < \chi(G) + 1 \text{ or } \bar{\chi}(H) < \bar{\chi}(G) + 1.$$

then $\chi(H) + \bar{\chi}(H) \leq P + 1$.

Suppose then that $\chi(H) = \chi(G) + 1$ and $\bar{\chi}(H) = \bar{\chi}(G) + 1$.

This implies that the removal of v from H, producing G, decreases the chromatic number so that $d \geq \chi(G)$.

Similarly $P - d - 1 \geq \bar{\chi}(G)$,

thus $\chi(G) + \bar{\chi}(G) \leq P - 1$

Therefore, we always have $\chi(H) + \bar{\chi}(H) \leq P + 1$

Finally, applying the inequality

$$4\chi\bar{\chi} \leq (\chi + \bar{\chi})^2 \quad \text{we see that} \quad \chi\bar{\chi} \leq \left[\frac{(P+1)}{2}\right]^2.$$

Theorem 5.16. *Every tree T with two or more vertices is 2-chromatic.*

Proof. Since Tree T is a bipartite graph.

The vertex set V of G can be partitioned into two subsets V_1 and V_2 such that no two vertices of the set V_1 are adjacent and two vertices of the set V_2 are adjacent.

Now color the vertices of the set V_1 by the color 1 and the vertices of the set V_2 by the color 2.

This coloring is a proper coloring.

Hence, chromatic number of $G \leq 2$, and since T contains atleast one edge chromatic number of $G \geq 2$.

Thus, chromatic number of G is 2.

Theorem 5.17. *A graph G is 2-chroamtic if and only if G is bipartite.*

Proof. Let chromatic index of a graph G be two.

Let G be properly colored with two colors 1 and 2. Consider the set of vertices colored with the color 1 and the set of all vertices colored with the color 2.

These sets are precisely partition of the vertex set such that no two of the vertices of the same set are adjacent.

Hence G is bipartite.

Conversely, G is not bipartite then G contains a odd cycle.

The chromatic number of a odd cycle is three.

Hence G contains a subgraph whose chromatic number is three.

Therefore, K(G) ≥ 3.

Theorem 5.18. *The chromatic number of a graph cannnot exceed one more than the maximum degree of a vertex of G.*

Proof. Since maximum degree of the graph is m, the graph cannot have a subgraph K_n, $n > m + 1$.

Thus $K(G) \leq m + 1$.

Corollary. The chromatic number of a graph cannot exceed maximum degree m of a vertex of G if and only if G does not have a subgraph isomorphic to K_{m+1}.

Theorem 5.19. *If d_{max} is the maximum degree of the vertices in a graph G, chromatic number of G $\leq 1 + d_{max}$.*

Theorem 5.20. (König's theorem)

A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof. Let G be a connected graph with circuits of only even lengths.

Consider a spanning tree T in G, let us properly color T with two colors. Now add the chords to T one by one.

Since G had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T.

Thus G is colored with two colors, with no adjacent vertices having the same color.

That is, G is 2-chromatic.

Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit.

Thus the theorem.

Theorem 5.21. *A graph G is 2-chromatic if and only if it is a non-null bipartite graph.*

Proof. Suppose a graph G is 2-chromatic. Then it is non-null, and some vertices of G have one color, say α, and the rest of the vertices have another color, say β.

Let V_1 be the set of vertices having color α and V_2 be the set of vertices having color β.

Then $V_1 \cup V_2 = V$, the vertex set of G, and $V_1 \cap V_2 = \phi$.

Also, no two vertices of V_1 can be adjacent and no two vertices of V_2 can be adjacent.

As such, every edge in G has one end in V_1 and the other end in V_2.

Hence G is a bipartite graph.

Conversely, suppose G is a non-null bipartite graph. Then the vertex set of G has two partitions V_1 and V_2 such that every edge in G has one end in V_1 and another end in V_2.

Consequently, G cannot be properly colored with one color, because then vertices in V_1 and V_2 will have the same color and every edge has both of its ends of the same color.

Suppose we assign a color α to all vertices in V_1 and a different color β to all vertices in V_2.

This will make a proper coloring of V.

Hence G is 2-chromatic.

Corollary. *Every three with two or more vertices is a bipartite graph.*

Proof. Every tree with two or more vertices is 2-chromatic. Therefore, it is bipartite, by the theorem.

Theorem 5.22. For a graph G, the following statements are equivalent :

(*i*) G is 2-chromatic

(*ii*) G is non-null and bipartite

(*iii*) G has no circuits of odd length.

Corollary. A graph G is a non-null bipartite graph if and only if it has no circuits of odd length.

Theorem 5.23. *If G is a graph with n vertices and degree δ, then $\chi(G) \geq \frac{n}{n-\delta}$.*

Proof. Recall that δ is the minimum of the degrees of vertices.

Therefore, every vertex v of G has atleast δ number of vertices adjacent to it.

Hence there are at most $n - \delta$ vertices can have the same color.

Let K be the least number of colors with which G can be properly colored.

Then $K = \chi(G)$.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ be these colors and let n_1 be the number of vertices having color α_1, n_2 be the number of vertices having color α_2 and so on, and finally n_K be the number of vertices having color α_K.

Then $$n_1 + n_2 + n_3 + \ldots\ldots n_k = n \qquad \ldots(1)$$

and $$n_1 \leq n - \delta,\ n_2 \leq n - \delta,\ \ldots\ldots\ n_k \leq n - \delta \qquad \ldots(2)$$

Adding the K in equalities in (2), we obtain

$$n_1 + n_2 + \ldots\ldots + n_k \leq K(n - \delta)$$

or $$n \leq K(n - \delta), \text{ using (1)}$$

Since $K = \chi(G)$, this becomes

$$\chi(G) \geq \frac{n}{n-\delta}$$

This is the required result.

Problem 5.19. *Write down chromatic polynomial of a given graph on n vertices.*

Solution. Let G be a graph on n vertices.

Let C_i denote the different ways of properly coloring of G using exactly i distinct colors.

These i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ distinct ways.

Thus total number of distinct ways a proper coloring to a graph with i colors out of λ colors is possible in $\binom{\lambda}{i}C_i$ ways.

Hence $\sum_{i=1}^{n} \binom{\lambda}{i} C_i$. Each C_i has to be evaluated individually for the given graph.

Problem 5.20. *Find all maximal independent sets of the following graph.*

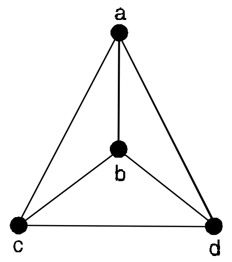


Fig. 5.22.

Solution. The maximal independent sets of G are $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$.

Problem 5.21. *Find all maximal independent sets of the following graph.*

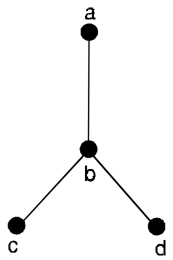


Fig. 5.23.

Solution. Maximal independent sets are $\{a, c, d\}$ and $\{b\}$.

Problem 5.22. *Find all possible maximal independent sets of the following graph using Boolean expression.*

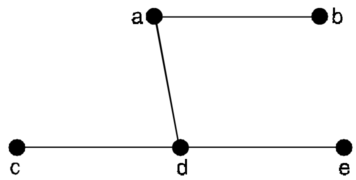


Fig. 5.24.

Solution. The Boolean expression for this graph

$$\phi = \Sigma xy = ab + ad + cd + de \text{ and}$$

$$\phi' = (a' + b')(a' + d')(c' + d')(d' + e')$$

$$= \{a'(a' + d') + b'(a' + d')\} \{c'(d' + e') + d'(d' + e')\}$$

$= \{a' + b'a' + b'd'\}\ \{c'd' + c'e' + d'\}$
$= \{a'(1 + b') + b'd'\}\ \{d'(c' + 1) + c'e'\}$
$= \{a' + b'd'\}\ \{d' + c'e'\}$
$= a'd' + a'c'e' + b'd' + b'c'd'e'$
$= a'd' + a'c'e' + b'd'\ (1 + c'e')$
$= a'd' + a'c'e' + b'd'$

Thus $f_1 = a'd'$, $f_2 = a'c'e'$ and $f_3 = b'd'$.

Hence maximal independent sets are $V - \{a, b\} = \{b, c, e\}$

$V - \{a, c, e\} = \{b, d\}$ and $V - \{b, d\} = \{a, c, e\}$.

Problem 5.23. *Find the chromatic polynomial of a connected graph on three vertices.*

Solution. Since the graph is connected it contains an edge, hence minimum two colors are required for any proper coloring of G.

Thus $C_1 = 0$.

Further the number of ways a graph on n vertices with n distinct colors can be properly assigned in n ! ways.

Hence for the graph on 3 vertices $C_3 = 3\,! = 6$.

If G is a triangle, then G cannot be labeled with two colors.

Hence $C_2 = 0$, thus

$$P_3(\lambda) = \sum_{i=1}^{3} \binom{\lambda}{i} C_i = 0 + 0 + \binom{\lambda}{3} 6$$

$$= \frac{\lambda(\lambda-1)(\lambda-2)}{3\,!}\ 6 = \lambda(\lambda - 1)(\lambda - 2)$$

If G is a path, then end vertices can be colored with only two ways with two colors and for each choice of end vertex only one choice of another color is possible for the middle vertex. Thus $C_2 = 2$ and similar to above argument $C_3 = 3\,!$.

Therefore, $$P_3(\lambda) = \sum_{i=1}^{3} \binom{\lambda}{i} C_i = 0 + \binom{\lambda}{2} 2 + \binom{\lambda}{3} 6$$

$$= \frac{\lambda(\lambda-1)}{2\,!} 2 + \frac{\lambda(\lambda-1)(\lambda-2)}{3\,!} 6$$

$$= \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2)$$

$$= \lambda(\lambda - 1)(1 + (\lambda - 2))$$

$$= \lambda(\lambda - 1)^2.$$

Theorem 5.24. *An n-vertex graph is a tree if and only if its chromatic polynomial* $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof. Let G be a tree on n vertices.

We prove the result by induction on n.

If $n = 1$, then G contains only one vertex which can be colored in λ distinct ways only.

Hence the result holds in this case.

If $n = 2$, then G contains one edge, so that exactly two colors are required for the proper coloring of the graph.

Hence $C_1 = 0$ and two colors can be assigned in two different ways for the vertices of the graph.

Therefore, $C_2 = 2$.

$$\text{Thus } P_n(\lambda) = 0 + \left[\frac{\lambda(\lambda-1)}{2\,!}\right]2 = \lambda(\lambda - 1)$$

Hence the result holds with $n = 2$.

Now assume the result for lesser values of n, $n \geq 2$.

Since the graph G is a tree, it contains a pendent vertex. Let v be a pendent vertex of the graph. Let G′ be the graph obtained by deleting the vertex v. Then by inductive hypothesis the chromatic polynomial of G′ is $\lambda(\lambda - 1)^{n-2}$.

Now for each proper coloring of G′ the given graph can be properly colored by painting the vertex v with the color other than vertex adjacent to the vertex v.

Thus we can choose $(\lambda - 1)$ colors to v for each proper coloring of G′.

Hence total $\lambda(\lambda - 1)^{n-2}(\lambda - 1) = \lambda(\lambda - 1)^{n-1}$ ways we can properly color the given tree.

Thus the result hold by induction.

Problem 5.24. *How many ways a tree on 5 vertices can be properly colored with at most 4 colors.*

Solution. We have a tree with n vertices can be colored with at most λ colors in $\lambda(\lambda - 1)^{n-1}$ ways.

Therefore a tree on $n = 5$ vertices can be properly colored with at most $\lambda = 4$ colors in $\lambda(\lambda - 1)^{n-1} = 4 \bullet 3^4 = 4 \bullet 81 = 324$ ways.

Problem 5.25. *Write down the chromatic polynomial of the graph $K_4 - e$.*

Solution.

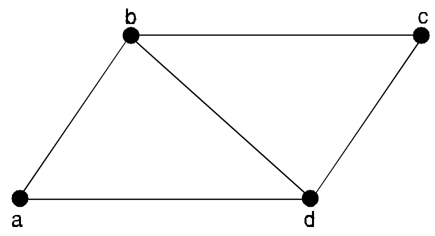


Fig. 5.25.

The graph $K_4 - e$ is shown below. It contains exactly two non-adjacent vertices.

Let G′ be a graph obtained by adding the edge between these non adjacent vertices.

Then G′ is a complete graph K_4.

Hence $P_4(\lambda)$ of $G' = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$

Let G″ be the graph obtained by fusing these vertices and replacing the parallel edges.

Then G″ is a complete graph K_3.

Hence $P_3(\lambda)$ of $G'' = \lambda(\lambda - 1)(\lambda - 2)$

Now, $P_4(\lambda)$ of $G = P_4(\lambda)$ of $G' + P_{4-1}(\lambda)$ of G''

$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)$$
$$= \lambda(\lambda - 1)(\lambda - 2)(1 + \lambda - 3)$$
$$= \lambda(\lambda - 1)(\lambda - 2)^2.$$

Problem 5.26. *Find the chromatic number of the following graphs*

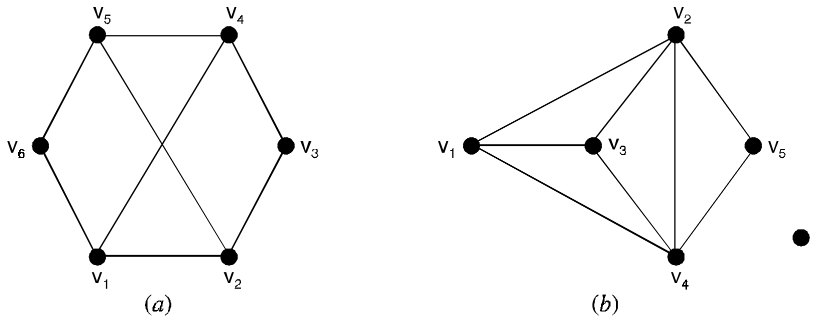


Fig. 5.26.

Solution. (*i*) For the graph in Figure 5.26(*a*), let us assign a color α to the vertex v_1.

Then, for a proper coloring, we have to assign a different color to its neighbours v_2, v_4, v_6.

Since v_2, v_4, v_6 are non adjacent vertices, they can have the same color, say β (which is different from α).

Since v_3, v_5 are not adjacent to v_1, these can have the same colors as v_1, namely α.

Thus, the graph can be properly colored with at least two colors, with the vertices v_1, v_3, v_5 having one color α and v_2, v_4, v_6 having a different color β.

Hence the graph is 2-chromatic

(*i.e.,* the chromatic number of the graph is 2).

(*ii*) For the graph in Figure 5.26(*b*), let us again the color α to the vertex v_1.

Then, for a proper coloring, its neighbours v_2, v_3 and v_4 cannot have the color α, but v_5 can have the color α.

Furthermore, v_2, v_3, v_4 must have different colors, say β, γ, δ.

Thus, at least four colors are required for a proper coloring of the graph.

Hence, the graph is 4-chromatic (*i.e.,* the chroamtic number of the graph is 4).

Problem 5.27. *Prove that a simple planar graph G with less than 30 edges in 4-colorable.*

Solution. If G has 4 or less number of vertices, the required result is true.

Assume that the result is true for any graph with n = K vertices.

Consider a graph G′ with K + 1 vertices and less than 30 edges.

Then, G′ has at least one vertex v of degree at most 4.

Now, considering the graph $G' - v$ we find that G' is 4-colorable.

Problem 5.28. *Prove that a graph of order n (≥ 2) consisting of a single circuit is 2-chromatic if n is even, and 3-chromatic if n is odd.*

Solution. The given graph is the cycle graph C_n, $n \geq 2$ as shown in figure below.

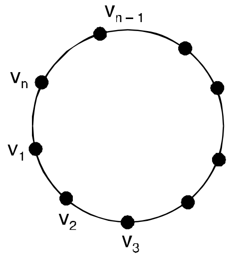


Fig. 5.27.

Obviously, the graph cannot be properly colored with a single color. Assign two colors alternatively to the vertices, starting with v_1.

That is, the odd vertices v_1, v_3, v_5 etc, will have a color α and the even vertices v_2, v_4, v_6 etc., will have a different color β.

Suppose n is even. Then the vertex v_n is an even vertex and therefore will have the color β, and the graph gets properly colored.

Therefore, the graph is 2-chormatic.

Suppose n is odd. Then the vertex v_n is an odd vertex and therefore will have the color α, and the graph is not properly colored. To make it properly colored. v_n should be assigned a third color γ. Thus, in this case, the graph is 3-chromatic.

Problem 5.29. *Prove that every tree with two or more vertices is 2-chromatic.*

Solution. Consider a tree T rooted at a vertex v as shown in figure below. Assign a color α to v and a different color β to all vertices adjacent to v. Then the vertices adjacent to those which have the color β are not adjacent to v (because a tree has no circuits) and are at a distance 2 from v. Assign the color α to these vertices. Repeat the process until all vertices are colored.

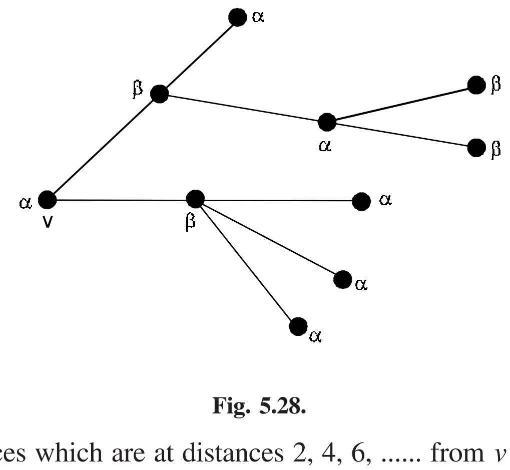


Fig. 5.28.

Thus, v and all vertices which are at distances 2, 4, 6, ... from v have α as their color and all vertices which are at distances 1, 3, 5, ... from v have β as their color.

Accordingly, along any path of T the vertices are of alternating colors.

Since there is one and only one path between any two vertices in a tree, no two adjacent vertices will have the same color.

Thus, T has been properly colored with 2 colors.

If T has two or more vertices, it has one or more edges. As such, it cannot be colored with 1 color. This proves that the chromatic number of T is 2, that is 2-chromatic.

Problem 5.30. *Find the chromatic number of a cubic graph with $p \geq 6$ vertices.*

Solution. Every cubic graph contains of odd degree and in which there exists at least one triangle.

Hence $\chi(G) = 3$, where G is a cubic graph.

The following Figure (5.29) gives the result :

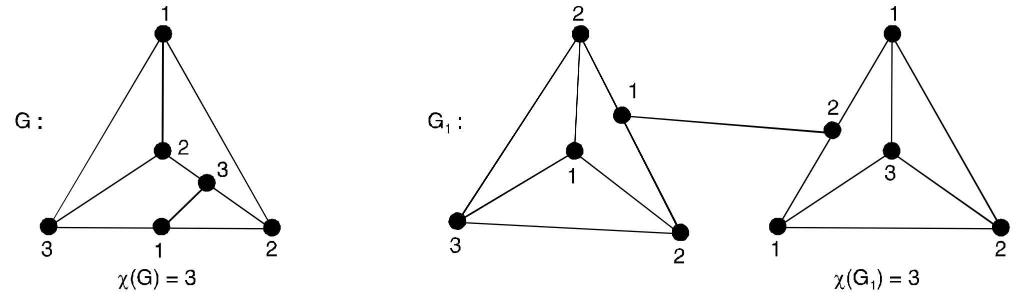


Fig. 5.29

Problem 5.31. *Find the chromatic polynomial of a complete graph on n vertices.*

Solution. Since minimum n colors required for the proper coloring of complete graph K_n on n vertices.

We have $\quad C_i = 0$ for all $i = 1, 2, \ldots\ldots n - 1$.

Further since the graph contains n vertices, n distinct colors can be assigned in n ! ways.

Thus $C_n = n\,!$.

$$\text{Therefore,} \quad P_n(\lambda) = \sum_{i=1}^{n} \binom{\lambda}{i} C_i = \binom{\lambda}{n} C_n$$

$$= \frac{\lambda(\lambda-1)(\lambda-2)\ldots\ldots\lambda-(n+1)}{n\,!}\, n\,!$$

$$= \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1).$$

Problem 5.32. *Show that the chromatic number of a graph G is $\lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1)$ if and only if G is a complete graph on n vertices.*

Solution. For a given λ, the first vertex of a graph can be colored in λ ways.

A second vertex can be colored properly with $\lambda - 1$ ways, the third vertex in only $\lambda - 2$ ways if and only if this vertex is adjacent to first two vertices. Continuing like this we have, the last vertex can be colored with $(\lambda - n + 1)$ ways if and only if the graph is complete.

Problem 5.33. *Prove that, for a graph G with n vertices*

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

Solution. Let K be the minimum number of colors with which G can be properly colored.

Then $K = \chi(G)$. Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ be these colors and let $n_1, n_2, \ldots\ldots n_K$ be the number of vertices having colors $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ respectively.

Then $n_1, n_2, \ldots\ldots n_K$ are the orders of the maximal independent sets, because a set of all vertices having the same color contain all vertices which are mutually non-adjacent.

Since $\beta(G)$ is the order of a maximal independent set with largest number of vertices, none of $n_1, n_2, \ldots\ldots n_K$ can exceed $\beta(G)$.

i.e., $n_1 \leq \beta(G), \quad n_2 \leq \beta(G), \ldots\ldots n_K \leq \beta(G)$

Adding these inequalities, we get

$$n_1 + n_2 + \ldots\ldots + n_K \leq K\beta(G)$$

Since $n_1 + n_2 + \ldots\ldots + n_K = n$ and $K = \chi(G)$, this becomes

$$n \leq \chi(G) . \beta(G)$$

or

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

Problem 5.34. *Show that the following graph is uniquely colorable.*

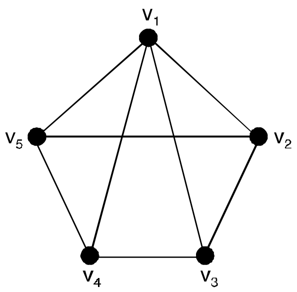


Fig. 5.30.

Solution. We check that the given graph G has only the following independent sets both of which are maximal.

$$W_1 = \{v_2, v_4\}, W_2 = \{v_3, v_5\}$$

Both of these have 2 vertices, and as such $\beta(G) = 2$.

The sets W_1 and W_2 are mutually disjoint and yield only one chromatic partition given below :

$$P = \{W_1, W_2, \{v_1\}\}$$

In view of this single possible chromatic partitioning of G, we infer that G is uniquely colorable.

5.4 COLOR PROBLEM

The most famous unsolved problem in graph theory and perhaps in all of Mathematics is the celebrated four color conjecture. This remarkable problem can be explained in five minutes by any mathematician to the so called man in the street. At the end of the explanation, both will understand the problem, but neither will be able to solve it.

The conjecture states that, any map on a plane or the surface of a sphere can be colored with only four colors so that no two adjacent countries have the same color. Each country must consist of a single connected region, and ajdacent countries are those having a boundary line in common. The conjecture has acted as a catalyst in the branch of mathematics known as combinatorial topology and is closely related to the currently fashionable field of graph theory. More than half a century of work by many mathematicians has yielded proofs for special cases ..... . The consensus is that the conjecture is correct but unlikely to be proved in general.

It seems destined to retain for some time the distinction of being both the simplest and most fascinating unsolved problem of mathematics.

The four color conjecture has an interesting history, but its origin remains some what vague. There have been reports that Möbius was familiar with this problem in 1840, but it is only definite that the problem was communicated to De Morgan by Guthrie about 1850.

The first of many erroneous proofs of the conjecture was given in 1879 by Kempe. An error was found in 1890 by Heawood who showed, however, that the conjecture becomes true when ‘four’ is replaced by ‘five’.

A counter example, if ever found, will necessarily be extremely large and complicated, for the conjecture was proved most recently by Ore and Stemple for all maps with fewer than 40 countries.

The four color conjecture is a problem in graph theory because every map yields a graph is which the countries are the points, and two points are joined by a line whenever the corresponding countries are adjacent. Such a graph obviously can be drawn in the plane without intersecting lines.

Thus, if it is possible to color the points of every **planar graph** with four or fewer colors so that adjacent points have different colors, then the four color conjecture will have been proved.

5.4.1. The Four color theorem

Every planar graph is 4-colorable.

Assume the four color conjecture holds and let G be any plane map.

Let G* be the underlying graph of the geometric dual of G.

Since two regions of G are adjacent if and only if the corresponding vertices of G* are adjacent, map G is 4-colorable because graph G* is 4-colorable.

Conversely, assume that every plane map is 4-colorable and let H be any planar graph.

Without loss of generality, we suppose H is a connected plane graph.

Let H* be the dual of H, so drawn that each region of H* encloses precisely one vertex of H. The connected plane pseudograph H* can be converted into a plane graph H′ by introducing two vertices into each loop of H* and adding a new vertex into each edge in a set of multiple edges.

The 4-colorability of H′ now implies that H is 4-colorable, completing the verification of the equivalence.

If the four color conjecture is ever proved, the result will be best possible, for it is easy to give examples of planar graphs which are 4-chromatic, such as K_4 and W_6 (see Figure 5.31 below).

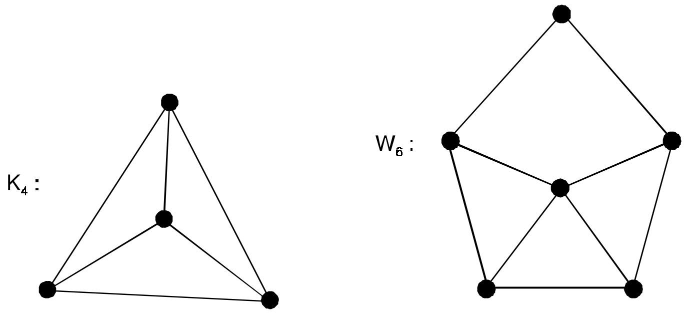


Fig. 5.31. Two 4-chromatic planar graphs.

Theorem 5.25. *Every planar graph with fewer than 4 triangles is 3-colorable.*

Corollary. Every planar graph without triangle is 3-colorable.

Theorem 5.26. *The four color conjecture holds if and only if every cubic bridgeless plane map is 4-colorable.*

Proof. We have already seen that every plane map is 4-colorable if and only if the four color conjecture holds.

This is also equivalent to the statement that every bridgeless plane map is 4-colorable since the elementary contraction of identifying the end vertices of a bridge affects neither the number of regions in the map nor the adjacency of any of the regions.

Certainly, if every bridgeless plane map is 4-colorable, then every cubic bridgeless plane map is 4-colorable.

In order to verify the converse, let G be a bridgeless plane map and assume all cubic bridgeless plane maps are 4-colorable.

Since G is bridgeless, it has no end vertices.

If G contains a vertex v of degree 2 incident with edges y and z, we subdivide y and z, denoting the subdivision vertices by u and w respectively.

We now remove v, identify u with one of the vertices of degree 2 in a copy of the graph $K_4 - x$ and identify w with the other vertex of degree 2 in $K_4 - x$.

Observe that each new vertex added has degree 3 (see Figure 5.32).

If G contains a vertex v_0 of degree $n \geq 4$ incident with edges x_1, x_2, ... x_n, arranged cyclically about v_0, we subdivide each x_i producing a new vertex v_i.

We then remove v_0 and add the new edges v_1v_2, v_2v_3, ..., $v_{n-1}v_n$, v_nv_1.

Again each of the vertices so added has degree 3.

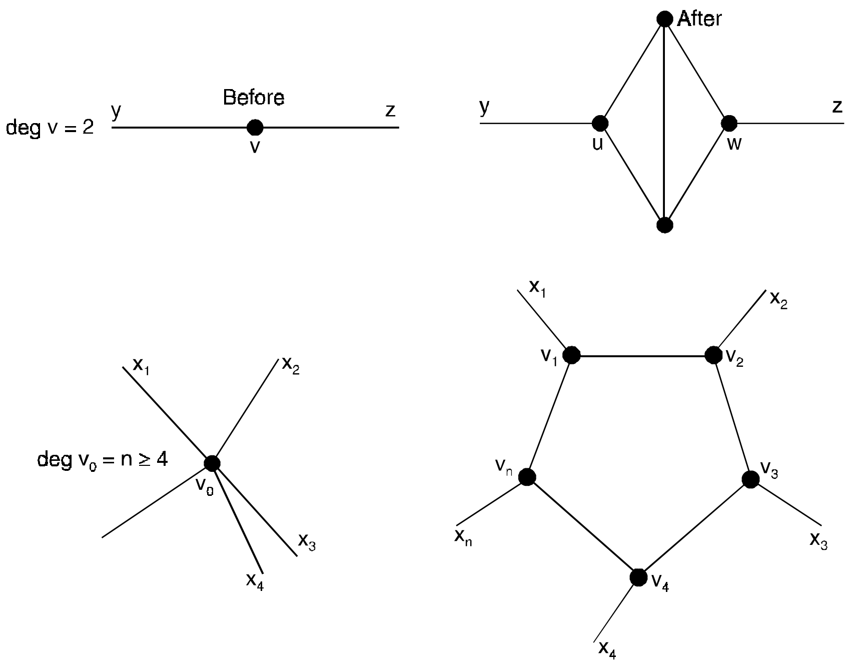


Fig. 5.32. Conversion of a graph into a cubic graph.

Denote the resulting bridgeless cubic plane map by G′, which, by hypothesis, is 4-colorable.

If for each vertex v of G with deg $v \neq 3$, we identify all the newly added vertices associated with v in the formation of G′, we arrive at G once again. Thus, let there be given a 4-coloring of G′. The above mentioned contradiction of G′ into G induces an m-coloring of G, $m \leq 4$, which completes the proof.

Theorem 5.27. *The four color conjecture holds if and only if every hamiltonian planar graph is 4-colorable.*

Theorem 5.28. *For any graph G, the line chromatic number satisfies the inequalties*

$$\Delta \leq \chi' \leq \Delta + 1.$$

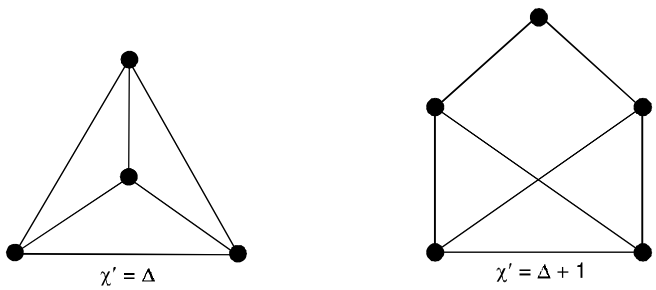


Fig. 5.33. The two values for the line-chromatic number.

5.4.2. The Five color theorem

Every planar graph is 5-colorable.

Proof. We proceed by induction on the number P of points. For any planar graph having $P \le 5$ points, the result follows trivially since the graph is P-colorable.

As the inductive hypothesis we assume that all planar graphs with P points, $P \ge 5$, are 5-colorable.

Let G be a plane graph with P + 1 vertices, G contains a vertex v of degree 5 or less.

By hypothesis, the plane graph $G - v$ is 5-colorable.

Consider an assignment of colors to the vertices of $G - v$ so that a 5-coloring results, when the colors are denoted by C_i, $1 \le i \le 5$.

Certainly, if some color, say C_j, is not used in the coloring of the vertices adjacent with v, then by assigning the color C_j to v, a 5-coloring of G results.

This leaves only the case to consider in which deg $v = 5$ and five colors are used for the vertices of G adjacent with v.

Permute the colors, if necessary, so that the vertices colored C_1, C_2, C_3, C_4 and C_5 are arranged cyclically about v,

Now label the vertex adjacent with v and colored C_i by v_i, $1 \le i \le 5$ (see Figure 5.34)

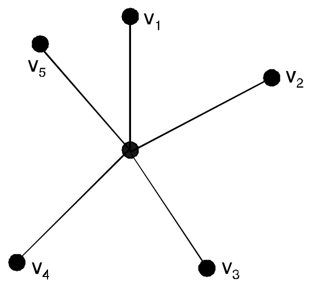


Fig. 5.34. A step in the proof of the five color theorem.

Let G_{13} denote the subgraph of $G - v$ induced by those vertices colored C_1 or C_3.

If v_1 and v_3 belong to different components of G_{13}, then a 5-coloring of $G - v$ may be accomplished by interchanging the colors of the vertices in the component of G_{13} containing v_1.

In this 5-coloring however, no vertex adjacent with v is colored C_1, so by coloring v with the color C_1, a 5-coloring of G results.

If, on the other hand, v_1 and v_3 belong to the same component of G_{13}, then there exists in G a path between v_1 and v_3 all of whose vertices are colored C_1 or C_3.

This path together with the path $v_1\, vv_3$ produces a cycle which necessarily encloses the vertex v_2 or both the vertices v_4 and v_5.

In any case, there exists no path joining v_2 and v_4, all of whose vertices are colored C_2 or C_4.

Hence, if we let G_{24} denote the subgraph of $G - v$ induced by the vertices colored C_2 or C_4, then v_2 and v_4 belong to different components of G_{24}.

Thus if we interchange colors of the vertices in the component of G_{24} containing v_2, a 5-coloring of $G - v$ is produced in which no vertex adjacent with v is colored C_2.

We may then obtain a 5-coloring of G by assigning to v the color C_2.

5.5 MATCHING THEORY

A matching in a graph is a set of edges with the property that no vertex is incident with more than one edge in the set. A vertex which is incident with an edge in the set is said to be saturated. A matching is perfect if and only if every vertex is saturated, that is ; if and only if every vertex is incident with precisely one edge of the matching.

Let G = (V, E) be a bipartite graph with V partitioned as $X \cup Y$. (each edge of E has the form $\{x, y\}$ with $x \in X$ and $y \in Y$).

(*i*) A matching in G is a subset of E such that no two edges share a common vertex in X or Y.

(*ii*) A complete matching of X into Y is a matching in G such that every $x \in X$ is the end point of an edge.

Let G = (V, E) be bipartite with V partitioned as $X \cup Y$. A maximal matching in G is one that matches as many vertices in X as possible with the vertices in Y.

Let G = (V, E) be a bipartite graph where V is partitioned as $X \cup Y$. If $A \subseteq X$, then $\delta(A) = |A| - |R(A)|$ is called the deficiency of A. The deficiency of graph G, denoted $\delta(G)$, is given by $\delta(G) = \max \{\delta(A)/A \subseteq X\}$.

For example, in the graph shown on the left in Fig. (5.35)

(*i*) the single edge *bc* is a matching which saturates *b* and *c*, but neither *a* nor *d* ;

(*ii*) the set $\{bc, bd\}$ is not a matching because vertex *b* belongs to two edges ;

(*iii*) the set $\{ab, cd\}$ is a perfect matching.

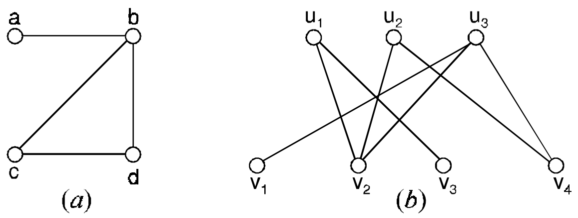


Fig. 5.35.

Edge set $\{ab, cd\}$ is a perfect matching in the graph on the left. In the graph on the right, edge set $\{u_1, v_2, u_2v_4, u_3v_1\}$ is a matching which is not perfect.

Note that, if a matching is perfect, the vertices of the graph can be partitioned into two sets of equal size and the edges of the matching provide a one-to-one correspondence between these sets. In the graph on the left in Fig. (5.35), for instance, the edges of the perfect matching $\{ab, cd\}$ establish a one-to-one correspondence between $\{a, c\}$ and $\{b, d\}$: $a \to b$, $c \to d$.

In the graph on the right of Fig. (5.35).

(*i*) the set of edges $\{u_1v_2, u_2v_4, u_3v_1\}$ is a matching which is not perfect but which saturates $v_1 = \{u_1, u_2, u_3\}$,

(*ii*) no matching can saturate $v_2 = \{v_1, v_2, v_3, v_4\}$ since such a matching would require four edges but then at least one u_i would be incident with more than one edge.

In the figure to the right, if $X = \{u_1, u_2, u_4\}$, then $A(X) = \{v_3, v_4\}$.

Since | X | ≤ | A(X) |, the workers in X cannot all find jobs for which they are qualified. There is no matching in this graph which saturates V_1.

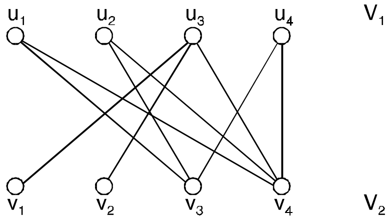


Fig. 5.36.

The bipartite graph shown in Fig. (5.37) has no complete matching. Any attempt to construct such a matching must include $\{x_1, y_1\}$ and either $\{x_2, y_3\}$ or $\{x_3, y_3\}$.

If $\{x_2, y_3\}$ is included, there is no match for x_3. Likewise, if $\{x_3, y_3\}$ is included, we are not able to match x_2.

If A = $\{x_1, x_2, x_3\} \subseteq$ X, then R(A) = $\{y_1, y_3\}$. With | A | = 3 > 2 = | R(A) |, it follows that no complete matching can exist.

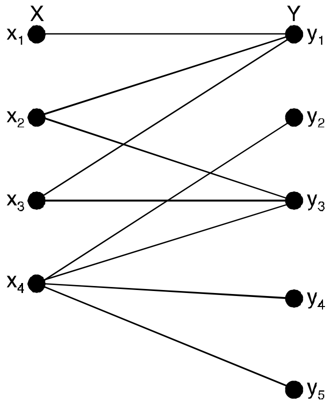


Fig. 5.37.

Theorem 5.29. *Let G = (V, E) be bipartite with V partitioned as X ∪ Y. A complete matching of X into Y exists if and only if for every subset of X, | A | ≤ | R(A) |, where R(A) is the subset of Y consisting of those vertices each of which is adjacent to at least one vertex in A.*

Proof. With V partitioned as X ∪ Y, let X = $\{x_1, x_2, ..., x_m\}$ and Y = $\{y_1, y_2, ..., y_n\}$

Construct a transport network N that extends graph G by introducing two new vertices *a* and *z* (the source and sink).

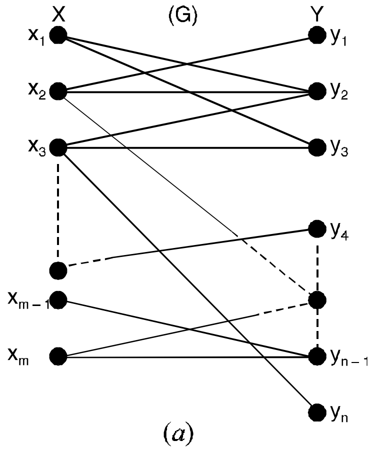
For each vertex x_i, $1 \le i \le m$, draw edge (a, x_i) ; for each vertex y_j, $1 \le j \le n$, draw edge (y_j, z).

Each new edge is given a capacity of 1. Let M be any positive integer that exceeds | X |. Assign each edge in G the capacity M.

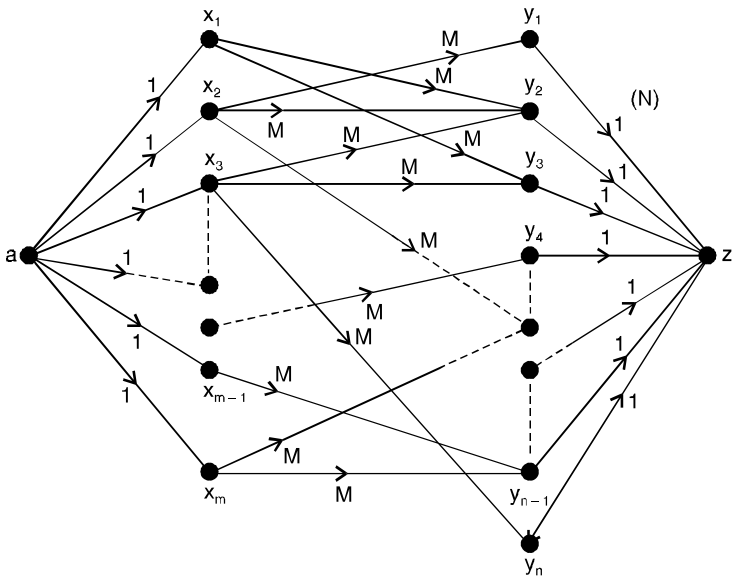
The original graph G and its associated network N appear as shown in Fig. (5.38).

It follows that a complete matching exists in G if and only if there is a maximum flow in N that uses all edges (a, x_i), $1 \le i \le m$.

Then the value of such a maximum flow is $m = |X|$.



(*a*)



(*b*)

Fig. 5.38.

We shall prove that there is a complete matching in G by showing that $C(P, \bar{P}) \ge |X|$ for each cut $(P, \bar{P})$ in N. So if $(P, \bar{P})$ is an arbitrary cut in the transport network N, let us define $A = X \cap P$ and $B = Y \cap P$.

Then $A \subseteq X$ where we shall write $A = \{x_1, x_2, ..., x_i\}$ for some $0 \le i \le m$.

Now P consists of the source a together with the vertices in A and the set $B \subseteq Y$, as shown in Fig. (5.39)(*a*).

In addition, $\overline{P} = (X - A) \cup (Y - B) \cup \{z\}$.

Since each of these edges has capacity 1, $C(P, \overline{P}) = |X - A| + |B| = |X| - |A| + |B|$, with $B \supseteq R(A)$, we have $|B| \ge R(A)$, and since $|R(A)| \ge |A|$, it follows that $|B| \ge |A|$.

Consequently, $c(P, \overline{P}) = |X| + (|B| - |A|) \ge |X|$.

Therefore, since every cut in network N has capacity at least $|X|$, such a flow will result in exactly $|X|$ edges from X to Y having flow 1, and this flow provides a complete matching of X into Y.

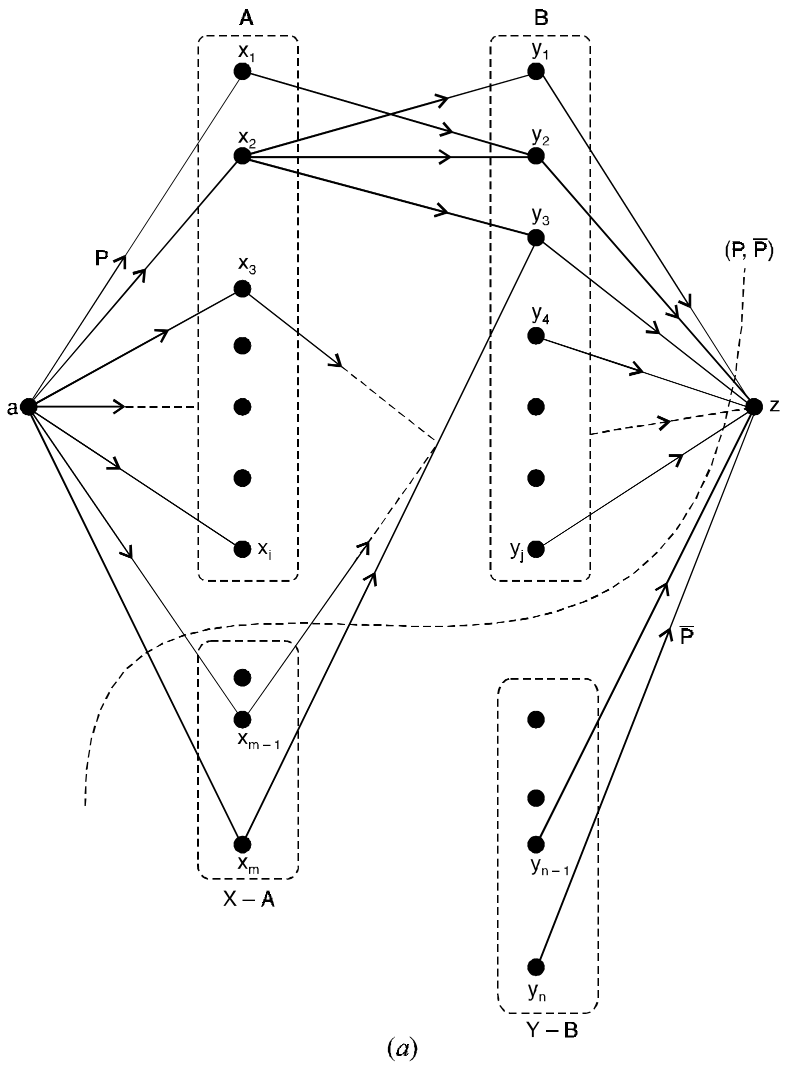
Conversely, suppose that there exists a subset A of X where $|A| > |R(A)|$.

Let $(P, \overline{P})$ be the cut shown for the network in Fig. 5.39(*b*), with $P = \{a\} \cup A \cup R(A)$ and $\overline{P} = (X - A) \cup (Y - R(A)) \cup \{z\}$. Then $C(P, \overline{P})$ is determined by (*i*) the edges from the source *a* to the vertices in X – A and (*ii*) the edges from the vertices in R(A) to the sink *z*.

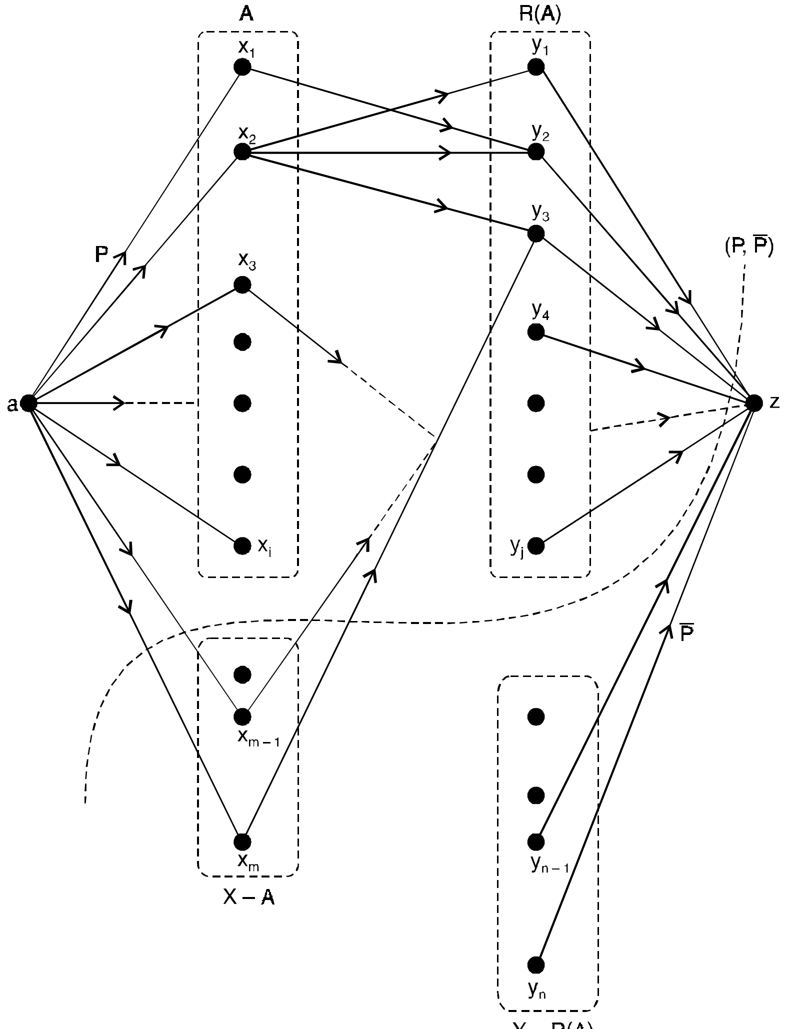
Hence $C(P, \overline{P}) = |X - A| + |R(A)| = |X| - (|A| - |R(A)|) < |X|$,

since $|A| > |R(A)|$. The network has a cut of capacity less than $|X|$, it follows that any maximum flow in the network has value smaller than $|X|$.

Therefore, there is no complete matching from X into Y for the given bipartite graph G.


A
B
x_1
x_2
x_3
x_i
x_{m-1}
x_m
X − A
y_1
y_2
y_3
y_4
y_j
y_{n-1}
y_n
Y − B
a
z
P
$\overline{P}$
$(P, \overline{P})$

(a)



(*b*)

Fig. 5.39

Theorem 5.30. *For any bipartite graph G with partition V_1 and V_2, if there exists a positive integer m satisfying the condition that $deg_G(v_1) \geq m \geq deg_G(v_2)$, for all vertices $v_1 \in V_1$ and $v_2 \in V_2$, then a complete matching of V_1 into V_2 exists.*

Proof. Let G be a bipartite graph with partition V_1 and V_2.

Let m be a positive integer satisfying the condition that $\deg_G(v_1) \geq m \geq \deg_G(v_2)$ for all vertices $v_1 \in V_1$ and $v_2 \in V_2$.

Consider an r-element subset S of the set V_1.

Since the $\deg(v_1) \geq m$, from each element of S, there are at least m edges incident to the vertices in V_1.

Thus there are $m\,r$ edges incident from the set S to the vertices in V_1, but degree of every vertex of V_2 cannot exceed m implies that these $m\,r$ edges are incident on at least $(mr)/r = r$ vertices in V_2.

Hence, there exists a complete matching of V_1 into V_2 exists.

5.5.1 Hall's Marriage Theorem

If G is a bipartite graph with bipartition sets V_1 and V_2, then there exists a matching which saturates V_1 if and only if, for every subset X of V_1, $|X| \leq |A(X)|$.

Proof. It remains to prove that the given condition is sufficient, so we assume that $|X| \leq |A(X)|$ for all subsets X of V_1.

In particular, this means that every vertex in V_1 is joined to at least one vertex in V_2 and also that $|V_1| \leq |V_2|$.

Assume that there is no matching which saturates all vertices of V_1. We derive a contradiction.

We turn G into a directed network in exactly the same manner as with the job assignment application.

Specifically, we adjoin two vertices s and t to G and draw directed arcs from s to each vertex in V_1 and from each vertex in V_2 to t.

Assign a weight of 1 to each of these new arcs. Orient each edge of G from its vertex in V_1 to its vertex in V_2, and assign a large integer $I > |V_1|$ to each of these edges.

As noted before, there is a one-to-one correspondence between matchings of G and (s, t)-flows in this network, and the value of the flow equals the number of edges in the matching.

Since we are assuming that there is no matching which saturates V_1, it follows that every flow has value less than $|V_1|$ and hence by Max-Flow-Mincut theorem, there exists an (s, t)-cut $\{S, T\}$ $\{s \in S, t \in T)$.

Whose capacity is less than $|V_1|$.

Now every original edge of G has been given a weight larger than $|V_1|$.

Since the capacity of our cut is less than $|V_1|$, no edge of G can join a vertex of S to a vertex of T.

Letting $X = V_1 \cap S$, we have $A(X) \subseteq S$.

Since each vertex in A(X) is joined to $t \in T$, each such vertex contributes 1 to the capacity of the cut.

Similarly, since s is joined to each vertex in $V_1 \backslash X$, each such vertex contributes 1. Since $|X| \leq |A(X)|$, we have a contradiction to the fact that the capacity is less than $|V_1|$.

Problem 5.35. *Let G be a bipartite graph with bipartition sets v_1, v_2 in which every vertex has the same degree k. Show that G has a matching which saturates v_1.*

Solution. Let X be any subset of v_1 and let A(X) be as defined earlier.

We count the number of edges joining vertices of X to vertices of A(X).

On the one hand (thinking of X), this number is $k\,|X|$.

On the otherhand (thinking of A(X)), this number is atmost $k\,|A(X)|$ since $k\,|A(X)|$ is the total degree of all vertices in A(X).

Hence, $k\,|X| \leq k\,|A(X)|$, so $|X| \leq |A(X)|$.

Problem 5.36. *Can you conclude from this problem that G also has a matching which saturates V_2 ? More generally, does G have a matching which saturates both V_1 and V_2 at the same time (a perfect matching) ?*

Solution. Yes, the same argument works. But more easily, note that since G is bipartite, the sum of the degrees of vertices in V_1 must equal the sum of degrees of vertices in V_2.

Since all vertices have the some degree, we conclude that $| V_1 | = | V_2 |$, so a matching which saturates V_1 must automatically saturate V_2 as well and vice versa.

Proposition : Let G be a graph with vertex set V.

1. If G has a perfect matching then $| V |$ is even.

2. If G has a Hamiltonian path or cycle then G has a perfect matching if and only if $| V |$ is even.

Theorem 5.30(*a*). *If G is a graph with vertex set V, $| V |$ is even, and each vertex has degree $d \geq \frac{1}{2} | V |$ then G has a perfect matching.*

Problem 5.37. *Given a set S and n subsets $A_1, A_2, \ldots, A_n$ of S, it is possible to select distinct elements $s_1, s_2, \ldots, s_n$ of S such that $S_1 \in A_1, S_2 \in A_2, \ldots, S_n \in A_n$ if and only if, for each subset X of $\{1, 2, \ldots, n\}$ the number of elements in $\bigcup_{x \in X} Ax$ is at least $| X |$. Why ?*

Solution. Construct a bipartite graph with vertex sets V_1 and V_2 where V_1, has n vertices corresponding to $A_1, A_2, \ldots, A_n$, V_2 has one vertex for each element of S and there is an edge joining A_i to s if and only if $s \in A_i$.

Given a subset X of V_1, the set A(X) is precisely the set of elements in $\bigcup_{x \in X} Ax$.

Thus this question is just a restatement of Hall's Marriage theorem.

Problem 5.38. *Determine necessary and sufficient conditions for the complete bipartite graph $K_{m, n}$ to have a perfect matching.*

Solution. $K_{m, n}$ has a perfect matching if and only if $m = n$. To see this, first assume that $m = n$ and let the vertex sets be $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$.

Then $\{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ is a perfect matching.

Conversely, say we have a perfect matching and $m \leq n$. Since each edge in a matching must join a vertex of V_1 to a vertex of V_2, there can be at most m edges.

If $m < n$, some vertex in V_2 would not be part of any edge in the matching, a contradiction.

Thus, $m = n$.

Problem 5.39. *Show that a complete matching of V_1 into V_2 exists in the following graph.*

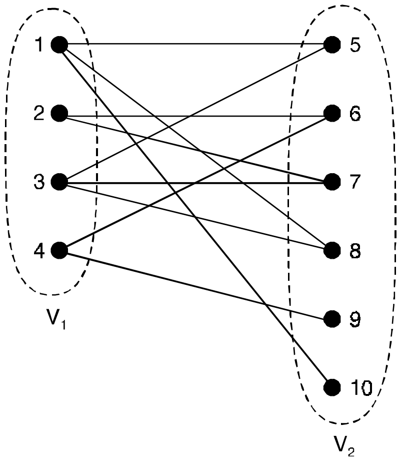


Fig. 5.50.

Solution. The minimum degree of a vertex of $V_1 = 2 \geq 2$

= Maximum degree of a vertex of V_2

By choosing $m = 2$, there exists a complete matching from the set V_1 into V_2.

Problem 5.40. *Find whether a complete matching of V_1 into V_2 exist for the following graph ? What can you say from V_2 into V_1.*

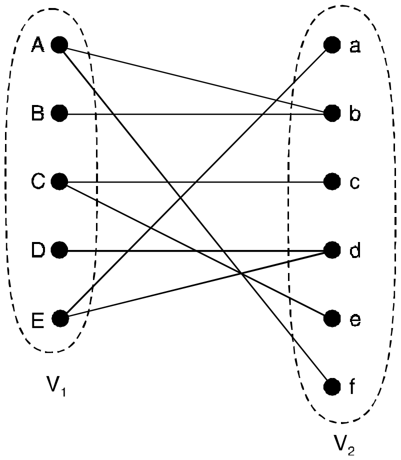


Fig. 5.51.

Solution. Yes, a complete matching exists from V_1 into V_2, which is {A*f*, B*b*, C*c*, D*d*, E*a*}.

This matching is not unique, because {A*f*, B*b*, C*e*, D*d*, E*a*} is also a complete matching from V_1 into V_2 complete matching from V_2 into V_1 is not exists because cordinality of V_2 is more than the cordinality of V_1.

Problem 5.41. *Find whether a complete matching of V_1 into V_2 exist for the following graph.*

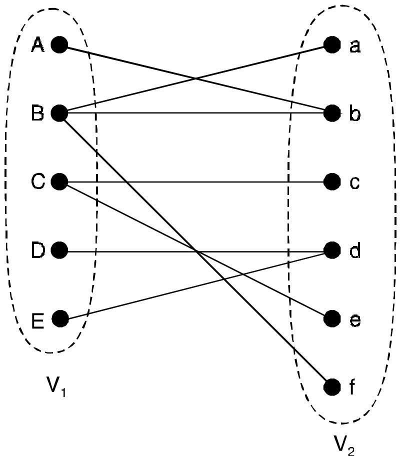


Fig. 5.52.

Solution. No, because if we take a subset {D, E} of V_1 having two vertices, then the elements of this set is collectively adjacent to only the subset {d} of V_2.

The cordinality of {d} is one that is less than the cordinality of the set {D, E}.

Problem 5.42. *Find a complete matching of the graph of Fig. (5.53).*

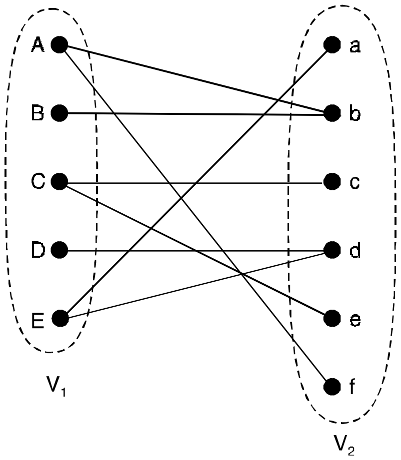


Fig. 5.53.

Solution. $X(G) = \begin{bmatrix} 0 & X_{n_1 \times n_2} \\ X^T_{n_1 \times n_2} & 0 \end{bmatrix}$ where $X_{n_1 \times n_2} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$

Here $n_1 = 5$, $n_2 = 6$ and $n = n_1 + n_2 = 11$ = total number of vertices of G.

Step 1 : Choose the row B and the column *b* (since B contains 1 in only one place in the entire row).

Step 2 : Discard the column *b* (since it is already chosen).

Step 3 : Choose the row D and the column *d* (since D contains 1 in only one place in the entire row).

Step 4 : Discard the column *d* (since it is already chosen).

Step 5 : Choose the row E and the column *a* (since E *a*th entry is one and which are not chosen earlier).

Step 6 : Discard the column *a* (the edge which is chosen in step 5).

Step 7 : Choose the row A and the column *f* (since the row A contains exactly one 1 in the column *f*).

Step 8 : Discard the column *f* (since it is chosen in step 7).

Step 9 : Choose the row C and the column the column *e* (or *c*) (since C is the final).

Step 10 : No row is left to choose and all the rows are able to choose, hence the matching is complete.

The resultant matrices after each step and the final matching is given below.

After the steps 1 and 2

$$\begin{array}{c} \\ A \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{matrix} a & c & d & e & f \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

After the steps 3 and 4

$$\begin{array}{c} \\ A \\ C \\ E \end{array}\begin{array}{c} \begin{matrix} a & c & e & f \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

After the steps 5 and 6

$$\begin{array}{c} \\ A \\ C \end{array}\begin{array}{c} \begin{matrix} c & e & f \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

After the steps 7 and 8

$$C\begin{bmatrix} c & e \\ 1 & 1 \end{bmatrix}$$

Resultant matrix and the corresponding matching are shown in Fig. (5.54).

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

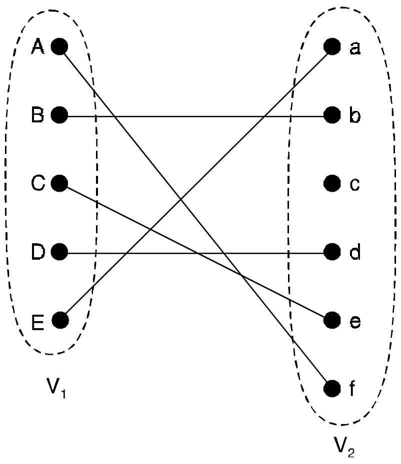


Fig. 5.54.

The complete matching is {A*f*, B*b*, C*e*, D*d*, E*a*}.

Problem 5.43. *Prove that the bipartite graph shown in Fig. 5.55. does not have a complete matching.*

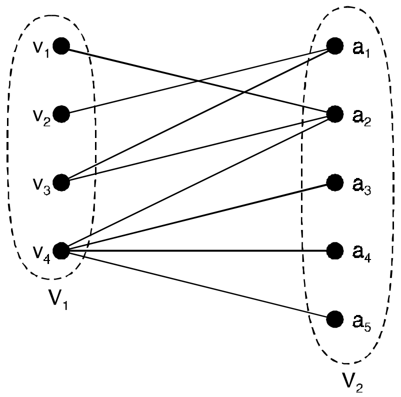


Fig. 5.55.

Solution. We observe that the three vertices v_1, v_2, v_3 in V_1 are together joined to two vertices a_1, a_2, in V_2. Thus, there is a subset of 3 vertices in V_1 which is collectively adjacent to 2 (< 3) vertices in V_2.

Hence, by Hall's theorem, there does not exist a complete matching from V_1 to V_2.

Problem 5.44. *Show that for the graph in Fig. (5.55) there does not exist a positive integer m such that the degree of every vertex in $V_1 \geq m \geq$ the degree of every vertex in V_2.*

Solution. From the graph, we find that degree of $v_1 = 1$ and degree of $a_2 = 3$

Therefore, the specified condition does not hold for any positive integer *m*.

That this is indeed the situation is confirmed by the fact that in this graph there is no complete matching from V_1 to V_2.

Problem 5.45. *Three boys b_1, b_2, b_3 and four girls g_1, g_2, g_3, g_4 are such that (i) b_1 is a cousin of g_1, g_3, g_4. (ii) b_2 is a cousin of g_2 and g_4 (iii) b_3 is a cousin of g_2 and g_3.*

Can every one of the boys marry a girl who is one of his cousins ? If so, find possible sets of such couples.

Solution. Let us draw a bipartite graph $G(V_1, V_2 ; E)$ in which V_1 consists of b_1, b_2, b_3 and V_2 consists of g_1, g_2, g_3, g_4 and E consists of edges representing the cousin relationship. The graph is as shown in Fig. (5.56).

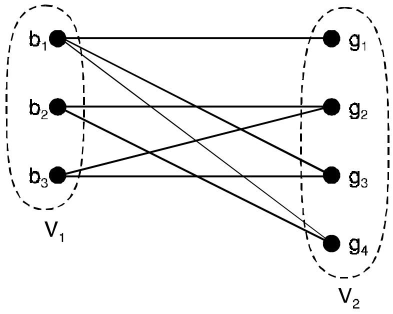


Fig. 5.56.

The problem is one of finding whether a complete matching exists from V_1 and V_2.

We have to consider every subset of V_1 with $k = 1, 2, 3$ elements and find whether each subset is collectively adjacent to k or more vertices in V_2. The subsets S_i of V_1 and their collective adjacent subsets S_i' in V_2 are shown in the following table :

K	S_i	S_i'
$k = 1$	$\{b_1\}$	$\{g_1, g_3, g_4\}$
	$\{b_2\}$	$\{g_2, g_4\}$
	$\{b_3\}$	$\{g_2, g_3\}$
$k = 2$	$\{b_1, b_2\}$	$\{g_1, g_2, g_3, g_4\}$
	$\{b_1, b_3\}$	$\{g_1, g_2, g_3, g_4\}$
	$\{b_2, b_3\}$	$\{g_2, g_3, g_4\}$
$k = 3$	$\{b_1, b_2, b_3\}$	$\{g_1, g_2, g_3, g_4\}$

We observe that, for each S_i, the number of elements in S_i' is greater than or equal to the number of elements in S_i.

Therefore, the graph has a complete matching. This means that each boy can marry a girl who is one of this cousins.

By examining the graph in Fig. (5.56) or the table above, we find the following five possible couple sets :

Step 1 : $(b_1, g_1), (b_2, g_2), (b_3, g_3)$

Step 2 : $(b_1, g_1), (b_2, g_4), (b_3, g_2)$

Step 3 : $(b_1, g_1), (b_2, g_4), (b_3, g_3)$

Step 4 : $(b_1, g_1), (b_2, g_4), (b_3, g_2)$

Step 5 : $(b_1, g_4), (b_2, g_2), (b_3, g_3)$.

5.6 COVERINGS

A point and a line are said to **cover** each other if they are **incident**. A set of points which covers all the lines of a graph G is called a **point cover** for G, while a set of lines which covers all the points is a **line cover**.

5.6.1. Point covering number and line covering number

The smallest number of points in any point cover for G is called its point **covering number** and is denoted by α_0 (G) or α_0. Similarly α_1 (G) or α_1 is the smallest number of lines in any line cover of G and is called its **line covering number**.

For example. $\alpha_0 (K_P) = P - 1$ and $\alpha (K_P) = [(P + 1)/2]$.

A point cover or line cover is called minimum if it contains α_0 (respectively α_1) dements.

We observe that a point cover may be minimum without being minimum, such a set of points is given by the 6 non cut points in Fig. 5.57 below. The same holds for line covers, the 6 lines incident with the cut point serve.

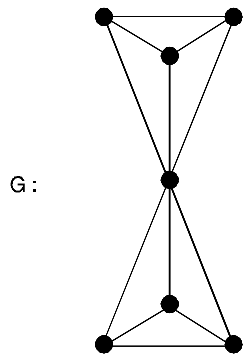


Fig. 5.57. The graph k_4.

5.7 INDEPENDENCE

A set of points in G is independent if no two of them are adjacent.

5.7.1. Point independence number

The largest number of points in such a set is called the point is independence number of G and is denoted by β_0 (G) or β_0.

5.7.2. Line independence number

An independent set of lines of G has no two of its lines adjacent and the maximum cardinality of such a set is the line independence number β_1 (G) or β_1.

For the complete graph, $\beta_0 (K_P) = 1$ and $\beta_1 (K_P) = [P/2]$.

From the above graph, $\beta_0 (G) = 2$ and $\beta_1(G) = 3$.

5.8 VERTEX COVERING

A subset W of V is called a vertex covering or a vertex cover of G if every edge in G is incident on at least one vertex in W.

5.8.1. Trivial vertex covering

A vertex cover of a graph is a subgraph of the graph, V itself is a vetex covering of G. This is known as the trivial vertex covering.

5.8.2. Minimal vertex covering

A vertex covering W of G is called a minimal vertex covering if no proper subset of W is a vertex covering of G.

For example. In the graph shown in Fig. 5.58 below, the set W = $\{v_2, v_4, v_6\}$ is a vertex covering.

We check that $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$ are not vertex coverings of the graph. Thus, no proper subset of W is a vertex covering. Hence W is a minimal vertex covering.

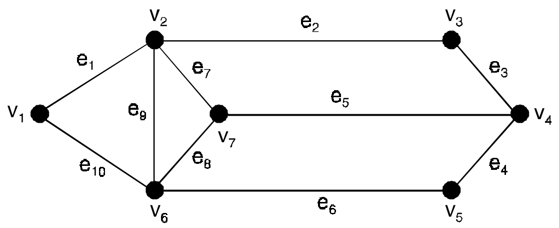


Fig. 5.58.

5.9 EDGE COVERING

A non empty subset S of E is called an edge covering or an edge cover of G if every non isolated vertex in G is incident with at least one edge in S.

5.9.1. Trivial edge covering

An edge cover of a graph is a subgraph of the graph, E itself is an edge covering of G. This is known as the trivial edge covering.

5.9.2. Minimal edge covering

An edge covering S of G is called a minimal edge covering if no proper subset of S is an edge covering of G.

For example. In Figure 5.58, the set S = $\{e_1, e_3, e_6, e_8\}$ is an edge covering.

5.10 CRITICAL POINTS AND CRITICAL LINES

If H is a subgraph of G, then α_0 (H) ≤ α_0 (G). In particular this inequality holds when H = G – *v* or H = G – *x* for any point *v* or line *x*.

If α_0 (G – *v*) < α_0 (G) then *v* is called a critical point, if α_0 (G – *x*) < α_0 (G) then *x* is a critical line of G.

If *v* and *x* are critical, it follows that

$$\alpha_0 (G - v) = \alpha_0 (G - x) = \alpha_0 - 1.$$

5.11 LINE-CORE AND POINT-CORE

The line-core C_1 (G) of a graph G is the subgraph of G induced by the union of all independent sets Y of lines (if any) such that |Y| = α_0 (G).

For example. Consider an odd cycle C_P. Here we find that α_0 (C_P) = (P + 1)/2 but that β_1 (C_P) = (P – 1)/2 so C_P has no line-core.

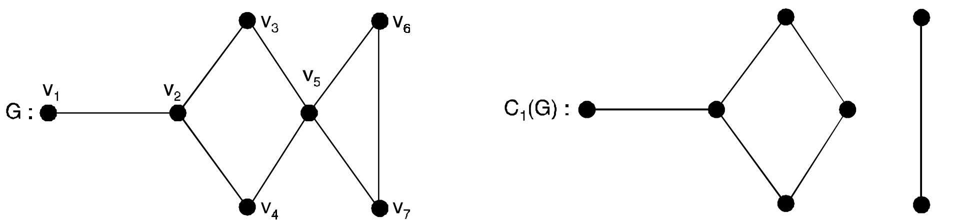


Fig. 5.59. A graph and its line-core.

A minimum point cover M for a graph G with point set V is said to be external if for each subset M′ of M, |M′| ≤ |U (M′)|, where U (M′) is the set of all points of V – W which are adjacent to a point of M′.

Observations

(*i*) A covering exists for a graph if and only if the graph has no isolated vertex.

(*ii*) A covering of an *n*-vertex graph will have at least [*n*/2] adges. ($\lceil x \rceil$ denotes the smallest integer not less than *x*)

(*iii*) Every pendent edge in a graph is include in every covering of the graph.

(*iv*) Every covering contains a minimal covering.

(*v*) If we denote the remaining edges of a graph by (G – *g*), the set of edges *g* is a covering if and only if, for every vertex *v*, the degree of vertex in (G – *g*) ≤ (degree of vertex *v* in G) – 1.

(*vi*) No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an *n*-vertex graph can contain no more than *n* – 1 edges.

(*vi*) A graph, in general, has many minimal coverings, and they may be of different sizes (*i.e,*. consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

5.12 DIGRAPH DEFINITION

A digraph D consists of a finite set V of points and a collection of ordered pairs of distinct points. Any such pair (u, v) is called an arc or directed line and will usually be denoted uv. The arc uv goes from u to v and is incident with u and v. We say that u is adjacent to v and v is adjacent from u.

In otherwords, A directed graph or a digraph G consists of a set of vertices $V = \{v_1, v_2, ...\}$, a set of edge $E = \{e_1, e_2, ...\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

For example, Fig. 5.60 below shows a digraph with five vertices and ten edges.

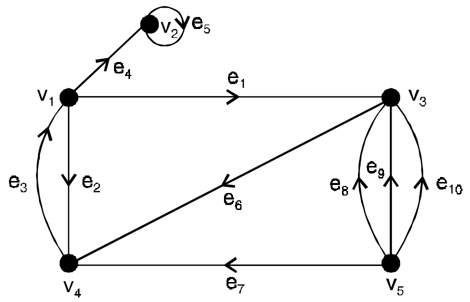


Fig. 5.60. Directed graph with 5 vertices and 10 edges.

5.12.1 Orientation of a Graph

Given a graph G, if there is a digraph D such that G is the underlying graph of D then D is called an orientation of G.

The digraphs in Fig. 5.61(*a*) and Fig. 5.61(*b*) are two different orientations of the graph in Fig. 5.61(*c*).

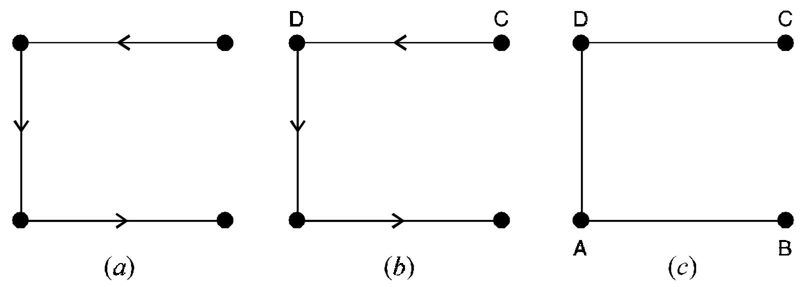


Fig. 5.61.

5.12.2 Underlying Graph

If D is a digraph, the graph obtained from D by 'removing the arrows' from the directed edges is called the underlying graph of D. This graph is also called the undirected graph corresponding to D.

The underlying graph of the digraph in Fig. 5.61(*a*) is shown in Fig. 5.61(*c*).

The graph in Fig. 5.61(*c*) is the underlying graph of the digraph shown in Fig. 5.61(*b*).

Note : Every digraph has a unique underlying graph.

5.12.3. Parallel Edges

Two (directed) edges e and e' of a digraph D are said to be parallel if e and e' have the same initial vertex and the same terminal vertex.

In the digraph in Fig. 5.62 the edges e_6 and e_7 are parallel edges whereas the edges e_1 and e_9 are not parallel. e_1 and e_6 are parallel edges in the underlying graph.

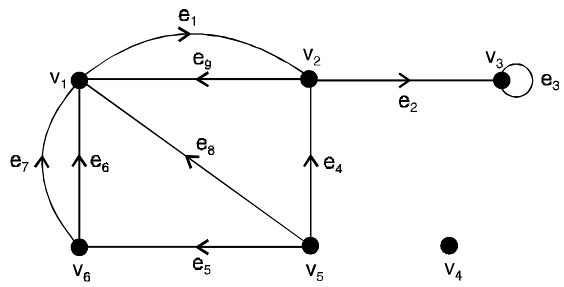


Fig. 5.62.

5.12.4. Incidence

In a digraph every edge has two end vertices, one vertex from which it begins and the other vertex at which it terminates. If an edge e begins at a vertex u and terminates at a vertex v, we say that e is incident out of u and incident into v. Here, u is called the initial vertex and v is called the terminal vertex of e.

For example, in the digraph in Fig. 5.62, the edge e_1 is incident out of the vertex v_1 and incident into the vertex v_2, v_1 is the initial vertex and v_2 is the terminal vertex of the edge e_1. For a self-loop in a digraph, the initial and terminal vertices are one and the same. In Fig. 5.62 the edge e_3 is a self-loop with v_3 as the initial and terminal vertex.

5.12.5. In-degree and Out-degree

If v is a vertex of a digraph D, the number of edges incident out of v is called the out-degree of v and the number of edges incident into v is called the in-degree of v. The out-degree of v is denoted by $d^+(v)$ and the in-degree of v is denoted by $d^-(v)$.

For example, the out-degrees and in-degrees of the six vertices of the digraph shown in Fig. 5.62. are as given below :

$d^+(v_1) = 1, \quad d^-(v_1) = 4$

$d^+(v_2) = 2, \quad d^-(v_2) = 2$

$d^+(v_3) = 1, \quad d^-(v_3) = 2$

$d^+(v_4) = 0, \quad d^-(v_4) = 0$

$d^+(v_5) = 3, \quad d^-(v_5) = 0$

$d^+(v_6) = 2, \quad d^-(v_6) = 1$

For example, in Fig. 5.60

$d^+(v_1) = 3, \quad d^-(v_1) = 1$

$d^+(v_2) = 1, \quad d^-(v_2) = 2$

$d^+(v_5) = 4, \quad d^-(v_5) = 0.$

5.12.6. Isolated Vertex

If v is a vertex of a digraph D then v is called an isolated vertex of D if $d^+(v) = d^-(v) = 0$.

5.12.7. Pendant Vertex

If v is a vertex of a digraph D then v is called a pendant vertex of D if $d^+(v) + d^-(v) = 1$.

5.12.8. Source

If v is a vertex of a digraph D then v is called a source of D if $d^-(v) = 0$.

5.12.9. Sink

If v is a vertex of a digraph D then v is called a sink of D if $d^+(v) = 0$.

The digraph is Fig. 5.60 has v_4 as an isolated vertex and v_5 as a source.

In the digraph in Fig. 5.60, the vertices B and C are pendant vertices, C is a source and B is a sink.

5.13 TYPES OF DIGRAPHS

5.13.1. Simple Digraphs

A digraphs that has no self-loop or parallel edges is called a simple digraph.

The digraph shown in Fig. 5.63(a) is simple, but its underlying graph shown in Fig. 5.63(b) is not simple.

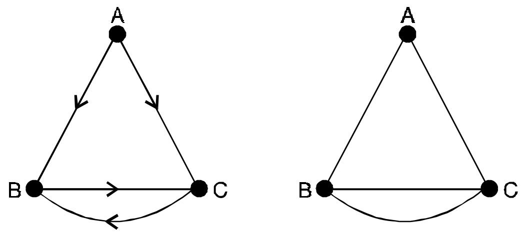


Fig. 5.63.(a), (b).

5.13.2. Asymmetric Digraphs

Digraphs that have atmost one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or antisymmetric digraph.

For example, the digraph in Fig. 5.64(a), is asymmetric.

The digraph in Fig. 5.64(b) is neither symmetric nor asymmetric.

The digraph in Fig. 5.64(b) is simple and asymmetric.

The digraph in Fig. 5.64(b) is simple but not asymmetric.

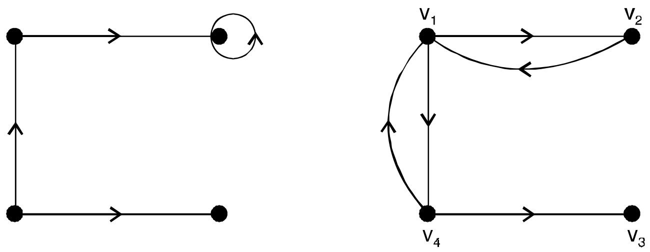


Fig. 5.64.(*a*), (*b*).

5.13.3. Symmetric Digraph

Digraphs in which for every edge (*a*, *b*) (*i.e.,* from vertex *a* to *b*) there is also an edge (*b*, *a*).

For example, the digraph in Fig. 5.65 is a symmetric digraph. The digraph in Fig. 5.64(*b*) is not symmetric.

This digraph has (v_4, v_3) as an edge but does not have (v_3, v_4) as an edge.

The digraph in Fig. 5.65 is simple also. Such a digraph is called a symmetric simple digraph. The digraph in Fig. 5.64(*b*) is simple and non-symmetric.

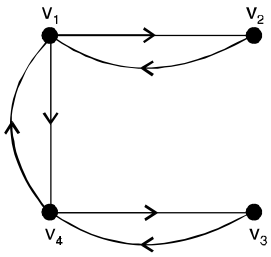


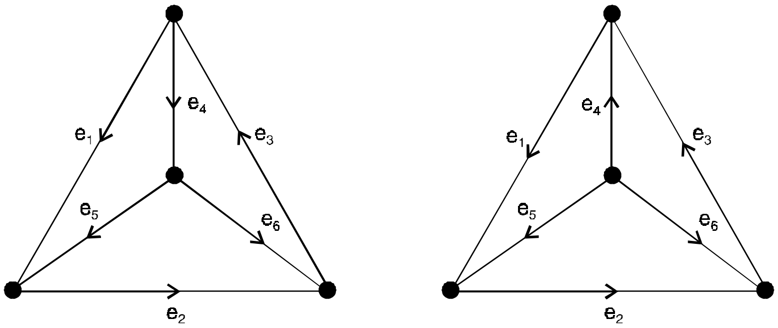
Fig. 5.65.

5.13.4. Isomorphic Digraphs

Isomorphic graphs were defined such that they have identical behaviour in terms of graph properties.

In otherwords, if their labels are removed, two isomorphic graphs are indistinguishable. For two digraphs to be isomorphic not only must their corresponding undirected graphs be isomorphic, but the directions of the corresponding edges must also agree.

For example, Fig. 5.66, shows two digraphs that are not isomorphic, although they are orientations of the same undirected graph.



(*a*) (*b*)

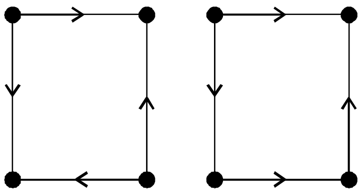
Fig. 5.66. Two nonisomorphic digraphs.

In otherwords, two digraphs D_1 and D_2 are said to be isomorphic if both of the following conditions hold :

(*i*) The underlying graphs of D_1 and D_2 are either identical or isomorphic.

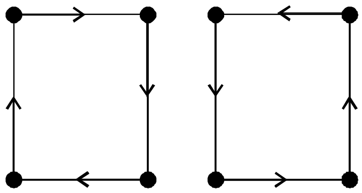
(*ii*) Under the one-to-one correspondence between the edges of D_1 and D_2 the directions of the corresponding edges are preserved.

The two digraphs in Fig. 5.67(*a*) and 5.67(*b*) are isomorphic, whereas the two digraphs in Fig.5.68(*a*) and 5.68(*b*) are not isomorphic.



(*a*) (*b*)

Fig. 5.67. Two isomorphic digraphs.



(*a*) (*b*)

Fig. 5.68. Two non-isomorphic digraphs.

5.13.5. Balanced Digraphs

A digraph D is said to be a balanced digraph or an isograph if $d^+(v) = d^-(v)$ for every vertex v of D.

5.13.6. Regular Digraph

A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

5.13.7. Complete Digraphs

A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge.

5.13.8. Complete Symmetric Digraph

A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex (see Fig. 5.69).

5.13.9. Complete Asymmetric Digraph

A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices (see Fig. 5.66).

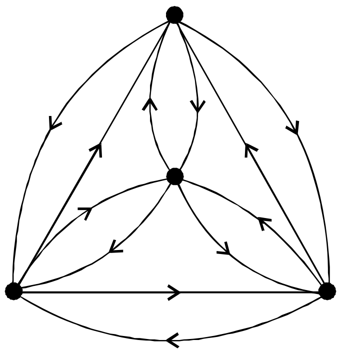


Fig. 5.69. Complete Symmetric Digraph of Four Vertices.

5.14 CONNECTED DIGRAPHS

5.14.1. Strongly Connected

A digraph G is said to be strongly connected if there is atleast one directed path from every vertex to every other vertex.

5.14.2. Weakly Connected

A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected.

Fig. 5.66, one of the digraphs is strongly connected, and the other one is weakly connected.

5.14.3. Component and Fragments

Each maximal connected (weakly or strongly) subgraph of a digraph G is called a component of G. But within each component of G the maximal strongly connected subgraphs are called the fragments (or strongly connected fragments) of G.

For example, the digraph in Fig. 5.70, consists of two components. The component g_1 contains three fragments $\{e_1, e_2\}$, $\{e_5, e_6, e_7, e_8\}$ and $\{e_{10}\}$.

We observe that e_3, e_4 and e_9 do not appear in any fragment of g_1.

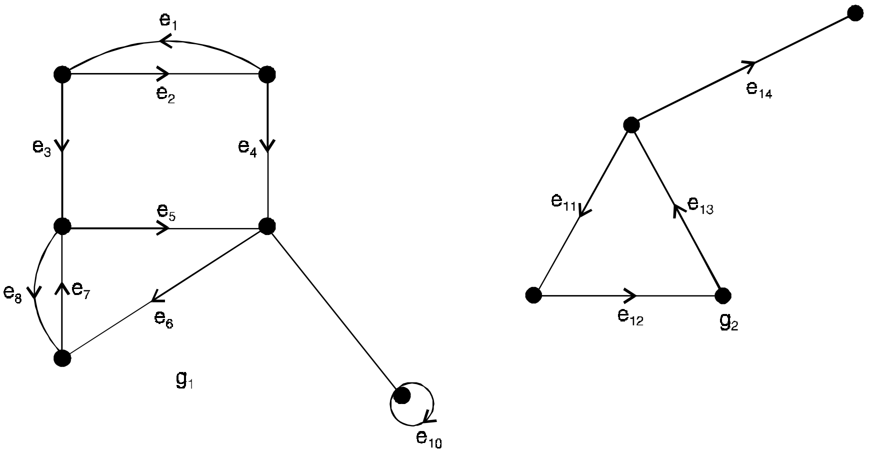


Fig. 5.70. Disconnected digraph with two components.

5.15 CONDENSATION

The condensation G_c of a digraph G is a digraph in which each strongly connected fragment is replaced by a vertex and all directed edges from one strongly connected component to another are replaced by a single directed edge.

The condensation of the digraph G in Fig. 5.70 is shown in Fig. 5.71.

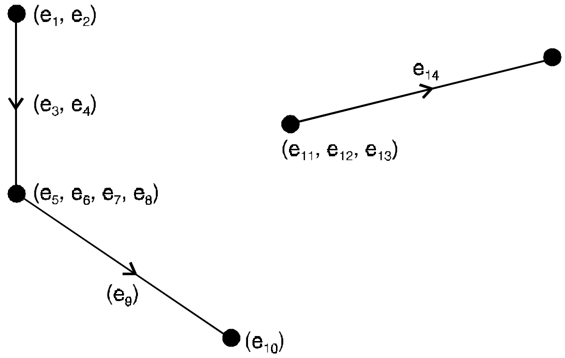


Fig. 5.71. Condensation of Fig. 5.70.

Observations :

(*i*) The condensation of a strongly connected digraph is simply a vertex.

(*ii*) The condensation of a digraph has no directed circuit.

5.16 REACHABILITY

Given two vertices u and v of a digraph D, we say that v is reachable (or accessible) from u if there exists atleast one directed path in D from u to v.

For example, in the digraph shown in Fig. 5.71, the vertex v_3 is reachable from the vertex v_5, but v_5 is not reachable from v_3.

5.17 ORIENTABLE GRAPH

A graph G is said to be orientable if there exists a strongly connected digraph D for which G is the underlying graph.

For example, the graph is Fig. 5.72(*a*) is orientable, a strongly directed digraph for which this graph is the underlying graph is shown in Fig. 5.72(*b*).

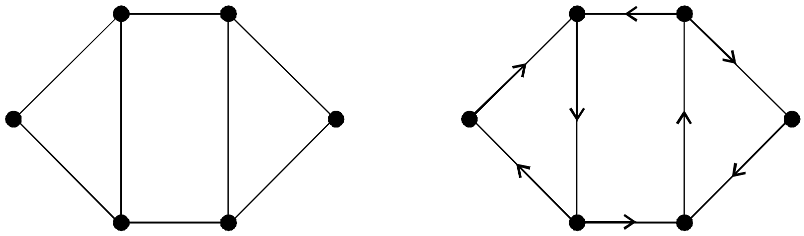


Fig. 5.72.(*a*) (*b*)

5.18 ACCESSIBILITY

In a digraph a vertex *b* is said to be accessible (or reachable) from vertex *a* if there is a directed path from *a* to *b*. Clearly, a digraph G is strongly connected if and only if every vertex in G is accessible from every other vertex.

5.19 ARBORESCENCE

A digraph G is said to be an arborescence if

(*i*) G contains no circuit, neither directed nor semi circuit.

(*ii*) In G there is precisely one vertex *v* of zero in-degree.

This vertex *v* is called the root of the arborescence.

An arborescence is shown in Fig. 5.73 below.

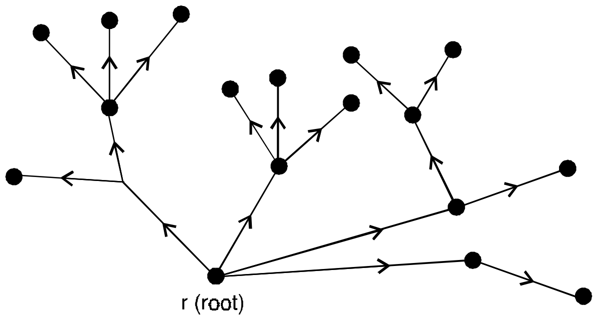


Fig. 5.73. Arborescence.

5.19.1. Spanning arborescence

A spanning tree in an *n*-vertex connected digraph, analogous to a spanning tree in an undirected graph, consists of $n - 1$ directed edges.

A spanning arborescence in a connected digraph is a spanning tree that is an arborescence.

For example, a spanning arborescence in Fig. 5.74, is {*f*, *b*, *d*}. There is a striking relationship between a spanning arborescence and an Euler line.

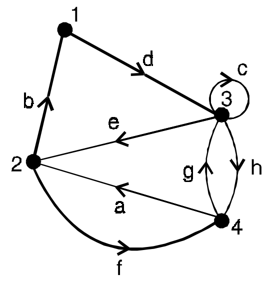


Fig. 5.74. Euler Digraph.

5.19.2. Euler digraphs

In a digraph G a closed directed walk (*i.e.,* a directed walk that starts and ends at the same vertex) which transverses every edge of G exactly once is called a directed Euler line.

A graph containing a directed Euler line is called an Euler digraph.

For example, the graph in Fig. 5.75, is an Euler digraph, in which the walk *a b c d e f* is an Euler line.

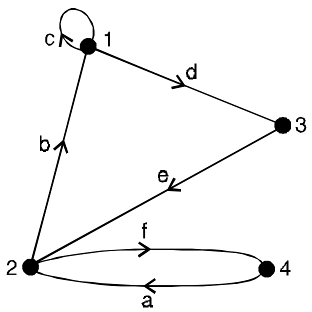


Fig. 5.75. Euler Digraph.

5.20 HAND SHAKING DILEMMA

In a digraph D, the sum of the out-degree of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.

For example, the digraph in Fig. 5.60 and 5.61, we note that the digraphs has 6 vertices and 9 edges and that the sums of the out-degrees and in-degrees of its vertices are

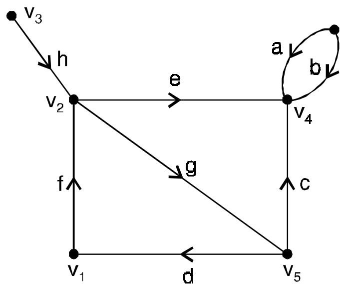
$$\sum_{i=1}^{6} d^{+}(v_i) = 9\,;\; \sum_{i=1}^{6} d^{-}(v_i) = 9$$

5.21 INCIDENCE MATRIX OF A DIGRAPH

The incidence matrix of a digraph with n vertices, e edges and no self-loops in an n by n matrix A = $[a_{ij}]$ whose rows correspond to vertices and columns correspond to edges such that

a_{ij} = 1, if j^{th} edge is incident out of i^{th} vertex
= – 1, if j^{th} edge is incident into i^{th} vertex
= 0, if j^{th} edge is not incident on i^{th} vertex.

For example, A digraph and its incidence matrix are shown in Fig. 5.76.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 5.76. Digraph and its incidence matrix.

5.22 CIRCUIT MATRIX OF A DIGRAPH

Let G be a digraph with e edges and q circuits. An arbitrary orientation is assigned to each of the q circuits. Then a circuit matrix B = $[b_{ij}]$ of the digraph G is a q by e matrix defined as

b_{ij} = 1, if i^{th} circuit includes j^{th} edge, and the orientations of the edge and circuit coincide
= – 1, if i^{th} circuit includes j^{th} edge, but the orientations of the two are opposite
= 0, if i^{th} circuit does not include the j^{th} edge.

For example, a circuit matrix of the digraph in Fig. 5.76 is

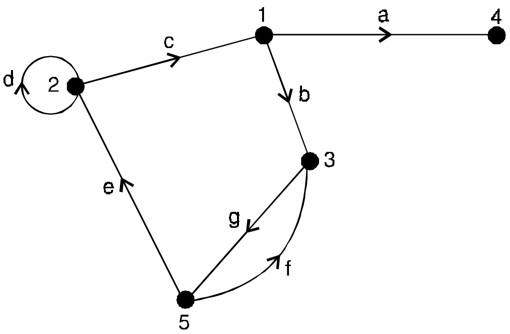
$$\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

5.23 ADJACENCY MATRIX OF A DIGRAPH

Let G be a digraph with n vertices, containing no parallel edges. Then the adjacency matrix X = $[x_{ij}]$ of the digraph G is an n by n (0, 1) matrix whose element.

x_{ij} = 1, if there is an edge directed from i^{th} vertex to j^{th} vertex
= 0, otherwise

For example, a digraph and its adjacency matrix are shown in Fig. 5.77.



$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 5.77. Digraph and its Adjacency Matrix.

Observations :

(*i*) X is a symmetric matrix if and only if G is a symmetric digraph.

(*ii*) Every non-zero element on the main diagonal represents a self-loop at the corresponding vertex.

(*iii*) There is no way of showing parallel edges in X. This is why the adjacency matrix is defined only for a digraph without parallel edges.

(*iv*) The sum of each row equals the out-degree of the corresponding vertex and the sum of each column equals the in-degree of the corresponding vertex. The number of non-zero entries in X equals the number of edges in G.

(*v*) If X is the adjacency matrix of a digraph G, then the transposed matrix X^T is the adjacency matrix of a digraph G^R obtained by reversing the direction of every edge in G.

(*vi*) For any square (0, 1)-matrix Q of order *n*, there exists a unique digraph G of *n* vertices such that Q is the adjacency matrix of G.

Theorem 5.31. *Let G be a connected graph. Then G is orientable if and only if each edge of G is contained in at least one cycle.*

Proof. The necessity of the condition is clear. To prove the sufficiency.

We choose any cycle C and direct its edges cyclically.

If each edge of G is contained in C, then the proof is complete. If not, we choose any edge *e* that is not in C but which is adjacent to an edge of C.

By hypothesis, *e* is contained in some cycle C′ whose edges we may direct cyclically, except for those edges that have already been directed, that is, those edges of C′ that also lie in C.

It is not difficult to see that the resulting digraph is strongly connected, the situation is illustrated in Fig. 5.78 below, with dashed lines denoting edges of C′.

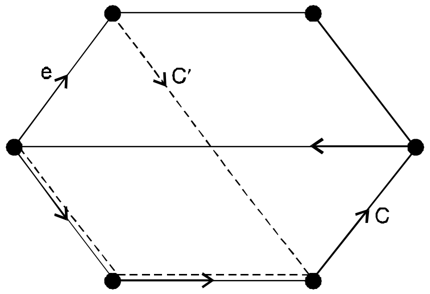


Fig. 5.78.

We proceed in this way, at each stage directing at least one new edge, until all edges are directed. Since the digraph remains strongly connected at each stage, the result follows.

Theorem 5.32. *A connected digraph is Eulerian if and only if for each vertex of D out deg(V) = in deg(v).*

Theorem 5.33. *Let D be a strongly connected digraph with n vertices. If out deg (v)* $\geq \frac{n}{2}$ *and in deg (v)* $\geq \frac{n}{2}$ *for each vertex v, then D is Hamiltonian.*

Theorem 5.34. (*i*) *Every non-Hamiltonian tournament is semi-Hamiltonian,*

(*ii*) *every strongly connected tournament is Hamiltonian.*

Proof. (*i*) The statement is clearly true if the tournament has fewer than four vertices.

We prove the result by induction on the number of vertices.

Assume that every non-Hamiltonian tournament on n vertices is semi-Hamiltonian.

Let T be a non-Hamiltonian tournament on $n + 1$ vertices, and let T′ be the tournament on n vertices obtained by removing from T a vertex v and its incident arcs.

By the induction hypothesis, T′ has a semi-Hamiltonian path $v_1 \to v_2 \to \ldots\ldots \to v_n$.

There are now three cases to consider

(1) if vv_1 is an arc in T, then the required path is $v \to v_1 \to v_2 \;\ldots\ldots\; \to v_n$.

(2) if vv_1 is not an arc in T, which means that v_1v is and if there exists an i such that vv_i is an arc in T, then choosing i to be the first such, the required path is (see Fig. 5.79(*a*) below)

$$v_1 \to v_2 \to \ldots\ldots \to v_{i-1} \to v \to v_i \to \ldots\ldots \to v_n.$$

(3) if there is no arc in T of the form vv_i, then the required path is $v_1 \to v_2 \to \ldots\ldots \to v_n \to v$.

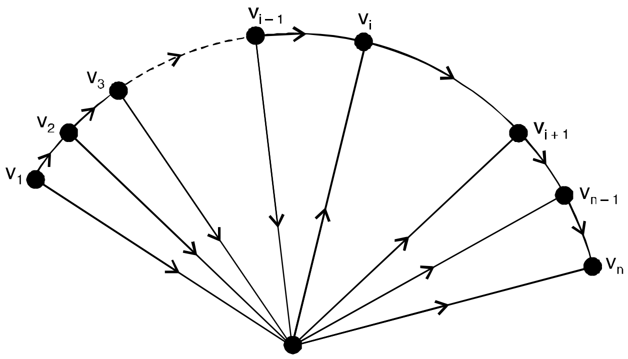


Fig. 5.79(*a*).

(*ii*) We prove the stronger result that a strongly connected tournament T on n vertices contains cycles of length 3, 4, ... n.

To show that T contains a cycle of length 3.

Let v be any vertex of T and let W be the set of all vertices W such that vW is an arc in T, and z be the set of all vertices z such that zv is an arc.

Since T is strongly connected W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is an W and z' is in Z (see Fig. 5.79(b) below). The required cycle of length 3 is then $v \to \omega' \to z' \to v$.

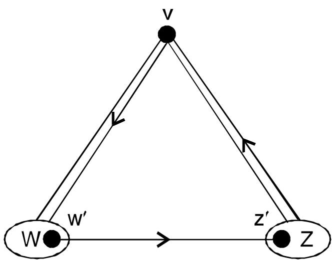


Fig. 5.79(*b*).

It remains only to show that, if there is a cycle of length k, where $k \leq n$, then there is one of length $k + 1$.

Let $v_1 \to$ ...... $v_k \to v_1$ be such that a cycle.

Suppose first that there exists a vertex v not contains in this cycle, such that there exist arcs in T of the form vv_i and of the form v_jv.

Then there must be a vertex v_i such that both $v_{i-1}v$ and vv_i are arcs in T. The required cycle is then $v_1 \to v_2 \to$ .... $\to v_{i-1} \to v \to v_i \to$ ...... $\to v_k \to v_1$ (see Fig. 5.79(c)).

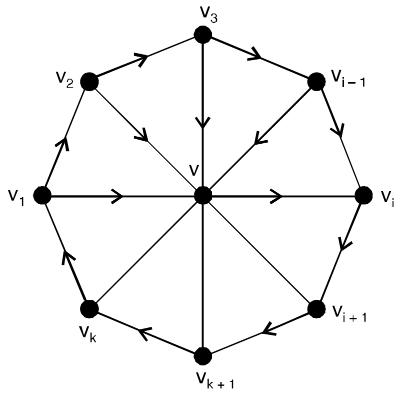


Fig. 5.79.(*c*)

If no vertex exists with the above-mentioned property, then set of vertices not contained in the cycle may be divided into two disjoint sets W and Z, where W is the set of vertices w such that vw_i is an arc for each i, and Z is the set of vertices z such taht zv_i is an arc for each i.

Since T is strongly connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is in W and z' is in Z.

The required cycle is then $v_1 \to w' \to z' \to v_3 \to ...... \to v_k \to v_1$. (See Fig. 5.79(*d*) below).

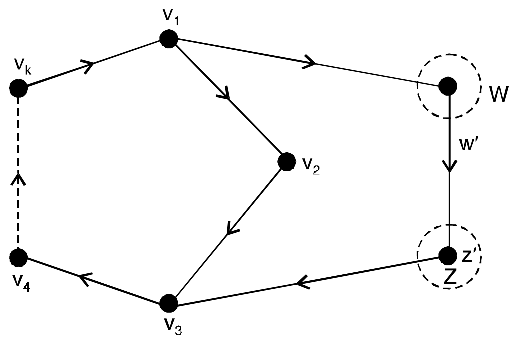


Fig. 5.79(*d*).

Theorem 5.35. *A digraph is strong if and only if it has a spanning closed walk, it is unilateral if and only if it has a spanning walk, and it is weak if and only if it has a spanning semi-walk.*

Theorem 5.36. *A weak digraph is an in-tree if and only if exactly one point has out degree 0 and all others have out degree 1.*

Theorem 5.37. *A weak digraph is an out-tree if and only if exactly one point has indegree 0 and all others have indegree 1.*

Theorem 5.38. *Every digraph with no odd cycles has a 1-basis.*

Corollary. Every acylic digraph has a 1-basis.

Theorem 5.39. *Every acyclic digraph has a unique point basis consisting of all points of indegree 0.*

Corollary. Every point basis of a digraph D consists of exactly one point from each of those strong components in D which form the point basis of D*.

Theorem 5.40. *An cyclic digraph D has at least one point of indegree zero.*

Theorem 5.41. *An acyclic digraph has at least one point of out degree zero.*

Proof. Consider the last point of any maximal path in the digraph. This point can have no points adjacent from it since otherwise there would be a cycle or the path would not be maximal.

The dual theorem follows immediately by applying the principle of Directional Duality. In keeping with the use of D′ to denote the converse of digraph D.

Theorem 5.42. *The following properties of a digraph D are equivalent.*

(*i*) D is a acyclic.

(*ii*) D* is isomorphic to D.

(*iii*) Every walk of D is a path.

(*iv*) It is possible to order the points of D so that the adjacency matrix A(D) is upper triangular.

Theorem 5.43. *The following are equivalent for a weak digraph D.*

(*i*) *D is functional.*

(*ii*) *D has exactly one cycle, the removal of whose arcs results in a digraph in which each weak component is an in-tree with its sink in the cycle.*

(*iii*) *D has exactly one cycle z, and the removal of any arc of Z results in an in-tree.*

Problem 5.46. *Teleprinter's Problem*

How long is a longest circular (or cycle) sequence of 1's and 0's such that no subsequence of r bits appears more than once in the sequence ? Construct one such longest sequence.

Solution. Since there are 2^r distinct r-tuples formed from 0 and 1, the sequence can be no longer than 2^r bits long. We shall construct a circular sequence 2^r bits long with the required property that no subsequence of r bits be repeated.

Construct a digraph G whose vertices are all $(r-1)$ tuples of 0's and 1's.

Clearly, there are 2^{r-1} vertices in G.

Let a typical vertex be $\alpha_1\alpha_2 \ldots\ldots \alpha_{r-1}$, where $\alpha_i = 0$ or 1.

Draw an edge directed from this vertex $(\alpha_1\alpha_2 \ldots\ldots \alpha_{r-1})$ to each of two vertices $(\alpha_2\alpha_3 \ldots\ldots \alpha_{r-1}0)$ and $(\alpha_2\alpha_3 \ldots\ldots \alpha_{r-1}1)$ label these directed edges $\alpha_1\alpha_2 \ldots\ldots \alpha_{r-1}0$ and $\alpha_1\alpha_2 \ldots\ldots \alpha_{r-1}1$ respectively.

Draw two such edges directed from each of the 2^{r-1} vertices. A self-loop will result in each of the two cases when $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{r-1} = 0$ or 1.

The resulting digraph is an Euler digraph because for each vertex the in-degree equals the out-degree (each being equal to two). A directed Euler line in G consists of the 2^r edges, each with a distinct r-bit label. The labels of any two consecutive edges in the Euler line are of the form $\alpha_1\alpha_2 \ldots\ldots \alpha_{r-1}\alpha_r$, $\alpha_2\alpha_3 \ldots\ldots\alpha_r\, \alpha_{r-1}$ that is ; the $r-1$ trailing bits of the first edge are identical to the $r-1$ leading bits of the second edge. Thus in the sequence of 2^r bits, made of the first bit of each of the edges in the Euler line, every possible subsequence of r bits occurs as the label of an edge, and since no two edges have the same label, no subsequence occurs more than once. The circular arrangement is achieved by joining the two ends of the sequence.

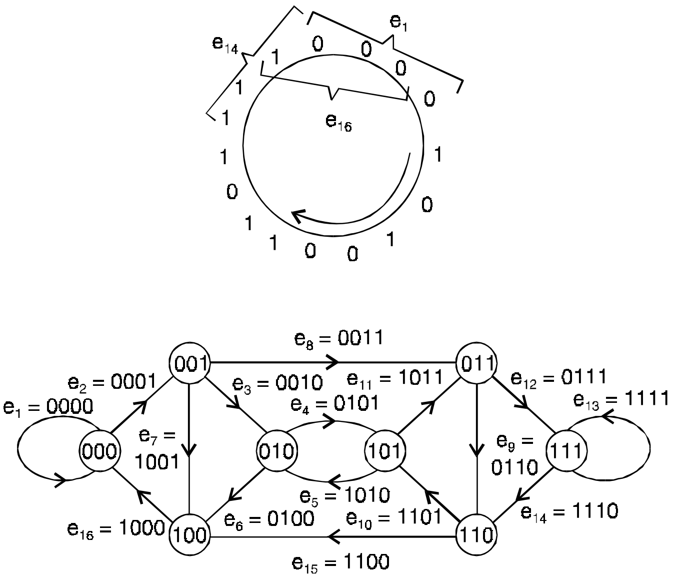


Fig. 5.80. Euler digraph for maximum-length sequence.

For $r = 4$, the graph in Fig. 5.80 above, illustrates the procedure of obtaining such a maximum length sequence one such sequence is 0000101001101111. Corresponding to the walk $e_1e_2e_3e_4e_5e_6e_7e_8e_9e_{10}\ e_{11}e_{12}e_{13}e_{14}e_{15}e_{16}$.

Problem 5.47. *Find the in-degrees and out-degrees of the vertices of the digraphs shown in Fig. 5.81 below. Also, verify the handshaking dilemma.*

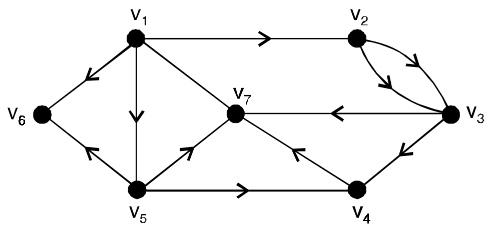


Fig. 5.81.

Solution. The given digraph has 7 vertices and 12 edges. The out-degree of a vertex is got by counting the number of edges that go out of the vertex and the in-degree of a vertex is got by counting the number of edges that end at the vertex. Thus, we obtain the following data.

Vertex	*Out-degree*	*In-degree*
v_1	4	0
v_2	2	1
v_3	2	2
v_4	1	2
v_5	3	1
v_6	0	2
v_7	0	4

This table gives the out-degrees and in-degrees of all vertices. We note that v_1 is a source and v_6 and v_7 are sinks.

Also, check that

sum of out-degrees = sum of in-degrees

= 12 = No. of edges.

Problem 5.48. *Let D be a digraph with an odd number of vertices prove that if each vertex of D has an odd out-degree then D has an odd number of vertices with odd in-degree.*

Solution. Let v_1, v_2, ... v_n be the n vertices of D, where n is odd. Also let m be the number of edges in D.

They by handshaking dilemma

$$d^+(v_1) + d^+(v_2) + ... + d^+(v_n) = m \quad ...(1)$$

$$d^-(v_1) + d^-(v_2) + ... + d^-(v_n) = m \quad ...(2)$$

If each vertex v_i has odd out-degree, then the left hand side of (1) is a sum of n odd numbers. Since n is odd, this sum must also be odd. Thus m is odd.

Let k be the number of vertics with odd in-degree. Then $n - k$ number of vertices have even in-degree. Without loss of generality, let us take v_1, v_2, ..., v_k to be the vertices with odd in-degree and v_{k+1}, v_{k+2}, ... v_n to be the vertices with even in-degree.

Then, (2) may be rewritten as

$$\sum_{i=1}^{k} d^-(v_i) + \sum_{i=k+1}^{n} d^-(v_i) = m \quad ...(3)$$

Now the second sum on the left hand side of this expression is even. Also, m is odd. Therefore, the first sum must be odd. That is, $d^-(v_1) + d^-(v_2) + ... + d^-(v_k) = \text{odd}$ ...(4)

But, each of $d^-(v_1)$, $d^-(v_2)$, ... $d^-(v_k)$ is odd.

Therefore, the number of terms in the left hand side of (4) must be odd, that is ; k is odd.

Theorem 5.44. *A digraph G is an Eulerian digraph if and only if G is connected and is balanced that is ; $d^-(v) = d^+(v)$ for every vertex v in G.*

Theorem 5.45. *An arborescence is a tree in which every vertex other than the root has an in-degree of exactly one.*

Proof. An arborescence with n vertices can have at most $n - 1$ edges because of condition (1).

Therefore, the sum of in-degree of all vertices in G

$$d^-(v_1) + d^-(v_2) + \ldots\ldots + d^-(v_n) \leq n - 1.$$

Of the n terms on the left-hand side of this equation, only one is zero because of condition (2), others must all be positive integers.

Therefore, they must all be 1's. Now since there are exactly $n - 1$ vertices of in-degree one and one vertex of in-degree zero, digraph G has exactly $n - 1$ edges. Since G is also circuitless, it must be connected, and hence a tree.

Example : In Fig. 5.82 below, W = (*b d c e f g h a*) is an Eulerian, starting and ending at vertex 2. The subdigraph {*b, d, f*} is a spanning arborescence rooted at vertex 2.

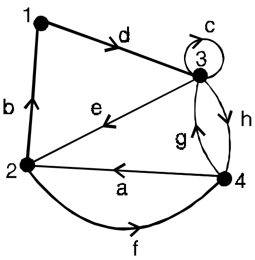


Fig. 5.82. Euler digraph.

Theorem 5.46. *In a connected, balanced digraph G of n vertices and m edges, let W = (e_1, e_2, ..., e_m) be an Euler line which starts and ends at a vertex v (that is, v is the initial vertex of e_1 and the terminal vertex of e_m). Among the m edges in W there are n – 1 edges that 'enter' each of n – 1 vertices, other than v, for the first time. The subdigraph g of these n – 1 directed edges together with the n vertices is a spanning arborescence of G, rooted at vertex v.*

Proof. In the subgraph g, vertex v is of in-degree zero, and every other vertex is of in-degree one ; for g includes exctly one edge going to each of the $n - 1$ vertices, and no edge going to v.

Moreover, the way g is defined in W, g is connected and contains $n - 1$ directed edges.

Therefore, g is a spanning arborescence in G and is rooted at v.

Theorem. 5.47. *In an arborescence there is a directed path from the root R to every other vertex. Conversely, a circuitless digraph G is an arborescence if there is a vertex v in G such that every other vertex is accessible from v, and v is not accessible from any other vertex.*

Proof. In an arborescence consider a directed path P starting from the root R and continuing as far as possible. P can end only at a pendant vertex, otherwise we get a vertex whose in-degree is two or more, a contradiction.

Since an arborescence is connected, every vertex lies on some directed path from the root R to each of the pendant vertices.

Conversely, since every vertex in G is accessible from v, and G has no circuit, G is a tree. Moreover, since v is not accessible from any other vertex, $d^-(v) = 0$.

Every other vertex is accessible from v and therefore the in-degree of each of these vertices must be at least one.

The in-degree cannot be greater than one because there are only $n - 1$ edges in G (n being the number of vertices in G.)

Theorem 5.48. *Let G be an Euler digraph and T be a spanning in-tree in G, rooted at a vertex R. Let e_1 be an edge in G incident out of the vertex R. Then a directed walk W = (e_1, e_2, ..., e_m) is a directed Euler line, if it is constructed as follows :*

(*i*) *No edge is included in W more than once.*

(*ii*) *In exiting a vertex the one edge belonging to T is not used until all other outgoing edges have been traversed.*

(*iii*) *The walk is terminated only when a vertex is reached from which there is no edge left on which to exit.*

Proof. The walk W must terminate at R, because all vertices must have been entered as often as they have been left(because G is balanced).

Now suppose there is an edge a in G that has not been included in W.

Let v be the terminal vertex of a. Since G is balanced v must also be the initial vertex of some edge b not included in W. Edge b going out of vertex v must be in T according to rule (*i*). Thus omitted edge leads to another omitted edge c in T, and so on.

Ultimately, we arrive at R, and find an outgoing edge there not included in W. This contradicts rule (*iii*).

Theorem 5.49. *If A(G) is the incidence matrix of a connected digraph of n vertices, the rank of A(G) = $n - 1$.*

Theorem 5.50. *The $(i, j)^{th}$ entry in X^r equals the number of different directed edge sequences of r edges from the i^{th} vertex to the j^{th}.*

Proof. (By induction)

The theorem is trivially true for $r = 1$.

As the inductive hypothesis, assume that the theorem holds for X^{r-1}. The $(i, j)^{\text{th}}$ entry in

$$X^r\ (= X^{r-1}.\ X) = \sum_{k=1}^{n} [(i, k)^{\text{th}} \text{ entry in } X^{r-1}]\ .\ x_{kj}$$

$$= \sum_{k=1}^{n} (\text{number of all directed edge sequences of length } r - 1 \text{ from vertex } i \text{ to } k).x_{kj}, \quad \text{...(1)}$$

according to the induction hypothesis. In (1), $x_{kj} = 1$ or 0 depending on whether or not there is a directed edge from k to j. Thus a term in the sum (1) is non zero if and only if there is a directed edge sequence of length r from i to j, whose last edge is from k to j.

If the term is non zero, its value equals the number of such edge sequences from i to j *via* k. This holds for every k, $1 \le k \le n$. Therefore (1) is equal to the number of all possible directed edge sequence from i to j.

Theorem 5.51. *Let B and A be respectively, the circuit matrix and incidence matrix of a self-loop-free digraph such that the columns in B and A are arranged using the same order of edges. Then*

$$A.B^T = B.A^T = 0.$$

Where superscript T denotes the transposed Matrix.

Proof. Consider the m^{th} row in B and the k^{th} row in A. If the circuit m does not include any edge incident on vertex k, the product of the two rows is clearly zero. If, on the other hand, vertex k is in circuit m, there are exactly two edges (say x and y) incident on k that are also in circuit m.

This situation can occur in only four different ways, as shown in Fig. 5.83 below.

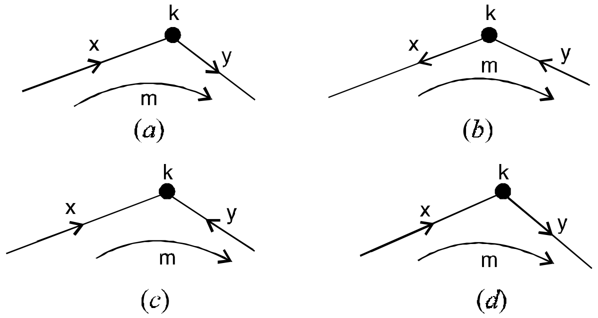


Fig. 5.83. Vertex k in circuit m.

The possible entries in row k of A and row m of B in column positions x and y are tabulated for each of these four cases.

Case	*Row k*		*Row m*		*Dot product*
	column x	*column y*	*column x*	*column y*	*Row k . Row m*
(*i*)	– 1	1	1	1	0
(*ii*)	1	– 1	– 1	– 1	0
(*iii*)	– 1	– 1	1	– 1	0
(*iv*)	1	1	– 1	1	0

In each case, the dot product is zero. Therefore, the theorem.

Theorem 5.52. *The i, j entry $a_{ij}^{(n)}$ of A^n is the number of walks of length n from v_i to v_j.*

Corollary (1) The entries of the reachability and distance matrices can be obtained from the powers of A as follows :

(*i*) for all i, $r_{ii} = 1$ and $d_{ii} = 0$

(*ii*) $r_{ij} = 1$ if and only if for some n, $a_{ij}^{(n)} > 0$

(*iii*) $d(v_i, v_j)$ is the least n(if any) such that $a_{ij}^{(n)} > 0$, and is ∞ otherwise.

Corollary (2) Let v_i be a point of a digraph D. The strong component of D containing v_i is determined by the entries of 1 in the ith row (or column) of the matrix $R \times R^T$.

Theorem 5.53. *The value of the cofactor of any entry in the j^{th} column of M_{id} is the number of spanning out-trees with v_j as source.*

Corollary. In an Eulerian digraph, the number of eulerian trails is $C. \prod_{i=1}^{P} (d_i - 1)!$

Where $d_i = id(v_i)$ and c is the common value of all the cofactors of M_{od}.

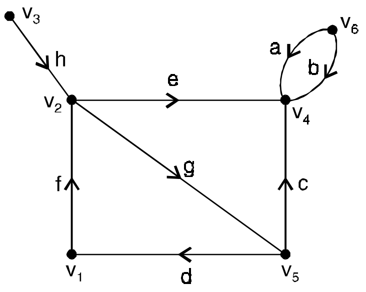
Theorem 5.54. *For any labeled digraph D, the value of the cofactor of any entry in the i^{th} row of M_{od} is the number of spanning in-trees with v_i as sink.*

Theorem 5.55. *The determinant of every square submatrix of A, the incident matrix of a digraph is 1, – 1 or 0.*

Proof. The theorem can be proved directly by expanding the determinant of a square submatrix of A.

Consider a k by k submatrix M of A.

If M has any column or row consisting of all zeros det M is clearly zero. Also det M = 0 if every column of M contains the two non zero entries, a^1 and a^{-1}.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(*a*) Digraph **(*b*) Incidence matrix**

Fig. 5.84. Digraph and its incidence matrix.

Now if det M ≠ 0 (*i.e.,* M is non singular), then the sum of entries in each column of M cannot be zero.

Therefore, M must have a column in which there is a single non zero element that either + 1 or – 1.

Let this single element be in the $(i, j)^{th}$ position in M.

Thus det M = ± 1. det M_{ij}, where M_{ij} is the sub matrix of M with its i^{th} row and j^{th} column deleted.

The $(k – 1)$ by $(k – 1)$ submatrix M_{ij} is also non singular. Therefore it too must have atleast one column with a single non zero entry, say, in the $(p, q)^{th}$ position.

Expanding det M_{ij} about this element in the $(p, q)^{th}$ position.

det M_{ij} = ± [determinant of a non sigular $(k – 2)$ by $(k – 2)$ submatrix of M]

Repeated application of this procedure yields det M = ± 1.

Hence the theorem.

Theorem 5.56. *Let A_f be the reduced incidence matrix of a connected digraph. Then the number of spanning trees in the graph equals the value of det $(A_f \,.\, T_f^T)$.*

Proof. According to the Binet-Cauchy theorem

det $(A_f.A_f^T)$ = sum of the products of all corresponding majors of A_f and A_f^T.

Every major of A_f or A_f^T is zero unless it corresponds to a spanning tree, in which case its value is 1 or – 1. Since both majors of A_f and A_f^T have the same value + 1 or – 1, the product is + 1 for each spanning tree.

Theorem 5.57. *Let k(G) be the Kirchhoff matrix of a simple digraph G. Then the value of the (p, q) cofactor of k(G) is equal to the number of arborescences in G rooted at the vertex* v_q.

Proof. The determinant of a square matrix is a linear function of its columns. Specifically, if p is a square matrix consisting of n column vectors, each of dimension n ; that is ;

$$P = [P_1, P_2, \ldots (P_i + P_i'), \ldots P_n]$$

then det P $$\det = [P_1, P_2, \ldots, P_i, \ldots, P_n] + \det [P_1, P_2, \ldots, P_i' \ldots P_n] \qquad \ldots(1)$$

In graph G suppose that vertex v_j has in-degree of d_j. The j^{th} column of k(G) can be regarded as the sum of d_j different columns, each corresponding to a graph in which v_j has in-degree one. And then (1) can be repeatedly applied. After this, splitting of columns can be carried out for each j, $j \neq q$, and det k_{qq}(G) can be expressed as a sum of determinants of subgraphs ; that is ; det $k_{qq}(G) = \sum_g \det k_{qq}(g)$,

$$\ldots(2)$$

Where g is a subgraph of G, with the following properties :

(*i*) Every vertex in g has an in-degree of exactly one, except v_q.

(*ii*) g has $n - 1$ vertices, and hence $n - 1$ edges

det $k_{qq}(g) = 1$, if and only if g is an arborescence rooted at q,

$= 0$, otherwise.

Thus the summation in (2) carried over all g's equals the number of arborescences rooted at v_q.

Theorem 5.58. *In an Euler digraph the number of Euler lines is*

$$\sigma . \prod_{i=1}^{n} [d^-(v_i) - 1]!$$

Theorem 5.59. *A simple digraph G of n vertices and n – 1 directed edges in an arborescence rooted at* v_1 *if and only if the (1, 1) cofactor of k(G) is equal to 1.*

Proof. Let G be an arborescence with n vertices and rooted at vertex v_1. Relabel the vertices as $v_1, v_2, \ldots v_n$ such that vertices along every directed path from the root v_1 have increasing indices.

Permute the rows and columns of k(G) to conform with this relabeling.

Since the in-degree of v_1 equals zero, the first column contains only zeros. Other entries in k(G) are

$k_{ij} = 0, i > j,$

$k_{ij} = -x_{ij}, i < j,$

$k_{ij} = 1, i > 1.$

Then the k matrix of an arborescence rooted at v_1 is of the form

$$k(G) = \begin{bmatrix} 0 & -x_{12} & -x_{13} & -x_{14} & \ldots\ldots & -x_{1n} \\ 0 & 1 & -x_{23} & -x_{24} & \ldots\ldots & -x_{2n} \\ 0 & 0 & 1 & -x_{34} & \ldots\ldots & -x_{3n} \\ 0 & 0 & 0 & 1 & \ldots\ldots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots\ldots & 1 \end{bmatrix}$$

Clearly, the cofactor of the (1, 1) entry is 1.

That is, det $k_{11} = 1$.

Conversely, let G be a simple digraph of n vertices and $n - 1$ edges and let the (1, 1) cofactor of its k matrix be equal to 1 : that is ; det $k_{11} = 1$.

Since det $k_{11} \neq 0$, every column in k_{11} has at least one non zero entry.

Therefore $d^-(v_i) \geq 1$, for $i = 2, 3, \ldots, n$.

There are only $n - 1$ edges to go around.

Therefore, $d^-(v_i) = 1$, for $i = 2, 3, \ldots, n$ and $d^-(v_1) = 0$.

Now since no vertex in G has an in-degree of more than one, if G can have any circuit at all, it has to be a directed circuit.

Suppose that such a directed circuit exists ; which passes through vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$.

Then the sum of the columns $i_1, i_2, \ldots\, i_r$ in k_{11} is zero.

Thus these r columns in k_{11} are linearly dependent.

Hence det $k_{11} = 0$, a contradiction.

Therefore, G has no circuits.

If G has $n - 1$ edges and no circuits, it must be a tree. Since in this tree $d^-(v_1) = 0$ and $d^-(v_i) = 1$ for $i = 2, 3, \ldots, n$.

G must be an arborescence rooted at vertex v_1.

The above arguments are valid for an arborescence at any vertex v_q. Any reordering of the vertices in G corresponds to identical permutations of rows and columns in k(G). Such permutations do not alter the value or sign of the determinant.

Problem 5.49. *Verify that the following two digraphs are isomorphic.*

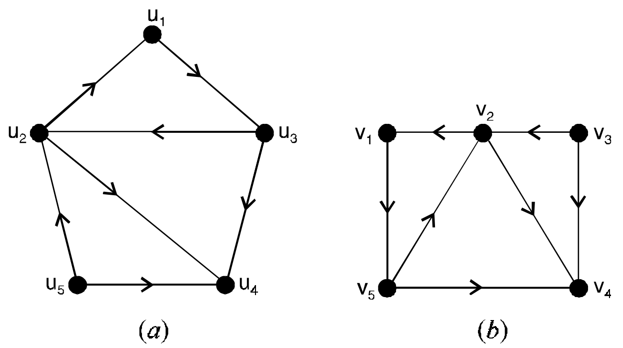


Fig. 5.85.

Solution. Let us consider the following one-to-one correspondence between the directed edges in the two digraphs :

$(u_2, u_1) \leftrightarrow (v_2, v_1)$,	$(u_5, u_2) \leftrightarrow (v_3, v_2)$
$(u_2, u_4) \leftrightarrow (v_2, v_4)$,	$(u_3, u_2) \leftrightarrow (v_5, v_2)$
$(u_3, u_4) \leftrightarrow (v_5, v_4)$,	$(u_5, u_4) \leftrightarrow (v_3, v_4)$
$(u_1, u_3) \leftrightarrow (v_1, v_5)$.	

These yield the following one-to-one correspondence between the vertices in the two digraphs :

$u_1 \leftrightarrow v_1, u_2 \leftrightarrow v_2, u_3 \leftrightarrow v_5, u_4 \leftrightarrow v_4, u_5 \leftrightarrow v_3.$

The above mentioned one-to-one correspondences between the vertices and the directed edges establish the isomorphism between the given digraphs.

Problem 5.50. *Prove that a complete symmetric digraph of n vertices contains n(n – 1) edges and a complete asymmetric digraph of n vertices contains* $\frac{n(n-1)}{2}$ *edges.*

Solution. In a complete asymmetric digraph, there is exactly one edge between every pair of vertices.

Therefore, the number of edges in such a digraph is precisely equal to the number of pairs of vertices. The number of pairs of vertices that can be chosen from n vertices is ${}^nC_2 = \frac{1}{2} n(n-1)$.

Thus, a complete asymmetric digraph with n vertices has exactly $\frac{1}{2} n(n-1)$ edges.

In a complete symmetric digraph there exist two edges with opposite directions between every pair of vertices.

Therefore, the number of edges in such a digraph with n vertices is $2 \times \frac{1}{2} n(n-1) = n(n-1)$.

Problem 5.51. *Let D be a connected simple digraph with n vertices and m edges. Prove that*

$$n - 1 \le m \le n(n-1).$$

Solution. Since D is connected, its underlying graph G is connected. Therefore, $m \ge n - 1$.

In a simple digraph, there exists at most two edges in opposite directions between every pair of vertices.

Therefore, the number of edges in such a digraph cannot exceed $2 \times {}^nC_2 = n(n-1)$.

i.e., $m \le n(n-1)$. Thus $n - 1 \le m \le n(n-1)$.

Problem 5.52. *Find the sequence of vertices and edges of the longest walk in the digraph shown in Fig. 5.86 below :*

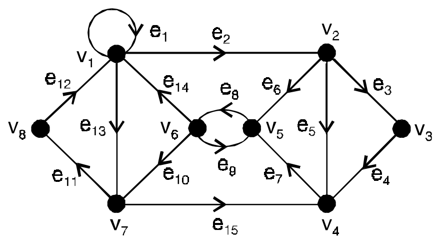


Fig. 5.86.

Solution. We check that in the given digraph, for each vertex, the in-degree is equal to the out-degree.

Therefore, the digraph is an Euler digraph. The longest walk in the digraph is a directed Euler line, a directed walk which includes all the edges of the digraph.

The digraph reveals that the directed Euler line is shown below :

$$v_1\ e_1\ v_1\ e_{13}\ v_7\ e_{11}\ v_8\ e_{12}\ v_1\ e_2\ v_2\ e_3\ v_3\ e_4\ v_4\ e_5\ v_2\ e_6\ v_5\ e_8\ v_6\ e_{10}\ v_7\ e_{15}\ v_4\ e_7\ v_5\ e_9\ v_6\ e_{14}\ v_1.$$

This is the required sequence.

Problem 5.53. *Show that the digraph shown in Fig. 5.87 below, is an Euler digraph. Indicate a directed Euler line in it.*

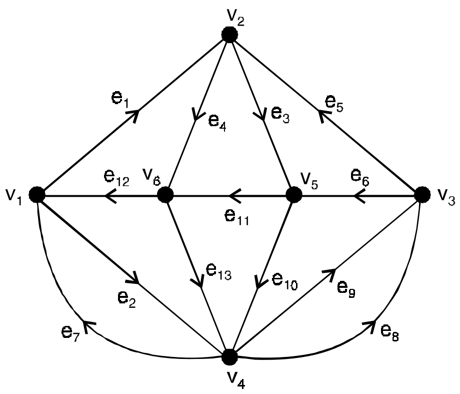


Fig. 5.87.

Solution. By examining the given digraph, we find that

$$d^-(v_1) = 2 = d^+(v_1),\ d^-(v_2) = 2 = d^+(v_2)$$
$$d^-(v_3) = 2 = d^+(v_3),\ d^-(v_4) = 3 = d^+(v_3)$$
$$d^-(v_5) = 2 = d^+(v_5),\ d^-(v_6) = 2 = d^+(v_6).$$

Thus, for every vertex the in-degree is equal to the out-degree.

Therefore the digraph is an Euler digraph.

By starting at v_1, we can obtain the following closed directed walk that includes all the thirteen edges :

$$v_1\ e_1\ v_2\ e_4\ v_6\ e_{12}\ v_1\ e_2\ v_4\ e_8\ v_3\ e_6\ v_5\ e_{11}\ v_6\ e_{13}\ v_4\ e_9\ v_3\ e_5\ v_2\ e_3\ v_5\ e_3\ v_5\ e_{10}\ v_4\ e_7\ v_1$$

This is a directed Euler line in the given digraph.

Problem 5.54. *Prove that a connected digraph D that does not contain a closed directed walk must have a source and a sink.*

Solution. Consider a directed walk q in D, which contains a maximum number of vertices.

Let u be the initial vertex of q and v be the terminal vertex. Suppose v is not a sink. Then there must be an edge that begins at v. Since D has no closed walk, this edge cannot end at u.

Hence it must end at some vertex v'.

Consequently, there is created a directed walk q' that contains all vertices of q and v'.

This contradicts the maximality of q. Hence v has to be a sink. Similarly, u has to be a source.

Thus, D contains at least one sink and at least one source.

Problem 5.55. *Consider the digraph D shown in Fig. 5.88 below. In this digraph find*

(*i*) *a directed walk of length 8. Is this walk a directed path ?*

(*ii*) *a directed trial of length 10*

(*iii*) *a directed trial of longest length*

(*iv*) *a directed path of longest length.*

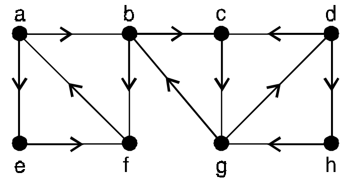


Fig. 5.88.

Solution. (*i*) *a e f a b c g d h* is a directed walk of length 8. This is not a directed path, because the vertex a appear twice.

(*ii*) *a e f a b c g d h g b* is a directed trial (walk) of length 10.

(*iii*) *a e f a b c g d h g b f* is a directed trial of length 11. This is the longest possible directed trial (walk).

(*iv*) *a b c g d h* is a directed path of length 5. This is the longest possible directed path.

Problem 5.56. *Show that the digraph given below is strongly connected.*

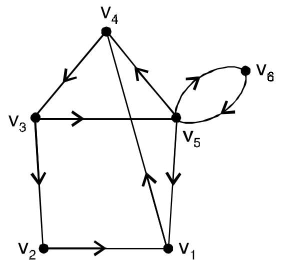


Fig. 5.89.

Solution. A digraph D is strongly connected if every vertex of D is reachable from every other vertex of D.

Let us verify wheather this condition holds for the given digraph.

The following table indicates the directed paths from each of the vertices v_1, v_2, v_3, v_4, v_5, v_6 to every other vertex. The existence of these directed paths proves that the given digraph is strongly connected.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	—	$v_1v_4v_3v_1$	$v_1v_4v_3$	v_1v_4	$v_1v_4v_3v_5$	$v_1v_4v_3v_5v_6$
v_2	v_2v_1	—	$v_2v_1v_4v_3$	$v_2v_1v_4$	$v_2v_1v_4v_3v_5$	$v_2v_1v_4v_3v_5$
v_3	$v_3v_2v_1$	v_3v_2	—	$v_3v_5v_4$	v_3v_5	$v_3v_5v_6$
v_4	$v_4v_3v_2v_1$	$v_4v_3v_2$	v_4v_3	—	$v_4v_3v_5$	$v_4v_3v_5v_6$
v_5	v_5v_1	$v_5v_4v_3v_2$	$v_5v_4v_3$	v_5v_4	—	v_5v_6
v_6	$v_6v_5v_1$	$v_6v_5v_4v_3v_2$	$v_6v_5v_4v_3$	$v_6v_5v_4$	v_6v_5	—

5.24. NULLITY OF A MATRIX

If Q is an n by n matrix then QX = 0 has a non trial solution $X \neq 0$ if and only if Q is singular, that is ; det Q = 0. The set of all vectors X that satisfy QX = 0 forms a vector space called the null space of matrix Q. The rank of the null space is called the nullity of Q.

Rank of Q + nullity of Q = n

When Q is not square but a k by n matrix, $k < n$.

Theorem 5.60. ***(Binet-Cauchy Theorem)***

If Q and R are k by m and m × k matrices respectively with k < m then the determinant of the product det (QR) = the sum of the products of corresponding major determinants of Q and R.

Proof. To evaluate det (QR), let us devise and multiply two $(m + k)$ by $(m + k)$ partitioned matrices

$$\begin{bmatrix} I_k & Q \\ O & I_m \end{bmatrix} \cdot \begin{bmatrix} Q & O \\ -I_m & R \end{bmatrix} = \begin{bmatrix} O & QR \\ -I_m & R \end{bmatrix}$$

where I_m and I_k are identity matrices of order m and k respectively.

Therefore $\det \begin{bmatrix} Q & O \\ -I_m & R \end{bmatrix} = \det \begin{bmatrix} O & QR \\ -I_m & R \end{bmatrix}$

that is, $$\det(QR) = \det \begin{bmatrix} Q & O \\ -I_m & R \end{bmatrix} \quad ...(1)$$

Let us now apply Cauchy's expansion method to the right-hand side of equation (1), and observe that the only non-zero minors of any order in matrix $-I_m$ are its principal minors of that order. We thus find that the Cauchy expansion consists of these minors of order $m - k$ multiplied by their cofactors of order k in Q and R together.

Theorem 5.61. ***(Sylvester's Law)***

If Q is a k by n matrix and R is an n by P matrix then the nullity of the product cannot exceed the sum of the nullities of the factors, that is ;

$$\textit{nullity of } QR \leq \textit{nullity of } Q + \textit{nullity of } R \quad ...(1)$$

Proof. Since every vector x that satisfies RX = 0 must certainly satisfy QRx = 0

We have nullity of QR ≥ nullity of R ≥ 0 ...(2)

Let s be the nullity of matrix R. Then there exists a set of s linearly independent vectors $\{x_1, x_2, ... x_s\}$ forming a basis of the null space of R.

Thus $Rx_i = 0$ for $i = 1, 2, ..., s$ ...(3)

Now let $s + t$ be the nullity of matrix QR. Then there must exist a set of t linearly independent vectors

$$\{x_{s+1}, x_{s+2}, ... x_{s+t}\} \text{ such that the set } \{x_1, x_2, ... x_s, x_{s+1}, x_{s+2}, x_{s+t}\}$$

forms a basis for the null space of matrix QR.

Thus $QRx_i = 0$, for $i = 1, 2, ... s, s + 1, s + 2, ... s + t$ ...(4)

In otherwords, of the $s + t$ vectors x_i forming a basis of the null space of QR, the first s vectors are sent to zero by matrix R and the remaining non-zero Rx_i's ($i = s + 1, s + 2, ... s + t$) are sent to zero by matrix Q.

Vectors $Rx_{s+1}, Rx_{s+2}, ... Rx_{s+t}$ are linearly independent ; for if

$$0 = a_1 Rx_{s+1} + a_2 Rx_{s+2} + ... + a_t Rx_{s+t}$$
$$= R(a_1 x_{s+1} + a_2 x_{s+2} + ... + a_t x_{s+t})$$

then vector $(a_1 x_{s+1} + a_2 x_{s+2} + ... + a_t x_{s+t})$ must be the null space of R, which is possible only if

$$a_1 = a_2 = ... = a_t = 0.$$

Thus we have found that there are at least t linearly independent vectors which are sent to zero by matrix Q and therefore nullity of Q ≥ t.

But since $t = (s + t) - s$

= nullity of QR – nullity of R, equation (1) follows.

Substituting equation, rank of Q + nullity of Q = n into equation (1), we find that

rank of QR ≥ rank of Q + rank of R – n ...(5)

Furthermore, in equation (5) if the matrix product QR is zero, then

rank of Q + rank of R ≤ n.

Theorem 5.62. *If G is a (p, q) graph whose points have degrees d_i, then L(G) has q points and q_L lines where $q_L = -q + \frac{1}{2}\sum d_i^2$.*

Proof. By the definition of line graph, L(G) has q points. The d_i lines incident with a point v_i contribute $\binom{d_i}{2}$ to q_L, so

$$q_L = \sum \binom{d_i}{2} = \frac{1}{2}\sum d_i(d_i - 1)$$
$$= \frac{1}{2}\sum d_i^2 - \frac{1}{2}\sum d_i$$
$$= \frac{1}{2}\sum d_i^2 - q .$$

Theorem 5.63. *Unless m = n = 4, a graph G is the line graph of* $k_{m,\,n}$ *if and only if*

(*i*) *G has mn points*

(*ii*) *G is regular of degree m + n – 2.*

(*iii*) *Every two non adjacent points are mutually adjacent to exactly two points*

(*iv*) *Among the adjacent pairs of points, exactly* $n\binom{m}{2}$ *pairs are mutually adjacent to exactly m – 2 points, and other* $m\binom{n}{2}$ *pairs to n – 2 points.*

Theorem 5.64. *Unless P = 8, a graph G is the line graph of* k_P *if and only if*

(*i*) *G has* $\binom{P}{2}$ *points,*

(*ii*) *G is regular of degree 2(P – 2),*

(*iii*) *Every two non adjacent points are mutually adjacent to exactly four points,*

(*iv*) *Every two adjacent points are mutually adjacent to exactly P – 2 points.*

Theorem 5.65. *The total graph T(G) is isomorphic to the square of the sub-division graph S(G).*

Corollary (1). If *v* is a point of G then the degree of point *v* in T(G) is 2 deg *v*. If *x* = *uv* is a line of G then the degree of point *x* in T(G) is deg *u* + deg *v*.

Corollary (2). If G is a (*p*, *q*) graph whose points have degrees d_i, then the total graph T(G) has $P_T = p + q$ points and $q_T = 2q + \frac{1}{2}\sum d_i^2$ lines.

Theorem 5.66. *If G is a non trivial connected graph with P points which is not a path, then* $L^n(G)$ *is hamiltonian for all* $n \geq P - 3$.

Theorem 5.67. *For n > 1, we always have* $r_1(2, n) = 3$. *For all other m and n,* $r_1(m, n) = (m - 1)(n - 1) + 2$.

Theorem 5.68. *A graph G is eulerian if and only if* $L_3(G)$ *is hamiltonian.*

Theorem 5.69. *Let G and G′ be connected graphs with isomorphic line graphs. Then G and G′ are isomorphic unless one is* k_3 *and the other is* $k_{1,\,3}$.

Proof. First note that among the connected graphs with up to four points, the only two different ones with isomorphic line graphs are k_3 and $k_{1,\,3}$.

Note further that if ϕ is an isomorphism of G and G′ then there is a derived isomorphism ϕ_1 of L(G) onto L(G′).

The theorem will be demonstrated when the following stronger result is proved.

If G and G′ have more than four points then any isomorphism ϕ_1 of L(G) and L(G′) is derived from exactly one isomorphism of G to G′.

We first show that ϕ_1 is derived from at most one isomorphism.

Assume there are two such, ϕ and ψ. We will prove that for any point *v* of G, $\phi(v) = \psi(v)$.

There must exist two lines $x = uv$ and $y = uw$ or vw.

If $y = vw$ then the points $\phi(v)$ and $\psi(v)$ are on both lines $\phi_1(x)$ and $\phi_1(y)$, so that since only one point is on both these lines, $\phi(v) = \psi(v)$.

By the same argument, when $y = uw$, $\phi(u) = \psi(u)$ so that since the line $\phi_1(x)$ contains the two points $\phi(v)$ and $\phi(u) = \psi(u)$, we again have $\phi(v) = \psi(v)$.

Therefore ϕ_1 is derived from at most one isomorphism of G to G′.

We now show the existence of an isomorphism ϕ from which ϕ_1 is derived.

The first step is to show that the lines $x_1 = uv_1$, $x_2 = uv_2$, and $x_3 = uv_3$ of a $k_{1,3}$ subgraph of G must go to the lines of a $k_{1,3}$ subgraph of G′ under ϕ.

Let y be another line adjacent with at least one of the x_i, which is adjacent with only one or all three. Such a line y must exist for any graph with $P \geq 5$ and the theorem is trivial for $P < 5$.

If the three lines $\phi_1(x_i)$ form a triangle instead of $k_{1,3}$ the $\phi_1(y)$ must be adjacent with precisely two of the three.

Therefore, every $k_{1,3}$ must go to a $k_{1,3}$.

Let $s(v)$ denote the set of lines at v. We now show that to each v in G, there is exactly one v' in G′ such that $S(v)$ goes to $S(v')$ under ϕ_1.

If deg $v \geq 2$, let y_1 and y_2 be lines at v and let v' be the common point of $\phi_1(y)$ and $\phi_1(y_2)$.

Then for each line x at v, v' is incident with $\phi_1(x)$ and for each line x' and v', v is incident with $\phi_1^{-1}(x')$.

If deg $v = 1$, let $x = uv$ be the line at v.

Then deg $u \geq 2$ and hence $s(u)$ goes to $s(u')$ and $\phi_1(x) = u'v'$.

Since for every line x' at v', the lines $\phi_1^{-1}(x')$ and x must have a common point, u is on $\phi_1^{-1}(x')$ and u' is on x'.

That is, $x' = \phi_1(x)$ and deg $v' = 1$. The mapping ϕ is therefore one-to-one from v to v' since $s(u) = s(v)$ only when $u = v$.

Now given v' in V′, there is an incident line x'.

Denote $\phi_1^{-1}(x')$ by uv. The either $\phi(u) = v'$ or $\phi(v)= v'$ so ϕ is onto.

Finally, we note that for each line $x = uv$ in G, $\phi_1(x) = \phi(u)\phi(v)$ and for each line $x' = u'v'$ in G′, $\phi_1^{-1}(x') = \phi^{-1}(u')\,\phi^{-1}(v')$, so that ϕ is an isomorphism from which ϕ_1 is derived.

This complete the proof.

Theorem 5.70. *A connected graph is isomorphic to its line graph if and only if it is a cycle.*

Theorem 5.71. *A sufficient condition for $L_2(G)$ to be hamiltonian is that G be hamiltonian and a necessary condition is that L(G) be hamiltonian.*

Theorem 5.72. *If G is eulerian then L(G) is both eulerian and hamiltonian. If G is hamiltonian then L(G) is hamiltonian.*

Theorem 5.73. *A graph is the line graph of a tree if and only if it is a connected block graph in which each cut point is on exactly two blocks.*

Proof. Suppose G = L(T), T some tree.

Then G is also B(T) since the lines and blocks of a tree coincide.

Each cut point x of G corresponds to a bridge uv to T, and is on exactly those two blocks of G which correspond to the stars at u and v. This proves the necessity of the condition.

Sufficient part

Let G be a block graph is which each cutpoint is on exactly two blocks.

Since each block of a block graph is complete, there exists a graph H such that L(H) = G.

If G = K_3, we can take H = $K_{1, 3}$.

If G is any other block graph, then we show that H must be a tree.

Assume that H is not a tree so that it contains a cycle. If H is itself a cycle then L(H) = H, but the only cycle which is a block graph is K_3, a case not under consideration.

Hence H must properly contain a cycle, there by implying that H has a cycle Z and a line *x* adjacent to two lines of Z, but not adjacent to some line *y* of Z. The points *x* and *y* of L(H) lie on a cycle of L(H) and they are not adjacent.

This contradicts that L(H) is a block graph.

Hence H is a tree and the theorem is proved.

Theorem 5.74. *The following statements are equivalent :*

(*i*) *G is a line graph*

(*ii*) *The lines of G can be partitioned into complete subgraphs in such a way that no point lies in more than two of the subgraphs.*

Proof. (*i*) implies (*ii*)

Let G be the line graph of H. Withoutloss of generality we assume that H has no isolated points. Then the lines in the star at each point of H induce a complete subgraph of G, and every line of G lies in exactly one such subgraph.

Since each line of H belongs to the stars of exactly to points of H, no points of G is in more than two of these complete subgraphs.

(*ii*) implies (*i*)

Given a decomposition of the lines of a graph G into complete subgraphs $S_1, S_2, ..., S_n$ satisfaying (*ii*), we indicate the construction of a graph H whose line graph is G.

The points of H correspond to the set S of subgraphs of the decomposition together with the set U of points of G belonging to only one of the subgraphs S_i.

Thus $S \cup U$ is the set of points of H and two of these points are adjacent whenever they have a non empty intersection ; that is ; H is the intersection graph Ω ($S \cup U$).

Theorem 5.75. *Every tournament has a spanning path.*

Proof. The proof is by induction on the number of points.

Every tournament with 2, 3, or 4 points has a spanning path, by inspection.

Assume the result is true for all tournament with *n* points, and consider a tournament T with *n* + 1 points.

Let v_0 be any point of T. Then T – v_0 is a tournament with *n* points, so it has a spanning path P, say $v_1v_2 .... v_n$.

Either arc v_0v_1 or arc v_1v_0 is in T. If v_0v_1 is in T, then $v_0v_1v_2 ..... v_n$ is a spanning path of T.

If v_1v_0 is in T, let v_i be the first point of P for which the arc v_0v_i is in T, if any.

Then $v_{i-1}v_0$ is in T, so that $v_1v_2 ..... v_{i-1}v_0v_i ..... v_n$ is a spanning path.

If no such point v_i exists, then $v_1v_2 \ldots\ldots v_nv_0$ is a spanning path. In any case, we have shown that T has a spanning path, completing the proof.

Theorem 5.76. *The distance from a point with maximum score to any other point is 1 or 2.*

Theorem 5.77. *The number of transitive triples in tournament with score sequence* $(S_1, S_2, \ldots, S_P)$ *is* $\sum \frac{S_i(S_i - 1)}{2}$.

Corollary. The maximum number of cycle triples among all tournaments with P points is

$$t(\mathrm{P}, 3) = \begin{cases} \dfrac{\mathrm{P}^3 - \mathrm{P}}{24} & \text{if P is odd,} \\ \dfrac{\mathrm{P}^3 - 4\mathrm{P}}{24} & \text{if P is even.} \end{cases}$$

Theorem 5.78. *Every strong tournament with P points has a cycle of length n, for* $n = 3, 4, \ldots$ *P.*

Proof. This proof is also by induction, but on the length of cycles.

If a tournament T is strong, then it must have a cycle triple.

Assume that T has a cycle $Z = v_1v_2 \ldots\ldots v_nv_1$ of length $n < \mathrm{P}$.

We will show that it has a cycle of length $n + 1$.

There are two cases : either there is a point u not in Z both adjacent to and adjacent from points of Z, or there is no such point.

Case (*i*) Assume there is a point u not in Z and points v and w in Z such that arcs uv and wu are in T. Without loss of generality, we assume that arc v_1u is in T.

Let v_i be the first point, going around Z from v_1, for which arc uv_i is in T. Then $v_{i-1}\,u$ is in T, and $v_1v_2 \ldots\ldots v_{i-1}\,uv_i \ldots\ldots v_nv_1$ is a cycle of length $n + 1$.

Case (*ii*) There is no such point u as in case (*i*). Hence, all points of T which are not in Z are partitioned into the two subsets U and W, where U is the set of all points adjacent to every point of Z and W is the set adjacent from every point of Z.

Clearly, these sets are disjoint, and neither set is empty since otherwise T would not be strong.

Furthermore, there are points u in U and w in W such that arc wu is in T.

Therefore $uv_1v_2 \ldots\ldots v_{n-1}\,wu$ is a cycle of length, $n + 1$ in T.

Hence, there is a cycle of length $n + 1$, completing the proof.

Corollary. A tournament is strong if and only if it has a spanning cycle.

5.25 TYPES OF ENUMERATION

Type 1. Counting the number of different graphs (or digraphs) of a particular kind.

For example, all connected, simple graphs with eight vertices and two circuits.

Type 2. Counting the number of subgraphs of a particular type in a given graph G, such as the number of edge-disjoint paths of length k between vertices a and b in G.

In problems of type 1 the word ‘different’ is of utmost importance and must be clearly understood. If the graphs are labeled, *i.e.,* each vertex is assigned a name distinct from all others, all graphs are counted on the otherhand, in the case of unlabeled graphs the word ‘different’ means non-isomorphic, and each set of isomorphic graphs is counted as one.

For example, let us consider the problem of constructing all simple graphs with n vertices and e edges. There are $\frac{n(n-1)}{2}$ unordered pairs of vertices. If we regard the vertices as distinguishable from one another *i.e.,* labeled graphs, there are $\begin{pmatrix} \frac{n(n-1)}{2} \\ e \end{pmatrix}$ ways of selecting e edges to form the graph.

Thus $\begin{pmatrix} \frac{n(n-1)}{2} \\ e \end{pmatrix}$ gives the number of simple labeled graphs with n vertices and e edges.

In the problems of type 2, usually involves a matrix representation of graph G and manipulations of this matrix. Such problems, although after encountered in practical applications, are not as varied and interesting as those in the first category.

5.26 LABELED GRAPHS

All of the labeled graphs with three points are shown in Figure 6.1 below. We see that the 4 different graphs with 3 points become 8 different labeled graphs. To obtain the number of labeled graphs with P points, we need only observe that each of the $\binom{P}{2}$ possible lines is either present or absent.

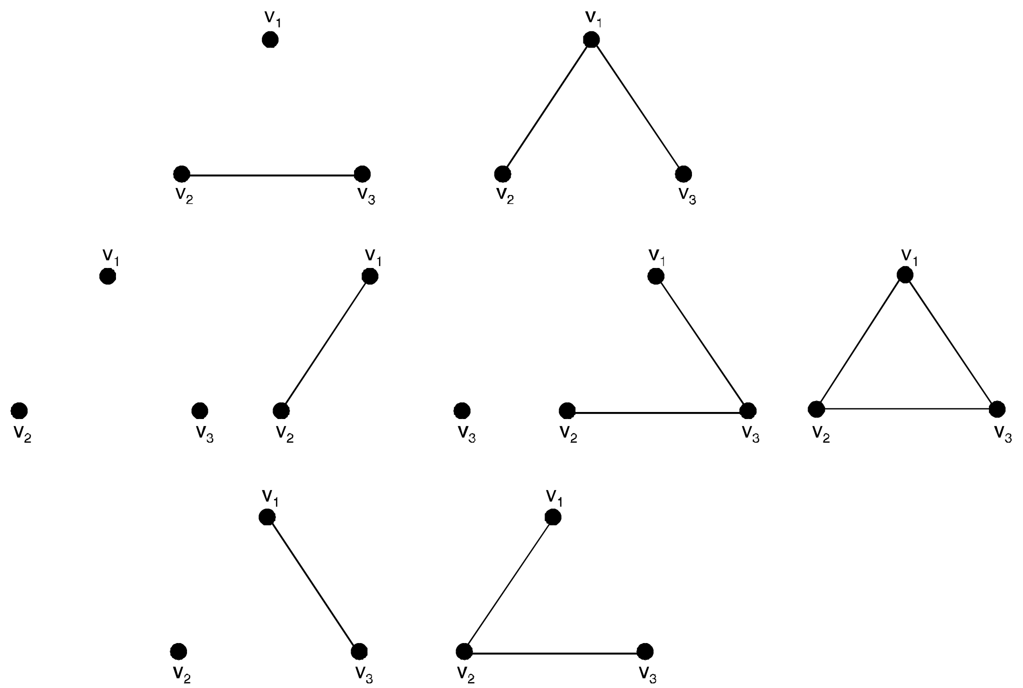


Fig. 5.90. The labeled graphs with 3 points.

5.26.1 Counting Labeled trees

Expression $\left(\begin{array}{c} \frac{n(n-1)}{2} \\ e \end{array}\right)$ can be used to obtain the number of simple labeled graphs of n vertices and $n - 1$ edges. Some of these are going to be trees and others will be unconnected graphs with circuits.

For example, In Figure 5.91 below are all the 16 labeled trees with 4 points. The labels on these trees are understood to be as in the first and last trees shown.

We note that among these 16 labeled trees, 12 are isomorphic to the path P_4 and 4 to $k_{1,\,3}$.

The order of $\Gamma(P_4)$ is 2 and that of $\Gamma(k_{1,\,3})$ is 6.

We observe that since P = 4 here, we have

$$12 = \frac{4\,!}{|\,\Gamma(P_4)\,|} \text{ and } 4 = \frac{4\,!}{|\,\Gamma(k_{1,\,3})\,|}$$

The expected generalization of these two observations holds not only for trees, but also for graphs, digraphs, and so forth.

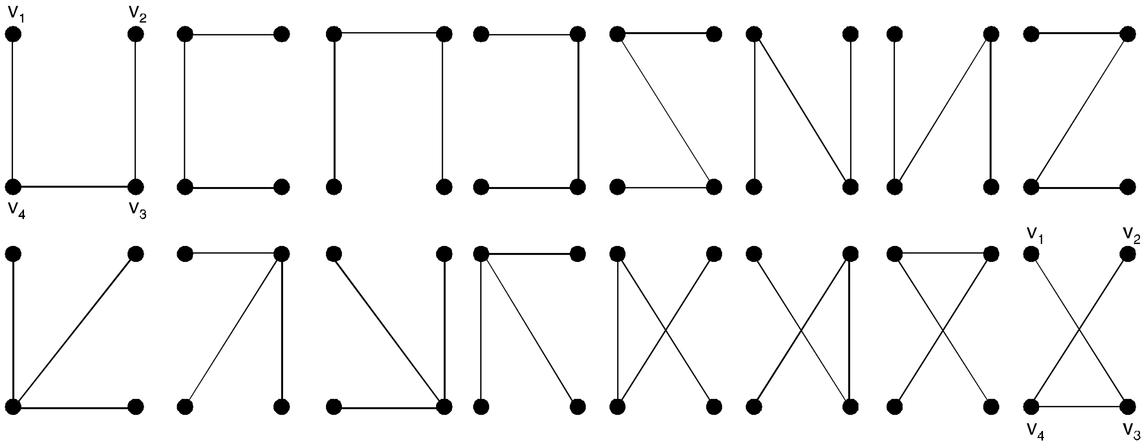


Fig. 5.91. The labeled trees with 4 points.

5.26.2 Rooted Labeled Trees

In a rooted graph one vertex is marked as the root. For each of the n^{n-2} labeled trees we have n rooted labeled trees, because any of the n vertices can be made a root. Therefore, the number of different rooted labeled trees with n vertices is n^{n-1}.

5.26.3 Enumeration of Graphs

To obtain the polynomial $g_P(x)$ which enumerates graphs with a given number P of points. Let g_{pq} be the number of (p, q) graphs and let $g_P(x) = \sum_q g_{pq} x^q$, all graphs with 4 points ; $g_4(x) = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6$.

5.26.4 Enumeration of Trees

To find the number of trees it is necessary to start by counting rooted trees. A rooted tree has one point, its root, distinguished from the others. Let T_P be the number of rooted trees with P points.

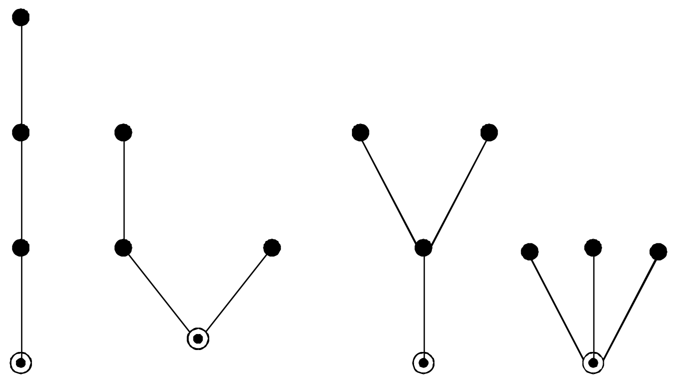


Fig. 5.92.

From figure 5.92 above, in which the root of each tree is visibly distinguished from the other points, we see $T_4 = 4$. The counting series for rooted trees is denoted by $T(x) = \sum_{P=1}^{\infty} T_P x^P$. We define t_P and $t(x)$ similarly for unrooted trees.

5.27 PARTITIONS

When a positive integer P is expressed as a sum of positive integers $P = \lambda_1 + \lambda_2 + \lambda_3 + \ldots\ldots \lambda_q$, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \lambda_q \geq 1$, the q-tuple is called a partition of integer P.

For example, (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), and (1 1 1 1 1) are the seven different partitions of the integer 5.

The integer's λ_i*'s* are called parts of the partitioned number P.

The number of partitions of a given integer P is often obtained with the help of generating function.

The coefficient of x^k in the polynomial

$$(1 + x)(1 + x^2)(1 + x^3) \ldots\ldots (1 + x^P)$$

gives the number of partitions, without repetition, of an integer $k \leq P$.

5.28 GENERATING FUNCTIONS

A generating function $f(x)$ is a power series

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots$$

is some dummy variable x. The coefficient a_k of x^k is the desired number, which depends on a collection of k objects being enumerated.

For example, in the generating function

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots\ldots \binom{n}{n}x^n$$

The coefficient of x^k gives the number of distinct combinations of n different objects taken k at a time.

Consider the following generating function :

$$(1-x)^{-n} = (1 + x + x^2 + x^3 + ...)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \quad ...(1)$$

The coefficient of x^k in (1) gives the ways of selecting k objects from n objects with unlimited repetitions.

5.29 COUNTING UNLABELED TREES

The problem of enumeration of unlabeled trees is more involved and requires familiarity with the concepts of generating functions and partitions.

5.30 ROOTED UNLABELED TREES

Let u_n be the number of unlabeled, rooted trees of n vertices and let $u_n(m)$ be the number of those rooted trees of n vertices in which the degree of the root is exactly m. Then

$$u_n = \sum_{m=1}^{n-1} u_n(m).$$

For example, In Fig. 5.93 below, an 11-vertex, rooted tree is composed of four rooted subtrees.

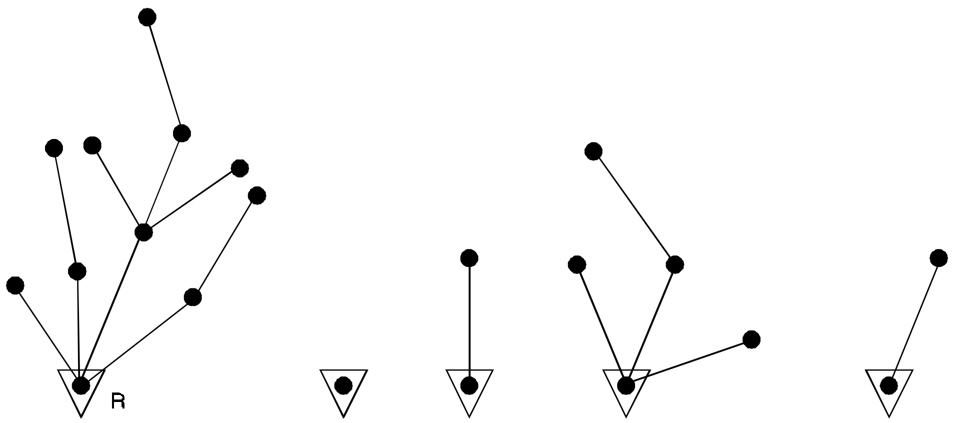


Fig. 5.93. Rooted tree decomposed into rooted subtrees.

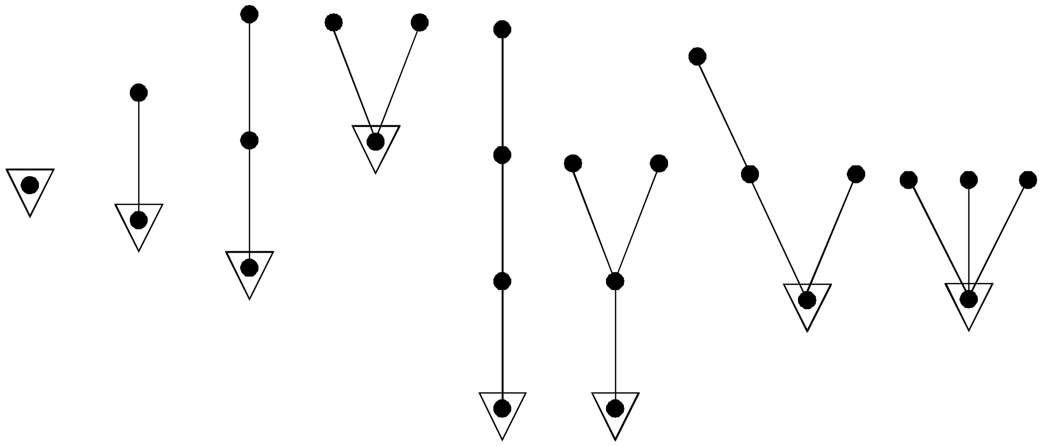


Fig. 5.94. Rooted, ublabeled trees of one, two, three, and four.

5.31 COUNTING SERIES FOR u_n

To circumvent some of these difficulties in computation of u_n, let us find its counting series, *i.e.*, the generating function, $u(x)$, where

$$u(x) = u_1x + u_2x^2 + u_3x^3 + ...$$

$$= \sum_{n=1}^{\infty} u_n x^n = x \sum_{n=1}^{\infty} u_n x^{n-1} .$$

5.32 FREE UNLABELED TREES

Let $t'(x)$ be the counting series for centroidal trees and $t''(x)$ be the counting series for bicentroidal trees. Then $t(x)$, the counting series for all trees, is the sum of the two. That is $t(x) = t'(x) + t''(x)$.

Thus the number of bicentroidal trees with $n = 2m$ vertices is given by $t''_n = \binom{u_m + 1}{2} =$

$$\frac{u_m(u_m+1)}{2}$$

and $$t''(x) = \sum_{m=1}^{\infty} \frac{u_m(u_m+1)}{2} x^{2m}$$

$$= \frac{1}{2} \sum_{m=1}^{\infty} u_m x^{2m} + \frac{1}{2} \sum_{m=1}^{\infty} (u_m x^m)^2$$

$$= \frac{1}{2}\ u(x^2) + \frac{1}{2} \ \sum_{m=1}^{\infty} (u_m x^m)^2 .$$

5.33 CENTROID

In a tree T, at any vertex v of degree d, there are d subtrees with only vertex v in common. The weight of each subtree at v is defined as the number of branches in the subtree. Then the weight of the vertex v is defined as the weight of the heaviest of the subtrees at v. A vertex with the smallest weight in the entire tree T is called a centroid of T.

Every tree has either one centroid or two centroids. If a tree has two centroids, the centroids are adjacent.

In Fig. 5.95 below, a tree with centroid, called a centroidal tree, and a tree with two centroids, called a bicentroidal tree. The centroids are shown enclosed in circles, and the numbers next to the vertices are the weights.

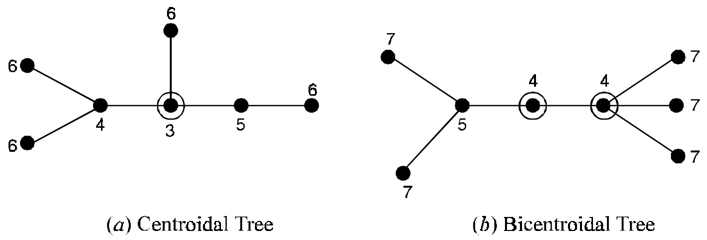


Fig. 5.95. Centroid and bicentroids.

5.34 PERMUTATION

On a finite set A of some objects, a permutation π is a one-to-one mapping from A onto intself.

For example, consider a set $\{a, b, c, d\}$.

A permutation $\pi_1 = \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$ takes a into b, b into d, c into c, and d into a.

Alternating, we could write $\pi_1(a) = b$, $\pi_1(b) = d$, $\pi_1(c) = c$, $\pi_1(d) = a$.

The number of elements in the object set on which a permutation acts is called the degree of the permutation.

For example, the permutation $\pi_1 = \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$ is represented diagrammatically by Figure 5.96 below :

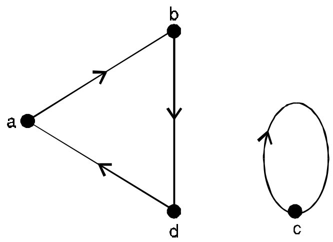


Fig. 5.96. Digraph of a permutation.

Permutation $\begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$ can be written as $(a\ b\ d)\ (c)$.

The number of edges in a permutation cycle is called the length of the cycle in the permutation.

A permutation π of degree k is said to be of type $(\sigma_1, \sigma_2, ... \sigma_k)$ if π has σ_i cycles of length i for $u = 1, 2, ... k$.

For example, permutation is of type (2, 0, 2, 0, 0, 0, 0, 0).

Clearly, $1\sigma_1 + 2\sigma_2 + 3\sigma_3 + ... + k\sigma_k = k$.

5.34.1. Composition of Permutation

Consider the two permutations π_1 and π_2 on an object set {1, 2, 3, 4, 5} : $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$

and $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$.

A composition of these two permutations $\pi_2\pi_1$ is another permutation obtained by first applying π_1 and then applying π_2 on the resultant.

That is,
$$\pi_2\pi_1(1) = \pi_2(2) = 4$$
$$\pi_2\pi_1(2) = \pi_2(1) = 3$$
$$\pi_2\pi_1(3) = \pi_2(4) = 2$$
$$\pi_2\pi_1(4) = \pi_2(5) = 5$$
$$\pi_2\pi_1(5) = \pi_2(3) = 1$$

Thus $\pi_2\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$.

5.34.2. Permutation Group

A collection of m permutations $P = \{\pi_1, \pi_2, ..., \pi_m\}$ acting on a set $A = \{a_1, a_2, ..., a_k\}$ forms a group under composition, if the four postulates of a group, that is, closure, associativity, identity, and inverse are satisfied. Such a group is called a permutation group.

The number of permutations m in a permutation group is called its order, and the number of elements in the object set on which the permutations are acting is called the degree of the permutation group.

For example, the set of four permutations

$\{(a)\ (b)\ (c)\ (d),\ (a\ c)\ (b\ d),\ (a\ b\ c\ d),\ (a\ d\ c\ b)\}$ acting on the object set $\{a, b, c, d\}$ forms a permutaion group.

5.34.3. Cycle Index of a Permutation Group

For a permutation group P, of order *m*, if we add the cycle structures of all *m* permutations in P and divide the sum by *m*, we get an expression called the cycle index Z(P) of P.

For example, the cycle index of S_3, the full symmetric group of degree three,

$$Z(S_3) = \frac{1}{6}(y_1^3 + 3y_1y_2 + 2y_3).$$

The cycle index of the permutation group is

$$\frac{1}{4}(y_1^4 + y_2^2 + 2y_4)$$

We have six permutations of type (2, 1, 0, 0) on the object set {*a*, *b*, *c*, *d*} :

(*a*) (*b*) (*c d*), (*a*) (*c*) (*b d*), (*a*) (*d*) (*b c*),
(*b*) (*c*) (*a d*), (*b*) (*d*) (*a c*), (*c*) (*d*) (*a b*).

The cycle index of S_4 : $Z(S_4) = \frac{1}{24}(y_1^4 + 6y_1^2y_2 + 8y_1y_3 + 3y_2^2 + 6y_4)$.

Permutation type	*Number of such permutations*	*Cycle structures*
(4, 0, 0, 0)	1	y_1^4
(2, 1, 0, 0)	6	$y_1^2y_2$
(1, 0, 1 , 0)	8	y_1y_3
(0, 2, 0, 0)	3	y_2^2
(0, 0, 0, 1)	6	y_4

5.34.4 Cycle Index of the Pair Group

When the *n* vertices of a group G are subjected to permutation, the $\frac{n(n-1)}{2}$ unordered vertex pair also get permuted.

For example, Let V = {*a*, *b*, *c*, *d*} be the set of vertices of a four-vertex graph. The permutation $\beta = \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$ on the vertices induces the following permutation on the six unordered vertex pairs :

$$\beta' = \begin{pmatrix} ab & ac & ad & bc & bd & cd \\ db & da & dc & ba & bc & ac \end{pmatrix}.$$

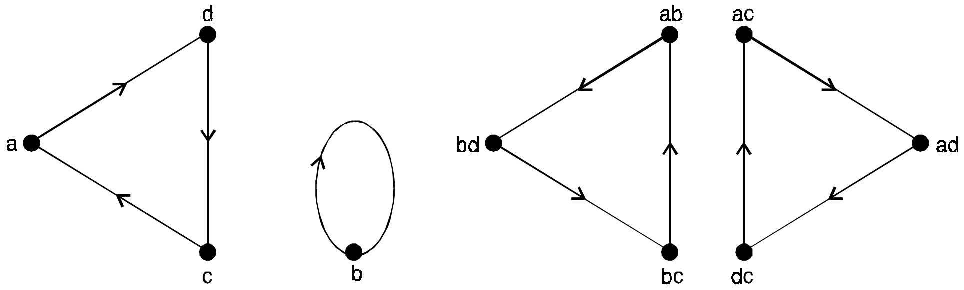


Fig. 5.97. Permutation on vertex set and the induced permutation on vertex-pair set.

5.35 EQUIVALENCE CLASSES OF FUNCTIONS

Consider two sets D and R, with the number of elements $|D|$ and $|R|$ respectively. Let f be a mapping or function which maps each element d from doamin D to a unique image $f(d)$ in range R. Since each of the $|D|$ elements can be mapped into any of $|R|$ elements, the number of different functions from D to R is $|R|^{|D|}$.

Let there be a permutation group P on the elements of set D. Then define two mappings f_1 and f_2 as P-equivalent if there is some permutation π in P such that for every d in D we have

$$f_1(d) = f_2[\pi(d)] \qquad ...(1)$$

The relationship defined by (1) is an equivalence relation can be shown as follows :

(*i*) *Since P is a permutation group, it contains the identity permutation and thus (1) is reflexive.*

(*ii*) *If P contains permutation π, it also contains the inverse permutation π^{-1}. Therefore, the relation is symmetric also.*

(*iii*) *Furthermore, if P contains permutations π_1 and π_2, it must also contain the permutation $\pi_1\pi_2$. This makes P-equivalence a transitive relation.*

The permutation group P on D is the set of all those permutations that can be produced by rotations of the cube. These permutations with their cycle structures are :

(*i*) *One identity permutation. Its cycle structure is y_1^8.*

(*ii*) *Three 180° rotations around lines connecting the centers of opposite faces. Its cycles structure is y_2^4.*

(*iii*) *Six 90° rotations (clockwise and counter clockwise) around lines connecting the centers of opposite faces. The cycle structure is y_4^2.*

(*iv*) *Six 180° rotations around lines connecting the mid-points of opposite edges. The corresponding cycle structure is y_2^4.*

(*v*) *Eight 120° rotations around lines connecting opposite corners in the cube. The cycle structure of the corresponding permutation is $y_1^2y_3^3$.*

The cycle index of this group consisting of these 24 permutations is, therefore,

$$Z(P) = \frac{1}{24}(y_1^8 + 9y_2^4 + 6y_4^2 + 8y_1^2y_3^2).$$

Theorem 5.79. *There are n^{n-2} labeled trees with n vertices ($n \geq 2$).*

Proof. Let the n vertices of a tree T be labeled 1, 2, 3, ..., n. Remove the pendant vertex (and the edge incident on it) having the smallest label, which is, say, a_1.

Suppose that b_1 was the vertex adjacent to a_1.

Among the remaining $n - 1$ vertices.

Let a_2 be the pendant vertex with the smallest label, and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2).

This operation is repeated on the remaining $n - 2$ vertices, and then on $n - 3$ vertices, and so on.

The process is terminated offer $n - 2$ steps, when only two vertices are left.

The tree T defines the sequence $(b_1, b_2, ..., b_{n-2})$ uniquely ...(1)

For example, for the tree in Fig. 5.98(a) below the sequence is (1, 1, 3, 5, 5, 5, 9).

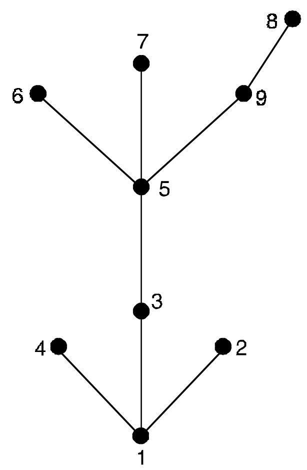


Fig. 5.98.(a) Nine vertex labeled tree, which yields sequence (1, 1, 3, 5, 5, 5, 9).

We note that a vertex i appears in sequence (1) if and only if it is not pendant.

Conversely, given a sequence (1) of $n - 2$ labels, an n-vertex tree can be constructed uniquely, as follows : Determine the first number in the sequence 1, 2, 3, ..., n .... (2) that does not appear in sequence (1).

This number clearly is a_1. And thus the edge (a_1, b_1) is defined. Remove b_1 from sequence (1) and a_1 from (2).

In the remaining sequence of (2) find the first number that does not appear in the remainder of (1). This would be a_2, and thus the edge (a_2, b_2) is defined.

The construction is continued till the sequence (1) has no element left.

Finally, the last two vertices remaining in (2) are joined.

For example, given a sequence (4, 4, 3, 1, 1).

We can construct a seven-vertex tree as follows : (2, 4) is the first edge. The second is (5, 4). Next, (4, 3). Then (3, 1), (6, 1), and finally (7, 1), as shown in Fig. 5.98(*b*) below :

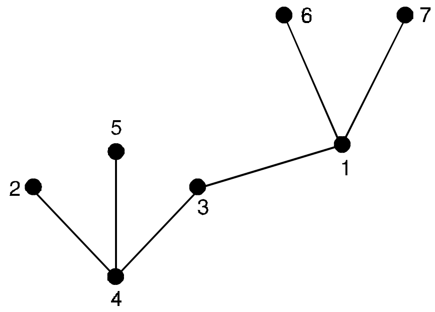


Fig. 5.98.(*b*) Three constructed from sequence (4, 4, 3, 1, 1).

For each of the $n - 2$ elements in sequence (1) we can choose any one of n numbers, thus forming n^{n-2} ...(3)

$(n - 2)$-tuples, each defining a distinct labeled tree of n vertices. And since each tree defines one of these sequences uniquely, there is a one-to-one correspondence between the trees and the n^{n-2} sequences. Hence the theorem.

Theorem 5.80. *The number of different rooted, labeled trees with n vertices is* n^{n-1}.

Theorem 5.81. ***Pólya's theorem***

The configuration-counting series B(x) is obtained by substituting the figure-counting $A(x^i)$ for each y_i in the cycle index $Z(P ; y_1, y_2, \ldots\ldots y_k)$ of the permutation group P.

That is, $B(X) = Z(P ; \Sigma a_q x^q, \Sigma a_q x^{2q}, \Sigma a_q x^{3q}, \ldots\ldots, \Sigma a_q x^{kq})$.

Theorem 5.82. *Let A be a permutation group acting on set X with orbits $\theta_1, \theta_2, \ldots\ldots, \theta_n$, and W be a function which assigns a weight to each orbit. Furthermore, W is defined on X so that $w(x) = W(\theta_i)$ whenever $x \in \theta_i$. Then the sum of the weights of the orbits is given by* $| A | \sum_{i=1}^{n} W(\theta_i) = \sum_{\alpha \in A} \sum_{x = \alpha x} W(x)$.

Proof. We have, the order | A | of the group A is the product | A(x) |. | θ(x) | for any x in X, where A(x) is the stabilizer of x.

Also, since the weight function is constant on the elements in a given orbit, we see that

$$| \theta_i | W(\theta_i) = \sum_{x \in \theta_i} W(x), \text{ for each orbit } \theta_i.$$

Combining these facts, we find that

$$| A | W(\theta_i) = \sum_{x \in \theta_i} | A(x) | W(x)$$

Summing over all orbits, we have

$$|A| \sum_{i=1}^{n} W(\theta_i) = \sum_{i=1}^{n} \sum_{x \in \theta_i} |A(x)| W(x)$$

Corollary (BURNSIDE'S LEMMA)

The number N(A) of orbits of the permutation group A is given by $N(A) = \dfrac{1}{|A|} \sum_{\alpha \in A} j_i(\alpha)$.

Proof. Let A be a permutation group of order m and degree d. The cycle index Z(A) is the polynomial in d variables $a_1, a_2, \ldots\ldots a_d$ given by the formula

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{d} a_k^{jk(\alpha)}$$

Since, for any permutation α, the numbers $j_k = j_k(\alpha)$ satisfy $1j_1 + 2j_2 + \ldots\ldots d_{jd} = d$ they constitute a partition of the integer d.

The vector notation $(j) = (j_1, j_2, \ldots\ldots, j_d)$ in describing α.

For example, the partition 5 = 3 + 1 + 1 corresponds to the vector (*j*) = (2, 0, 1, 0, 0).

Theorem 5.83. *The number of labeled graphs with P points is* $2^{\binom{p}{2}}$.

Corollary. The number of labeled (*p*, *q*) graphs is $\binom{\binom{p}{2}}{q}$.

Theorem 5.84. *The number of ways in which a given graph G can be labeled is* $\dfrac{P!}{|\Gamma(G)|}$.

Proof. Let A be a permutation group acting on the set X of objects. For any element *x* in X, the orbit of *x*, denoted θ(*x*), is the subset of X which consists of all elements *y* in X such that for some permutation α in A, α*x* = *y*.

The stabilizer of *x*, denoted A(*x*), is the subgraph of A which consists of all the permutations in A which leaves *x* fixed.

The result follows from an application of the well-known formula $|\theta(x)| \,.\, |A(x)| = |A|$.

Theorem 5.85. ***Pólya's Enumeration theorem***

The configuration counting series is obtained by substituting the figure counting series into the cycle index of the configuration group, C(x, y) = Z(c(x, y)).

Proof. Let α be a permutation in A, and let $\tilde{\alpha}$ be the corresponding permutation in the power group E^A.

Assume first that *f* is a configuration fixed by $\tilde{\alpha}$ and that ζ is a cycle of length *k* in the disjoint-cycle decomposition of α.

Then $f(b) = f(\zeta b)$ for every element b in the representation of ζ, so that all elements permuted by ζ must have the same image under f.

Conversely, if the elements of each cycle of the permutation α have the same image under a configuration f, then $\tilde{\alpha}$ fixes f.

Therefore, all configurations fixed by $\tilde{\alpha}$ are obtained by independently selecting an element r in R for each cycle ζ of α and setting $f(b) = r$ for all b permuted by ζ. Then if the weight W(r) is (m, n) where $m = W_1 r$ and $n = W_2 r$ and ζ has length k, the cycle ζ contributes a factor of $\sum_{r \in R} (x^m y^n)^k$ to the sum

$\Sigma_f = \tilde{\alpha} f \; W(f)$.

Therefore, since $\sum_{r \in R} (x^m \; y^n)^k = c(x^k, y^k)$.

We have, for each α in A,

$$\sum_{f = \tilde{\alpha} f} W(f) = \prod_{k=1}^{s} C(x^k, y^k)^{jk(\alpha)} .$$

Summing both sides of this equation over all permutations α in A (or equivalently over all $\tilde{\alpha}$ in E^A) and dividing both sides by $|A| = |E^A|$,

We obtain,

$$\frac{1}{|E^A|} \sum_{\tilde{\alpha} \in E^A} \sum_{f = \tilde{\alpha} f} W(f) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{S} C(x^k, y^k)^{jk(\alpha)} .$$

The right hand side of this equation is Z(A, C(x, y)).

To see that the left hand side is C(x, y).

Corollary. If A is a permutation group acting on X, then the number of orbits of n-subsets of X induced by A is the coefficient of x^n in Z(A, 1 + x).

Theorem 5.86. *The counting polynomial for graphs with P points is*

$$g_p(x) = Z(S_P^{(2)}, 1 + x),$$

where $$Z(S_P^{(2)}) = \frac{1}{P!} \sum_{(j)} \frac{P!}{\prod_{k=1}^{P} j_k ! k^{jk}} \prod_{k=1}^{[P/2]} (a_k a_{2k}^{k-1})^{j_{2k}} .$$

$$\prod_{k=0}^{[(P-1)/2]} a_{2k+1}^{kj_{2k+1}} \prod_{k=1}^{[P/2]} a_k^{k\binom{jk}{2}} \prod_{1 \le r < s < P-1} a_{m(r,s)}^{d(r,s) j_r j_s} .$$

Corollary 1. The counting polynomial for rooted graphs with P points is

$$r_P(x) = Z((S_1 + S_{P-1})^{(2)}, 1 + x).$$

When there are atmost two lines joining each pair of points, we need only replace the figure counting series for graphs by $1 + x + x^2$.

Corollary 2. The counting polynomial for multigraphs with at most two lines joining each pair of points is

$$g_P''(x) = Z(S_P^{(2)}, 1 + x + x^2)$$

For arbitrary multigraphs, the figure counting series becomes

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}.$$

Corollary 3. The counting polynomial for multigraphs with P points is $m_P(x) = Z\left(S_P^{(2)}, \frac{1}{1-x}\right)$.

Theorem 5.87. *The counting polynomial for digraphs with P points is* $d_P(x) = Z\left(S_P^{(2)}, 1 + x\right)$.

where
$$Z\left(S_P^{(2)}\right) = \frac{1}{P!}\sum_{(j)} \frac{P!}{\prod_{k=1}^{P} j_k!\,k^{j_k}} \prod_{k=1}^{P} a_k^{(k-1)j_k + 2k\binom{j_k}{2}} \cdot \prod_{1 \le r \le S \le P-1} a_{m(r,S)}^{2 j_r j_s d(rS)}.$$

Theorem 5.88. *The number S_P of self-complementary graphs on P points is $S_P = Z(S_P^{(2)}; 0, 2, 0, 2, ...)$.*

Theorem 5.89. *Identity trees are counted by the equations*

$$U(x) = x \exp \sum_{n=1}^{\infty} (-1)^{n+1} \frac{U(x^n)}{n}$$

$$u(x) = U(x) - \frac{1}{2}[U^2(x) + U(x^2)]$$

The number of identity trees through 12 points is given by

$$u(x) = x + x^7 + x^8 + 3x^9 + 6x^{10} + 15x^{11} + 29x^{12} + ...$$

Theorem 5.90. *The counting series for rooted trees is given by*

$$T(x) = x \prod_{r=1}^{\infty} (1 - x^r)^{-T_r}.$$

Theorem 5.91. *The counting series for rooted trees satisfies the functional equation*

$$T(x) = x \exp \prod_{r=1}^{\infty} \frac{1}{r}(x^r) \qquad ...(1)$$

Proof. Let $T^{(n)}(x)$ be the generating function for those rooted trees in which the root has degree n, so that

$$T(x) = \prod_{n=0}^{\infty} T^{(n)}(x) \qquad ...(2)$$

For example, $T^{(0)}(x) = x$ counts the rooted trivial graph, while the planted trees (rooted at an end point) are counted by $T^{(1)}(x) = xT(x)$.

In general a rooted tree with root degree n can be regarded as a configuration whose figures are the n rooted trees obtained on removing the root. Fig. 5.99 below, illustrates this for $n = 3$.

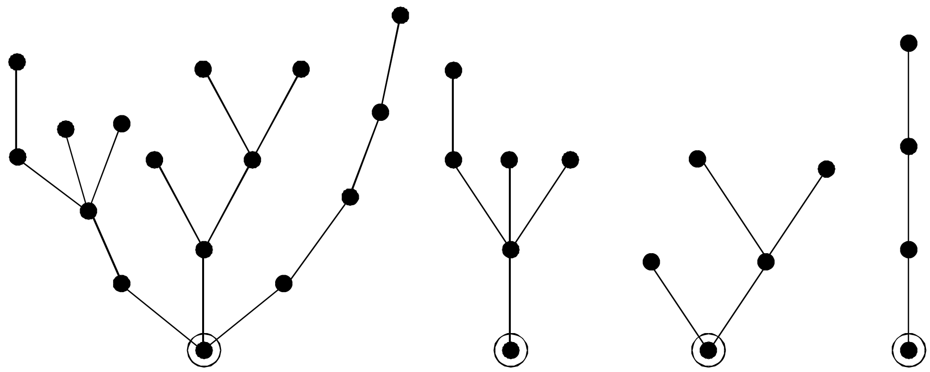


Fig. 5.99. A given rooted tree T and its constituent rooted trees.

Since these n rooted trees are mutually interchangeable without altering the isomorphism class of the given rooted tree, the figure counting series is $T(x)$ and the configuration graph is S_n, giving $T^{(n)}(x) = xZ(S_n, T(x))$ ...(3)

The factor x accounts for the removal of the root of the given tree since the weight of a tree is the number of points.

Fortunately, there is a well-known and easily derived identity which may now be invoked (where $Z(S_0)$ is defined as 1)

$$\sum_{n=0}^{\infty} Z(S_n, h(x)) = \exp \sum_{r=1}^{\infty} \frac{1}{r} h(x^r) \qquad ...(4)$$

On combining the last three equations, we obtain (1)

Theorem 5.92. *Homeomorphically irreducible trees are counted by the three equations,*

$$\overline{H}(x) = \frac{x^2}{1+x} \exp \sum_{r=1}^{\infty} \frac{\overline{H}(x^r)}{rx^r} \qquad ...(1)$$

$$H(x) = \frac{1+x}{x} \overline{H}(x) - \frac{1}{2x}[\overline{H}^2(x) - \overline{H}(x)^2] \qquad ...(2)$$

$$h(x) = \mathrm{H}(x) - \frac{1}{x^2}\ [\overline{\mathrm{H}}^2(x) - \overline{\mathrm{H}}(x^2)] \qquad ...(3)$$

The number of homeomorphically irreducible trees through 12 points is found to be :

$$h(x) = x + x^2 + x^4 + x^5 + 2x^6 + 2x^7 + 4x^8 + 5x^9 + 10x^{10} + 14x^{11} + 26x^{12} + ... \qquad ..(4)$$

Theorem 5.93. ***Power Group Enumeration Theorem***

The number of equivalence classes of functions in R^D determined by the power group B^A is

$$N(B^A) = \frac{1}{|B|} \sum_{\beta \in B} (A\,; m_1(\beta), m_2(\beta), ..., m_l(\beta)) \text{ where } m_k(\beta) = \sum_{s/k} S j_S(\beta).$$

Theorem 5.94. *The configuration counting series C(x) for 1 – 1 functions from a set of n interchangeable elements into a set with figure counting series C(x) is obtained by substituting C(x) into $Z(A_n - S_n)$:*

$$C(x) = Z(A_n - S_n\,,\ C(x)).$$

Theorem 5.95. *For any tree T, let p* and q* be the number of similarity classes of points and lines, respectively, and let S be the number of symmetry lines. Then S = 0 or 1 and*

$$p^* - (q - S) = 1 \qquad ...(1)$$

Proof. Whenever T has one central point or two dissimilar central points, there is no symmetry line, so S = 0.

In this case there is a subtree of T which contains exactly one point from each similarity class of points in T and exactly one line from each class of lines.

Since this subtree has p^* points and q^* lines, we have $p^* - q^* = 1$.

The other possibility is that T has two similar central points and hence S = 1.

In this case there is a subtree which contains exactly one point from each similarity class of points in T and, except for the symmetry line, one line from each class of lines.

Therefore, this subtree has p^* points and $q^* - 1$ lines and so $p^* - (q^* - 1) = 1$.

Thus, in both cases (1) holds.

Theorem 5.96. *The counting series for trees in terms of rooted trees is given by the equation*

$$t(x) = T(x) - \frac{1}{2}\,[T^2(x) - T(x^2)] \qquad ...(1)$$

Proof. For $i = 1$ to t_n, let p_i^*, q_i^* and S_i be the numbers of similarity classes of points, lines, and symmetry lines for the ith tree with n points.

Since $1 = p_i^* - (q_i^* - S_i)$ for each i, by $p^* - (q^* - S) = 1$, we sum over i to obtain

$$t_n = T_n - \sum_i (q_i^* - S_i) \qquad ...(2)$$

Furthermore $\Sigma(q_i^* - S_i)$ is the number of trees having n points which are rooted at a line, not a symmetry line. Consider a tree T and take any line y of T which is not a symmetry line.

Then T – y may be regarded as two rooted trees which must be non isomorphic.

Thus each non-symmetry line of a tree corresponds to an unordered pair of different rooted trees.

Problem Set 5.1

1. Define
(*i*) Graph coloring (*ii*) Chromatic number (*iii*) Chromatic polynomial

2. Write a short note on :
(*i*) Color problem (*ii*) Matching theory

3. Define
(*i*) Covering (*ii*) Independence (*iii*) Edge covering
(*iv*) Vertex covering

4. Define
(*i*) Critical points and critical lines (*ii*) Line-core and point-core

5. Define
(*i*) Digraph (*ii*) Orientation of a graph (*iii*) Underlying graph
(*iv*) Parallel edges (*v*) Incidence (*vi*) In-degree and out-degree

6. Define
(*i*) Pendant vertex (*ii*) Isolated vertex

7. Define
(*i*) Simple digraphs (*ii*) Isomorphic digraphs (*iii*) Regular digraphs
(*iv*) Complete digraphs

8. Define
(*i*) Connected digraphs (*ii*) Strongly and weakly connected
(*iii*) Components and Fragments (*iv*) Reachability
(*v*) Arborescence (*vi*) Accessibility

9. Define
(*i*) Euler digraphs (*ii*) Hand shaking dilemma
(*iii*) Incidence and circuit matrix of a digraph

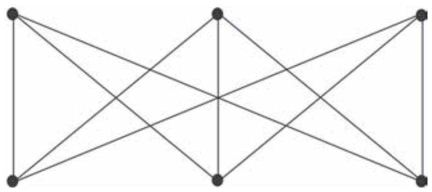
10. Define
(*i*) Types of enumeration (*ii*) Enumeration of a graphs and trees
(*iii*) Labelled graphs (*iv*) Rooted labelled trees

11. Define
(*i*) Counting unlabelled trees and rooted unlabelled trees (*ii*) Centroid
(*iii*) Free unlabelled trees (*iv*) Permutation group (*v*) Composition of permutation

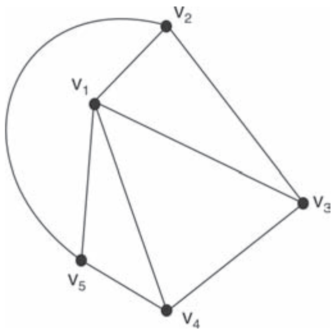
12. Define
(*i*) Cycle index of a permutation group (*ii*) Cycle index of a pair group
(*iii*) Equivalence classes of functions

13. If G = (V, E) is a connected graph and $e = \{a, b\} \in E$ then prove that $P(Ge, \lambda) = P(G, \lambda) + P(Ge', \lambda)$.

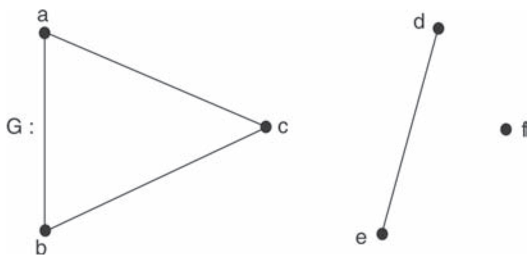
14. Find the chromatic polynomial and chromatic number for the graph $K_{3, 3}$.



15. Find the chromatic polynomial of the graph given in figure below



16. Find the chromatic polynomial and hence the chromatic number for the graph shown below



17. If G is a simple graph with maximum vertex degree Δ then show that $\Delta \leq \chi'(G) \leq \Delta + 1$.

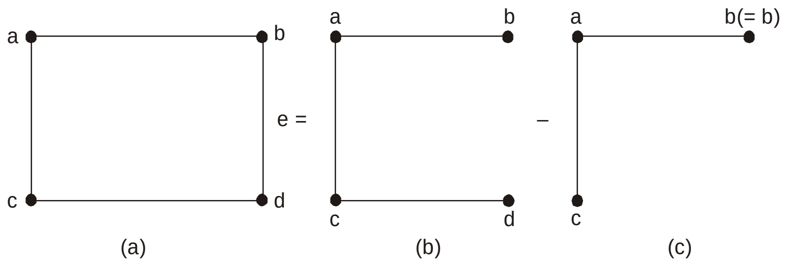
18. Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G then show that $\chi(G) \leq 1 + \Delta(G)$.

19. If G is a planar graph then show that $\chi(G) \leq 5$.

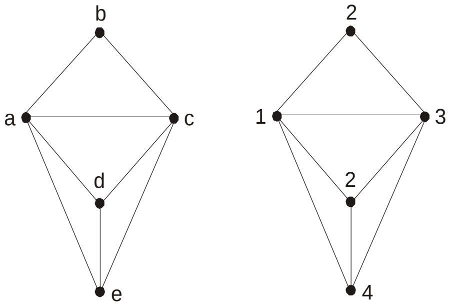
20. Let $G = (V, E)$ with $|E| > 0$ then show that the sum of the coefficients in $P(G, \lambda)$ is 0.

21. Show that, for each graph G, the constant term in $P(G, \lambda)$ is 0.

22. Using the decomposition theorem find the chromatic polynomial and hence the chromatic number for the graph below in figure.



23. Show that $\chi(G) = 4$ for the graph of G of figure below



24. What is the chromatic number of the graph C_n ?

25. In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed ?

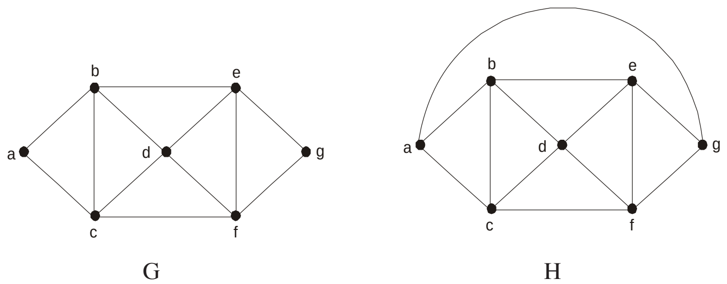
26. Television channels 2 through 13 are assigned to stations in New Delhi so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring ?

27. How can the final exams at a university be scheduled so that no student has two exams at the same time ?

28. What is the chromatic number of K_n ?

29. What is the chromatic number of the complete bipartite graph $K_{m,\, n}$ where m and n are positive integers ?

30. What is the chromatic numbers of the graph's G and H shown in figure below.



31. Prove that, A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \,.......\, (\lambda - n + 1)$$

32. Let a and b be two non adjacent vertices in a graph G. Let G′ be a graph obtained by adding an edge between a and b. Let G″ be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges then show that

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

33. Prove that, for any graph G, the sum and product of χ and $\bar{\chi}$ satisfy the inequalities

$$2\sqrt{P} \leq \chi + \bar{\chi} \leq P + 1$$

$$P \leq \chi\bar{\chi} \leq \left(\frac{P+1}{2}\right)^2$$

34. Prove that, for any graph G, $\dfrac{P}{\beta_0} \leq \chi \leq P - \beta_0 + 1$.

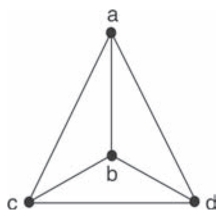
35. Prove that, for any graph G, $\chi(G) \leq 1 + \max\ \delta(G')$.

36. Show that, A graph G is 2-chromatic if and only if it is a non-null bipartite graph.

37. Prove that, A graph G is 2-chromatic if and only if G is bipartite.

38. If G is a graph with n vertices and degree δ then show that $\chi(G) \geq \dfrac{n}{n - \delta}$.

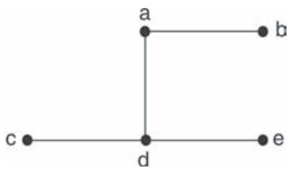
39. Find all maximal independent sets of the following graph



40. Prove that ; An n-vertex graph is a tree if and only if its chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

41. How many ways a tree on 5 vertices can be properly colored with atmost 4 colors.

42. Find all possible maximal independent sets of the following graphs using Boolean expression.



43. Write down the chromatic polynomial of the graph $K_4 - e$.

44. Prove that a simple planar graph G with less than 30 edges in 4-colorable.

45. Prove that, for a graph G with n vertices

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

46. Find the chromatic number of a cubic graph with $P \geq 6$ vertices.

47. Show that the chromatic number of a graph G is $\lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1)$ if and only if G is a complete graph on n vertices.

48. Prove that every planar graph is 4-colorable.

49. Prove that every planar graph is 5-colorable.

50. Prove that every planar graph with fewer than 4 triangles is 3-colorable.

51. Let D be a strongly connected digraph with n vertices. If out degree $(v) \geq \frac{n}{2}$ and in deg $(v) \geq \frac{n}{2}$ for each vertex v then show that D is Hamiltonian.

52. Let D be a digraph with an odd number of vertices prove that if each vertex of D has an odd out-degree then D has an odd number of vertices with odd in-degree.

53. Prove that, an arboresence is a tree in which every vertex other than the root has an indegree of exactly one.

54. Prove that, a simple digraph G of n vertices and $n - 1$ directed edges in an arboresence rooted at v_1 if an only if the (1, 1) cofactor of K(G) is equal to 1.

55. Prove that a complete symmetric digraph of n vertices contains $n(n - 1)$ edges and a complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$ edges.

56. Prove that a connected digraph D that does not contain a closed directed walk must have a source and a sink.

57. If Q amd R are K by M and M × K matrices respectively with K < M then prove that the determinant of the product det (QR) = the sum of the products of corresponding major determinants of Q and R.

58. If Q is a K by n matrix and R is any n by P matrix then prove that the nullity of the product cannot exceed the sum of the nullities of the factors.

i.e., nullity of QR ≤ nullity of Q + nullity of R.

59. If G is a (p, q)-graph whose points have degrees d_i then show that L(G) and q points and q_L lines where $q_L = -q + \frac{1}{2} \Sigma d_i^2$.

60. Prove that there are n^{n-2} labelled trees with n vertices $(n \geq 2)$.

61. Prove that, the configuration counting series is obtained by substituting the figure counting series into the cycle index of the configuration group

$$C(x, y) = Z\langle C(x, y)\rangle.$$

62. Prove that, the number N(A) of orbits of the permutation group A is given by N(A) = $\frac{1}{|A|}\sum_{\alpha \in A} j_i(\alpha)$.

63. Let A be a permutation group acting on set X with orbits $\theta_1, \theta_2, \ldots\ldots \theta_n$ and W be a function which assigns a weight to each orbit. Furthermore, W is a defined on X so that $w(x) = W(\theta_i)$ whenever $x \in \theta_i$ then prove that the sum of the weights of the orbits is given by

$$|A| \sum_{i=1}^{n} W(\theta_i) = \sum_{\alpha \in A} \sum_{x = \alpha x} W(x).$$

64. Prove that, the counting series for rooted trees satisfies the functional equation

$$T(x) = x \exp \prod_{r=1}^{\infty} \frac{1}{r} (x^r)$$

65. Prove that, the counting series for trees in terms of rooted trees is given by the equation

$$t(x) = T(x) - \frac{1}{2} [T^2(x) - T(x^2)]$$

66. For any tree T, let p^* and q^* be the number of similarity classes of points and lines respectively, and let S be the number of symmetry lines then show that S = 0 or 1 and $P^* - (q - S) = 1$.

Problem Set 5.2

1. Prove that every edge in a digraph belongs to either a directed circuit or a directed cut-set.

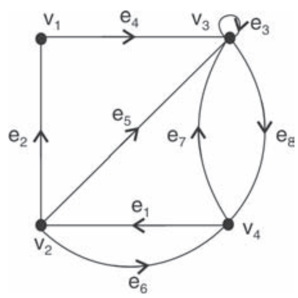
2. Let D be the digraph whose vertex is

$V = \{v_1, v_2, v_3, v_4, v_5\}$ and the edge set is

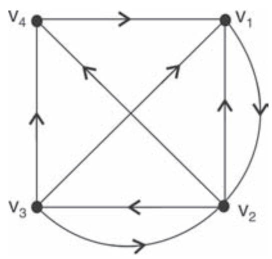
$E = \{(v_1, v_4), (v_2, v_3), (v_3, v_5), (v_4, v_2), (v_4, v_4), (v_4, v_5), (v_5, v_1)\}$

Write down a diagram of D and indicate the out degree and indegree of vertices.

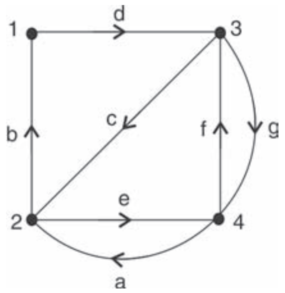
3. Find a directed Eulerian line and a spanning arborescence in the digraph shown in figure below



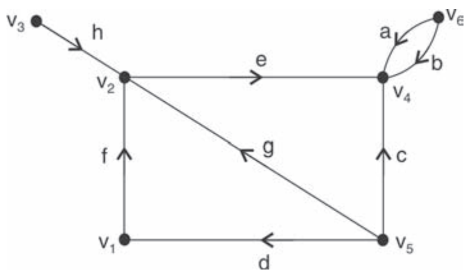
4. Find the adjacency matric for the digraph shown in figure below



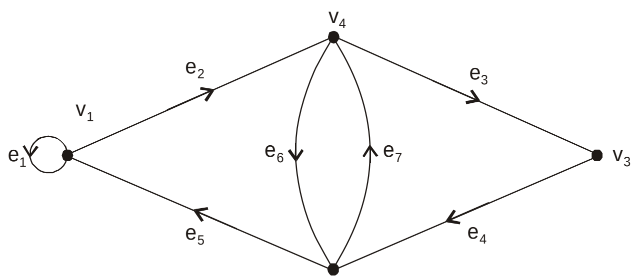
5. For the digraph in figure below, find all arborescence rooted at the vertex 4.



6. For the digraph in figure below, find the incidence matrix.

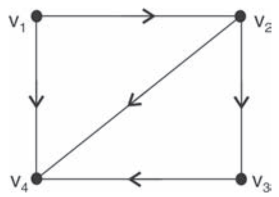


7. Show that the following is an Euler digraph. Find a directed Euler line in it.

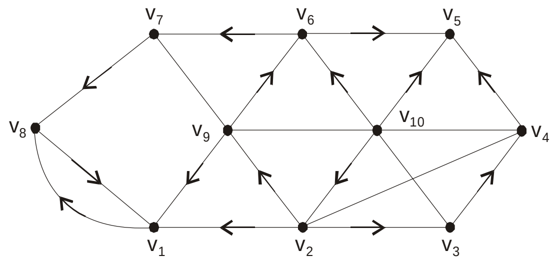


8. Prove that a connected digraph D is an Euler digraph if and only if $d^-(v) = d^+(v)$ for every vertex v of D.

9. For the digraph shown in figure below, find a path of maximum length.



10. Find the fragments and condensation of the digraph shown in figure below.



11. Let B and A be respectively the circuit matrix and incidence matrix of a selt-loop free digraph such that the columns in B and A are arranged using the same order of edges then show that $A \cdot B^T = B \cdot A^T = 0$.

12. Prove that digraph G is a cyclic if and only if det (I – X) is not equal to zero, where I is the identity matrix of the same size as X.

13. Prove that an n-vertex digraph is strongly connected if and only if the matrix M defined by $M = x + x^2 + x^3 + \ldots\ldots + x^n$, has no zero entry, x is the adjacency matrix.

14. Prove that, there exists a digraph with outdegree sequence $(S_1, S_2, \ldots\ldots S_P)$ where $P - 1 \geq S_1 \geq S_2 \geq S_P$ and indegree sequence $(t_1, t_2, \ldots\ldots t_n)$ where every $t_j \geq P - 1$ if and only if $\Sigma S_i = \Sigma t_i$ and

for each integer $K < P$, $\sum_{i=1}^{k} S_i \leq \sum_{i=1}^{k} \min\{K-1, t_i\} + \sum_{i=K+1}^{P} \min\{K, t_i\}$.

15. Prove that the determinant of every square submatrix of A, the incidence matrix of a digraph 1, – 1 or 0.

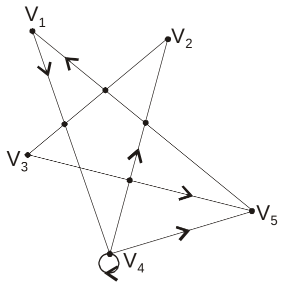
16. Let A be the adjacency matrix of the line digraph of a complete symmetric digraph then show that $A^2 + A$ has all entries 1.

17. Let D be a primitive digraph, if n is the smallest integer such that $A^n > 0$ then show that $n \geq (P - 1)^2 + 1$.

18. Let D be a primitive digraph, if n has the maximum possible value $(P - 1)^2 + 1$, then show that there exists a permutation matrix P such that PAP^{-1} has the form $[a_{ij}]$ where $a_{ij} = 1$ whenever $j = i + 1$ and $a_{P,1} = 1$ but $a_{ij} = 0$ otherwise.

19. Show that if a set of permutation P on an object set S forms a group, the set R of all permutations induced by P on set S × S along forms a group.

Answers 5.2

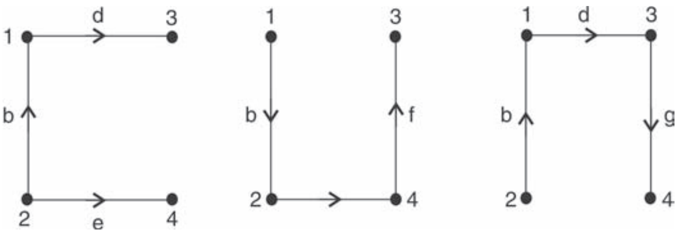
2. 

Vertices	v_1	v_2	v_3	v_4	v_5
d^+	1	1	1	3	1
d^-	1	1	1	2	2

3. Directed Euler line = $e_2e_4e_3e_5e_6e_7e_8e_1$

Spanning arborescence : $\{e_2, e_4, e_6\}$ rooted at v_2.

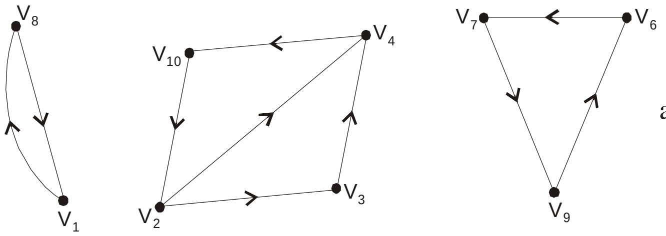
4. $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

5. 

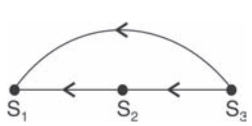
6. $$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. $v_1e_1v_1e_2v_4e_6v_2e_7v_4e_3v_3e_4v_2e_5v_1$

8. $v_1v_2v_4v_3$

10. 

s_1 s_2 s_3

and 

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